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A Generalisation of an Expansion for the Riemann Zeta Function Involving Incomplete Gamma Functions

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Abstract

We derive an expansion for the Riemann zeta function $\zeta(s)$ involving incomplete gamma functions with their second argument proportional to n^{2p}, where n is the summation index and p is a positive integer. The possibility is examined of reducing the number of terms below the value $N_t \simeq (t/2\pi)^{1/2}$ in the finite main sum appearing in asymptotic approximations for $\zeta(s)$ on the critical line $s = \frac{1}{2} + it$ as $t \to \infty$. It is shown that the expansion corresponding to quadratic dependence on n ($p = 1$) is the best possible representation of this type for $\zeta(s)$.

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1. Introduction

A representation of the Riemann zeta function $\zeta(s)$ valid for all values of s ($\neq 1$) is given by [6]

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} Q(\tfrac{1}{2}s, \pi n^2 \eta) + \chi(s) \sum_{n=1}^{\infty} n^{s-1} Q(\tfrac{1}{2} - \tfrac{1}{2}s, \pi n^2/\eta) + \Xi(s) \qquad (1.1)$$

where η is a parameter satisfying $|\arg \eta| \leq \frac{1}{2}\pi$, $Q(a,z) = \Gamma(a,z)/\Gamma(a)$ is the normalised incomplete gamma function defined by

$$Q(a,z) = \frac{1}{\Gamma(a)} \int_z^{\infty} u^{a-1} e^{-u}\, du \qquad (|\arg z| < \pi) \qquad (1.2)$$

and

$$\chi(s)=\pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}, \qquad \Xi(s)=\frac{(\pi\eta)^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}\left(\frac{\eta^{-\frac{1}{2}}}{s-1}-\frac{1}{s}\right). \tag{1.3}$$

From the standard asymptotic behaviour for fixed a [1, p. 263]

$$\Gamma(a,z)\sim z^{a-1}e^{-z} \qquad (z\to\infty \text{ in } |\arg z|<\tfrac{3}{4}\pi), \tag{1.4}$$

the late terms ($n\gg 1$) in the sums in (1.1) behave like $n^{-2}\exp(-\pi n^2\gamma)$, where $\gamma=\eta$ for the first sum and $\gamma=1/\eta$ for the second sum. Both sums therefore converge absolutely provided $|\arg\eta|\leq\frac{1}{2}\pi$. The particular case of this expansion with $\eta=1$ was effectively embodied in Riemann's 1859 paper [2, p. 299], though he did not explicitly identify the incomplete gamma functions. The result (1.1) is the expansion given in [3] for the Dirichlet L-function specialised to $\zeta(s)$.

To preserve the symmetry of (1.1) we take $|\eta|=1$ and set $\eta=e^{i\phi}$, with $|\phi|\leq\frac{1}{2}\pi$. In numerical computations on the upper half of the critical line $s=\frac{1}{2}+it$ ($t\geq 0$), the term $\Xi(s)$ has the controlling behaviour $\exp\{(\frac{1}{4}\pi-\frac{1}{2}\phi)t\}$ for large t. When $\phi<\frac{1}{2}\pi$, this represents a numerically large factor which would require the evaluation of the terms of (1.1) to exponentially small accuracy[1]. Thus, we are effectively forced to set $\phi=\frac{1}{2}\pi$ to avoid such numerically undesirable terms when computing $\zeta(s)$ high up on the critical line. It is customary to define the real function $Z(t)=e^{i\Theta(t)}\zeta(\frac{1}{2}+it)$, where $\Theta(t)=\arg\{\pi^{-\frac{1}{2}it}\Gamma(\frac{1}{4}+\frac{1}{2}it)\}$. From the result $\chi(\frac{1}{2}+it)=\exp\{-2i\Theta(t)\}$, it then follows from (1.1) that

$$Z(t)=2\mathrm{Re}\left\{e^{i\Theta(t)}\sum_{n=1}^{\infty}n^{-s}Q(\tfrac{1}{2}s,\pi n^2 i)-\frac{\pi^{\frac{1}{4}}e^{\frac{1}{4}\pi is}}{s|\Gamma(\frac{1}{2}s)|}\right\}, \tag{1.5}$$

where in this expression $s=\frac{1}{2}+it$, $t\geq 0$.

An asymptotic formula for $Z(t)$ as $t\to+\infty$ based on (1.5) has been given in [6] by exploiting the uniform asymptotics of the incomplete gamma function. The main characteristic of (1.5) is the smoothing property of the incomplete gamma function on the original Dirichlet series $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$ in $\mathrm{Re}(s)>1$. This can be seen from the integral defining $Q(a,z)$ in (1.2) which, for large a, has a saddle point at $u=a$. This saddle point coalesces with the lower endpoint of integration when $a\simeq z$, so that the behaviour of $Q(a,z)$ can then be expected to change suddenly. For the incomplete gamma functions appearing in (1.5), this occurs when $\frac{1}{2}s\simeq\pi n^2 i$; that is, when n attains the

[1]There are compensating exponentially large terms present in the incomplete gamma functions in (1.1) when $\phi<\frac{1}{2}\pi$.

critical value $N_t = [\surd(t/2\pi)]$, where the square brackets denote the integer part. More specifically, the uniform asymptotic expansion of the incomplete gamma functions [8] shows that for $n \simeq N_t$ on the critical line [6]

$$Q(\tfrac{1}{2}s, \pi n^2 i) \sim \frac{1}{2}\text{erfc}\left\{\frac{\pi}{\surd t}\left(n^2 - \frac{t}{2\pi}\right)e^{\frac{1}{4}\pi i}\right\}, \qquad t \to +\infty,$$

where erfc denotes the complementary error function. Thus for large t, the function $Q(\frac{1}{2}s, \pi n^2 i) \simeq 1$ when $n \lesssim N_t$ and decays algebraically to zero when $n \gtrsim N_t$. Loosely speaking therefore, the terms in the sum in (1.5) effectively "switch off" when $n \simeq N_t$. To illustrate this smoothing behaviour we show in Fig. 1 a plot of $|Q(\frac{1}{2}s, \pi n^2 i)|$ as a function of n when $s = \frac{1}{2} + it$ with $t = 100$.

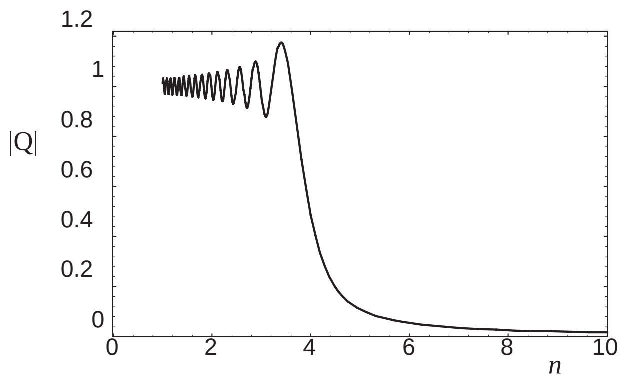

Figure 1: The behaviour of the normalised incomplete gamma function $|Q(\frac{1}{2}s, \pi n^2 i)|$ when $s = \frac{1}{2} + 100i$ as a function of n (regarded as a continuous variable for clarity).

The asymptotic formula that results from (1.5) for $Z(t)$ on the critical line consists of the finite-main sum contribution over N_t terms

$$2\sum_{n=1}^{N_t} n^{-\frac{1}{2}}\cos\{\Theta - t\log n\} \tag{1.6}$$

together with a sequence of asymptotic correction terms, and so is of the computationally more powerful Riemann-Siegel type [6]. In this paper we generalise the expansion (1.1) to examine the possibility of reducing the number of terms in the finite-main sum by allowing the dependence on the index n in the incomplete gamma functions to be proportional to n^{2p}, where p is a positive integer, instead of being quadratic. This generalisation will involve the introduction of a *generalised incomplete gamma function* $\mathcal{Q}_p(a, z)$ which we discuss in §2. Our analysis will demonstrate that (1.1) is the best possible representation of this type for $\zeta(s)$.

2. A generalised incomplete gamma function

Let p be a positive integer. We define the generalised incomplete gamma function $\Gamma_p(a,z)$, for $|\arg z|<\pi$, by

$$\Gamma_p(a,z)=\int_z^\infty \tau^{a-1}F_{2p}(\tau)\,d\tau \qquad (p=1,2,3,\ldots), \tag{2.1}$$

where

$$\begin{aligned} F_{2p}(\tau) &= \sum_{k=0}^\infty \frac{(-)^k\tau^{k/p}}{k!\,\Gamma(k+\frac{1}{2})}\Gamma\left(\frac{2k+1}{2p}\right) \qquad (|\tau|<\infty)\\ &= \frac{p}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{\Gamma(sp)}{\Gamma(\frac{1}{2}-sp)}\Gamma\left(\frac{1}{2p}-s\right)\tau^{-s}ds \qquad (c>0); \end{aligned} \tag{2.2}$$

see [5, p. 56]. When $p=1$, it is seen that $F_2(\tau)=e^{-\tau}$ and $\Gamma_1(a,z)$ reduces to the standard incomplete gamma function $\Gamma(a,z)$. Henceforth we exclude the case $p=1$ from our deliberations.

The function $F_{2p}(\tau)$ is related to the generalised hypergeometric function (or the Wright function)

$$\Psi_m(z)=\sum_{k=0}^\infty \frac{z^k}{k!\,\Gamma(k+\frac{1}{2})}\Gamma\left(\frac{2k+1}{m}\right) \qquad (|z|<\infty;\quad m>1).$$

The asymptotics of this function for large z can be obtained from the asymptotic theory of hypergeometric functions; see, for example, [5, §2.3]. We define the parameters

$$\kappa=2(1-\frac{1}{m}),\quad h=(2/m)^{2/m},\quad \vartheta'=\frac{1}{m}-\frac{1}{2},\quad X=\kappa(hz)^{1/\kappa} \tag{2.3}$$

and note that $\kappa>0$ for $m>1$. The expansion of $\Psi_m(z)$ consists of an exponentially large expansion together with an algebraic expansion in the sector $|\arg z|\le\frac{1}{2}\pi\kappa$ given by

$$\Psi_m(z)\sim E(z)+H(ze^{\mp\pi i}) \qquad (|z|\to\infty \text{ in } |\arg z|\le\tfrac{1}{2}\pi\kappa),$$

where the upper or lower sign[2] is chosen according as $\arg z>0$ or $\arg z<0$, respectively. The expansions $E(z)$ and $H(z)$ denote the formal asymptotic sums

$$E(z):=X^{\vartheta'}e^X\sum_{j=0}^\infty A_jX^{-j}, \qquad A_0=\kappa^{-\frac{1}{2}}(\tfrac{1}{2}m\kappa)^{-\vartheta'}, \tag{2.4}$$

[2]The choice of the signs in the algebraic expansion is a consequence of $\arg z=0$ being a Stokes line for $\Psi_m(z)$.

with the A_j $(j \geq 1)$ being computable coefficients that we do not discuss here, and

$$H(z) := \frac{1}{2}m \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \frac{\Gamma(\frac{1}{2}mk + \frac{1}{2})}{\Gamma(-\frac{1}{2}mk)} z^{-k-\frac{1}{2}}.$$

In the sector $|\arg(-z)| < \frac{1}{2}\pi(2-\kappa)$, the expansion $E(z)$ becomes exponentially small and the dominant expansion of $\Psi_m(z)$ in this sector is then $H(ze^{\mp\pi i})$.

If $m = 2p$ then $H(z) \equiv 0$. In this case the expansion of $\Psi_m(z)$ is exponentially small in the sector $|\arg(-z)| < \frac{1}{2}\pi(2-\kappa)$ and we find [10]

$$\Psi_m(z) \sim \begin{cases} E(z) & (|\arg z| < \pi) \\ \\ E(z) + E(ze^{\mp 2\pi i}) & (|\arg z| \leq \pi), \end{cases} \tag{2.5}$$

where the upper or lower sign is chosen according as $\arg z > 0$ or $\arg z < 0$, respectively. From (2.5) and the fact that $F_{2p}(\tau) = \Psi_{2p}(-\tau^{1/p})$, it then follows that $F_{2p}(\tau)$ is exponentially small for large positive τ with the leading behaviour

$$F_{2p}(\tau) \sim 2A_0 T^{\vartheta'} e^{T\cos(\pi/\kappa)} \cos\{T\sin(\pi/\kappa) + \pi\vartheta'/\kappa\} \tag{2.6}$$

as $\tau \to +\infty$ for $p \geq 2$, where $T = \kappa(\tau/p)^{1/\kappa p}$ and the parameters κ, ϑ' and A_0 are given by (2.3) and (2.4) with $m = 2p$. Since $1 < \kappa < 2$, this result shows that the integrand in (2.1) decays exponentially for large τ and hence that, provided $z \neq 0$, $\Gamma_p(a, z)$ is defined without restriction on the parameter a.

The asymptotic expansion of $\Gamma_p(a, z)$ for large z has been discussed in [4] where it was shown to consist of p exponential expansions, all of which are exponentially small (of different degrees of subdominance) in the sector $|\arg z| < \frac{1}{2}\pi$. From Eq. (2.10) of this reference, the dominant behaviour[3] is given by

$$z^{-a}\Gamma_p(a, z) \sim -2B_0 Z^{\vartheta} e^{Z\cos(\pi/\kappa)} \cos\{Z\sin(\pi/\kappa) + \pi\vartheta/\kappa\}, \tag{2.7}$$

for $z \to \infty$ in $|\arg z| \leq \pi$ and $p \geq 2$, where

$$Z = \kappa(z/p)^{1/\kappa p}, \qquad \vartheta = \frac{1-3p}{2p}, \qquad B_0 = \kappa^{-\frac{1}{2}}(\kappa p)^{-\vartheta}.$$

As $|z| \to \infty$ on the rays $\arg z = \pm\frac{1}{2}\pi$, the dominant behaviour of $|\Gamma_p(a, z)|$ is therefore algebraic of $O(z^{a+\vartheta/\kappa p})$.

The normalised incomplete gamma function is defined by

$$\mathcal{Q}_p(a, z) = \Gamma_p(a, z)/\Lambda_p(a), \tag{2.8}$$

[3] The dominant expansions in [4, Eq. (2.10)] correspond to $r = N - \lambda$ and $r = -N$, where $N = [\frac{1}{2}p]$ and $\lambda = 0$ (p odd), 1 (p even).

where the normalising factor $\Lambda_p(a)$ is specified by the requirement that $\mathcal{Q}_p(a,0) = 1$ (when $\mathrm{Re}(a) > 0$). This yields

$$\Lambda_p(a) = \int_0^\infty \tau^{a-1} F_{2p}(\tau)\, d\tau, \qquad \mathrm{Re}(a) > 0,$$

so that $\Lambda_p(a)$ can be considered as the Mellin transform of $F_{2p}(\tau)$. It then follows upon inversion that

$$F_{2p}(\tau) = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \Lambda_p(s) \tau^{-s} ds \qquad (c > 0).$$

Comparison with (2.2) then shows that

$$\Lambda_p(a) = \frac{p\Gamma(ap)}{\Gamma(\frac{1}{2} - ap)} \Gamma\left(\frac{1}{2p} - a\right), \tag{2.9}$$

which correctly reduces to $\Gamma(a)$ when $p = 1$.

3. A generalised expansion for $\zeta(s)$

We have the identity

$$P(a, z) + Q(a, z) = 1,$$

where $P(a, z) = \gamma(a, z)/\Gamma(a)$ is the complementary normalised incomplete gamma function defined by

$$P(a, z) = \frac{1}{\Gamma(a)} \int_0^z u^{a-1} e^{-u} du \qquad (\mathrm{Re}(a) > 0).$$

We choose the parameter $a = s/m$ and the variable $z = \pi^{m/2} n^m \eta$, where $m > 1$ is (for the moment) arbitrary and η is a complex parameter satisfying $|\phi| = |\arg \eta| < \frac{1}{2}\pi$. Then, from the Dirichlet series representation for $\zeta(s)$ valid in $\mathrm{Re}(s) > 1$ we find

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^\infty n^{-s} = \sum_{n=1}^\infty n^{-s} \left\{ P\left(\frac{s}{m}, (\pi^{\frac{1}{2}} n)^m \eta\right) + Q\left(\frac{s}{m}, (\pi^{\frac{1}{2}} n)^m \eta\right) \right\} \\ &= \sum_{n=1}^\infty n^{-s} P\left(\frac{s}{m}, (\pi^{\frac{1}{2}} n)^m \eta\right) + \sum_{n=1}^\infty n^{-s} Q\left(\frac{s}{m}, (\pi^{\frac{1}{2}} n)^m \eta\right). \end{aligned} \tag{3.1}$$

From (1.4), the late terms ($n \gg 1$) in the second sum in (3.1) involving Q possess the behaviour $n^{-m} \exp\{-(\pi^{\frac{1}{2}} n)^m \eta\}$ so that this sum converges absolutely for all s when $|\phi| \leq \frac{1}{2}\pi$.

For Re(s) > 1, the first sum in (3.1) can be written as

$$S = \pi^{-\frac{1}{2}s}\Gamma(\frac{s}{m})\sum_{n=1}^\infty n^{-s}P\Big(\frac{s}{m},(\pi^{\frac{1}{2}}n)^m\eta\Big) = \pi^{-\frac{1}{2}s}\sum_{n=1}^\infty n^{-s}\int_0^{(\pi^{\frac{1}{2}}n)^m\eta} u^{(s/m)-1}e^{-u}du$$

$$= \int_0^\eta x^{(s/m)-1}\psi_m(x)\,dx, \qquad \psi_m(x) = \sum_{n=1}^\infty \exp\{-(\pi^{\frac{1}{2}}n)^m x\} \tag{3.2}$$

upon reversal of the order of summation and integration. In [5, §8.1] it is shown that, for arbitrary $m > 1$ and Re(x) > 0, the function $\psi_m(x)$ satisfies the relation

$$2\psi_m(x) + 1 = \frac{2x^{-1/m}}{m}\left\{2\sum_{n=1}^\infty F_m(\pi^{m/2}n^m/x) + \pi^{-\frac{1}{2}}\Gamma(\frac{1}{m})\right\}, \tag{3.3}$$

where $F_m(u)$ denotes the generalised hypergeometric function

$$F_m(u) = \sum_{k=0}^\infty \frac{(-)^k u^{2k/m}}{k!\,\Gamma(k+\frac{1}{2})}\Gamma\left(\frac{2k+1}{m}\right) \qquad (|u| < \infty;\quad m > 1).$$

When $m = 2$, $F_2(u) = e^{-u}$ and (3.3) reduces to the well-known Poisson summation formula [9, p. 124]

$$2\psi_2(x) + 1 = x^{-\frac{1}{2}}\{2\psi_2(1/x) + 1\}$$

relating the behaviour of $\psi_2(x)$ to that of $\psi_2(1/x)$ when Re(x) > 0. It then follows from (3.2) and (3.3) that, provided $|\phi| < \frac{1}{2}\pi$ and Re(s) > 1,

$$S = \eta^{s/m}\left\{\pi^{-\frac{1}{2}}\Gamma(\frac{1}{m})\frac{\eta^{-1/m}}{s-1} - \frac{m}{2s}\right\} + \frac{2}{m}\int_{1/\eta}^\infty \tau^{(1-s)/m-1}\sum_{n=1}^\infty F_m(\pi^{m/2}n^m\tau)\,d\tau.$$

Now we set $m = 2p$, where p is a positive integer, so that from (2.6) $F_{2p}(\pi^p n^{2p}\tau)$ is exponentially small as $\tau \to +\infty$, and define the quantity

$$\Xi_p(s) = \frac{\pi^{\frac{1}{2}s}\eta^{s/2p}}{\Gamma(\frac{s}{2p})}\left\{\pi^{-\frac{1}{2}}\Gamma(\frac{1}{2p})\frac{\eta^{-1/2p}}{s-1} - \frac{p}{s}\right\}.$$

Then we obtain upon reversal of the order of summation and integration (justified by absolute convergence) and with the change of variable $u = (\pi n^2)^p\tau$

$$S = \pi^{-\frac{1}{2}s}\Gamma(\frac{s}{2p})\,\Xi_p(s) + \frac{\pi^{\frac{1}{2}s-\frac{1}{2}}}{p}\sum_{n=1}^\infty n^{s-1}\int_{(\pi n^2)^p/\eta}^\infty u^{(1-s)/2p-1}F_{2p}(u)\,du$$

$$= \pi^{-\frac{1}{2}s}\Gamma(\frac{s}{2p})\,\Xi_p(s) + \frac{\pi^{\frac{1}{2}s-\frac{1}{2}}}{p}\sum_{n=1}^\infty n^{s-1}\Gamma_p\Big(\frac{1-s}{2p},(\pi n^2)^p/\eta\Big),$$

where $\Gamma_p(a,z)$ is the generalised incomplete gamma function defined in (2.1). It then follows that

$$\sum_{n=1}^\infty n^{-s}P\Big(\frac{s}{2p},(\pi n^2)^p\eta\Big)=\Xi_p(s)+\frac{\pi^{s-\frac{1}{2}}}{p\Gamma(\frac{s}{2p})}\sum_{n=1}^\infty n^{s-1}\Gamma_p\Big(\frac{1-s}{2p},(\pi n^2)^p/\eta\Big)$$

$$=\Xi_p(s)+\chi(s)\sum_{n=1}^\infty n^{s-1}\mathcal{Q}_p\Big(\frac{1-s}{2p},(\pi n^2)^p/\eta\Big), \tag{3.4}$$

where $\mathcal{Q}_p(a,z)=\Gamma_p(a,z)/\Lambda_p(a)$ is the normalised incomplete gamma function and, from (2.9), we have used the fact that

$$\frac{\pi^{s-\frac{1}{2}}}{p\Gamma(\frac{s}{2p})}\Lambda_p\Big(\frac{1-s}{2p}\Big)=\chi(s).$$

Then, from (3.1) and (3.4), we finally obtain the desired generalised expansion given by

$$\zeta(s)=\sum_{n=1}^\infty n^{-s}Q\Big(\frac{s}{2p},(\pi n^2)^p\eta\Big)+\chi(s)\sum_{n=1}^\infty n^{s-1}\mathcal{Q}_p\Big(\frac{1-s}{2p},(\pi n^2)^p/\eta\Big)+\Xi_p(s) \tag{3.5}$$

valid for $\mathrm{Re}(s)>1$, $|\phi|<\frac{1}{2}\pi$ and integer $p\geq 2$. From the asymptotic behaviour of $\Gamma_p(a,z)$ in (2.7) we have

$$\Gamma_p\Big(\frac{1-s}{2p},(\pi n^2)^p/\eta\Big)=O\Big(n^{-s-(1/\kappa)}\exp\{An^{2/\kappa}\cos\omega\}\Big) \tag{3.6}$$

as $n\to\infty$ and $p\geq 2$, where

$$A=\kappa(\pi h|\eta|^{-1/p})^{1/\kappa},\qquad \omega=\frac{\pi p\mp\phi}{\kappa p}=\frac{1}{2}\pi+\frac{\frac{1}{2}\pi\mp\phi}{\kappa p}$$

and the parameters κ and h are defined in (2.3) with $m=2p$. The upper or lower sign in ω is chosen according as $\phi>0$ or $\phi<0$, respectively. Since $\frac{1}{2}\pi<\omega<\frac{3}{2}\pi$ when $p\geq 2$ and $|\phi|<\frac{1}{2}\pi$, the second sum in (3.5) is seen to be absolutely convergent for all s. When $\phi=\pm\frac{1}{2}\pi$, we have $\omega=\frac{1}{2}\pi$ and the exponential in (3.6) is oscillatory; the late terms in the second sum in (3.5) then behave like $n^{-1-1/\kappa}$ as $n\to\infty$ and so this latter sum therefore converges absolutely for all s when $|\phi|\leq\frac{1}{2}\pi$. Since the first sum in (3.5) has been shown to converge absolutely for all s when $|\phi|\leq\frac{1}{2}\pi$, it follows by analytic continuation that the expansion (3.5) holds for all s ($\neq 1$) and $|\arg\eta|\leq\frac{1}{2}\pi$. When $p=1$ it is seen that (3.5) reduces to (1.1).

4. Discussion

Initially we put $|\eta| = 1$ in (3.5) and let $\eta = e^{i\phi}$. The late terms in the first sum in (3.5) then behave ultimately like $n^{-2p} \exp\{-(\pi n^2)^p \cos\phi\}$. As explained in §1, we are forced to set $\phi = \arg\eta = \frac{1}{2}\pi$ on the critical line $s = \frac{1}{2} + it$ $(t \geq 0)$; these terms then possess the algebraic decay n^{-2p}. But more importantly the "cut-off" in $Q(s/2p, (\pi n^2)^p i)$ now occurs when $s/2p \simeq \pi^p n^{2p} i$; that is, when $n = n_1^*$ where

$$n_1^* \simeq (p\pi^{p-1})^{-1/2p} \left(\frac{t}{2\pi}\right)^{1/2p}. \tag{4.1}$$

Thus the first sum in (3.5) "switches off" more rapidly when $p \geq 2$ with the terms in the tail of this sum decaying more rapidly.

The decay of the generalised incomplete gamma function in (3.5) is similarly no longer exponential, but algebraic given by $n^{-s-1/\kappa}$, when $\phi = \frac{1}{2}\pi$. In Fig. 2 we show a plot of $|\mathcal{Q}_p((1-s)/2p, -(\pi n^2)^p)i)|$ when $p = 2$ and $s = \frac{1}{2} + it$ with $t = 50$, which clearly shows that this function also exhibits a cut-off structure. We note that a convenient way of numerically computing $\mathcal{Q}_p(a, z)$

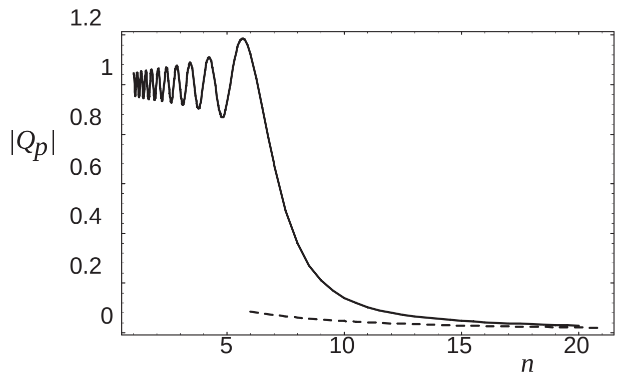

Figure 2: The behaviour of the generalised incomplete gamma function $|\mathcal{Q}_p(a, z)|$, where $a = (1-s)/2p$, $z = -i(\pi n^2)^p$, for $p = 2$ and $s = \frac{1}{2} + 50i$ as a function of n (regarded as a continuous variable for clarity). The dashed curve denotes the asymptotic approximation in (2.7) and (2.8).

when $\mathrm{Re}(a) > 0$ follows from the result

$$\begin{aligned}\mathcal{Q}_p(a, z) &= 1 - \frac{1}{\Lambda_p(a)} \int_0^z \tau^{a-1} F_{2p}(\tau)\, d\tau \\ &= 1 - \frac{pz^a}{\Lambda_p(a)} \sum_{k=0}^{\infty} \frac{(-)^k z^{k/p}}{k!\,\Gamma(k+\frac{1}{2})} \frac{\Gamma((k+\frac{1}{2})/p)}{k+ap},\end{aligned} \tag{4.2}$$

obtained by substitution of the series expansion for $F_{2p}(\tau)$ in (2.2) followed by term-by-term integration. The late terms in the second sum in (3.5) therefore behave like $n^{-1-1/\kappa}$, so that when $p \geq 2$, the convergence of this sum is *weakened*: the late terms behave like n^{-2} when $p = 1$ and approach the behaviour $n^{-3/2}$ for large p.

Of greater importance, however, is the location of the transition point of the function $\mathcal{Q}_p(a, z)$ for large values of a and z, which determines the "cut-off" point in the second sum in (3.5). This corresponds to the saddle point of the integrand in (2.1). From the first equation in (2.5) combined with $F_{2p}(u) = \Psi_{2p}(-u^{1/p})$, the leading asymptotic behaviour of $F_{2p}(u)$ is described by the single term

$$F_{2p}(u) \sim C u^{\vartheta/\kappa p} \exp\{\kappa(u/p)^{1/\kappa p} e^{\pi i/\kappa}\}$$

as $u \to \infty$ in the sector $|\arg(e^{\pi i} u^{1/p})| < \pi$, where C is a constant that we do not need to specify here. For large a, the location of the saddle point u_s is then controlled by the term $u^a \exp\{\kappa(u/p)^{1/\kappa p} e^{\pi i/\kappa}\}$ in the integrand and so is given by

$$u_s = p(ap)^{\kappa p} e^{\pi i(p-1)}.$$

When $a = (1-s)/2p$, this corresponds approximately to $u_s \simeq -ip(t/2)^{\kappa p}$ for large positive t on the critical line. Hence the behaviour of $|\mathcal{Q}_p((1-s)/2p, -(\pi n^2)^p i)|$ can be expected to change from oscillating about the value unity to decaying to zero when $u_s \simeq z = -(\pi n^2)^p i$; that is, when n has the value

$$n_2^* \simeq (p\pi^{p-1})^{1/2p} \left(\frac{t}{2\pi}\right)^{\kappa/2}. \tag{4.3}$$

Thus the "cut-off" in the second sum in (3.5) occurs for an n value scaling like $t^{\kappa/2}$. Since $1 < \kappa < 2$, this indicates that the number of terms from this sum that will make a contribution to the finite main sum (1.6) will increase with increasing p.

We observe that the introduction of a more rapid growth in the argument of the incomplete gamma functions has resulted in a certain asymmetry in the expansion (3.5): the first sum effectively switches off after $n_1^* \ll N_t$ terms whereas the second sum switches off after $n_2^* \gg N_t$ terms. This imbalance can be restored if we take $|\eta| \neq 1$. Thus, if we set $\eta = i/K$, where K denotes a positive constant, it is easily seen that the corresponding "cut-off" values in (4.1) and (4.3) now become

$$n_1^* \simeq (p\pi^{p-1})^{-1/2p} \left(\frac{t}{2\pi}\right)^{1/2p} K^{1/2p}$$

and

$$n_2^* \simeq (p\pi^{p-1})^{1/2p} \left(\frac{t}{2\pi}\right)^{\kappa/2} K^{-1/2p}.$$

By appropriate choice of K it is then possible to increase n_1^* and decrease n_2^*. The maximum amount by which we can meaningfully decrease n_2^* occurs when $n_1^* \simeq n_2^*$. This is readily seen to arise when K has the particular scaling with t given by $K = p(t/2)^{p-1}$, whereupon $n_1^* \simeq n_2^* \simeq (t/2\pi)^{1/2} \simeq N_t$. Thus, it does not appear possible to simultaneously reduce *both* n_1^* and n_2^* below the value N_t, thereby showing that the representation (1.1) is optimal in this sense.

Finally, we remark that a representation for $\zeta(\frac{1}{2}+it)$ involving the the original Dirichlet series $\sum_{n=1}^{\infty} n^{-s}$ smoothed by a simple Gaussian exponential $\exp\{-(n/N)^2\}$ has been given in [7]. This was shown to result in a computationally less powerful Gram-type expansion since the index N had to be chosen to satisfy $N \gtrsim t/(2\pi)$ for the correction terms to possess an asymptotic character.

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