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On Minimal Surfaces of Axially Symmetric Planal Riemannian Metrics in 3-Space

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Abstract

We write and give some minimal surfaces which are solutions of the minimal surfaces equation in the domain $\mathcal{B}_{\mu,\xi}$ of the Cartesian 3-space $\mathbb{R}^3$ equipped with the 2-parameters family of Riemanian metrics

$$g_{\mathcal{B}\mu,\xi} = \frac{1}{1-(\mu+\xi^2)(x^2+y^2)}(dx^2+dy^2-\mu\omega^2+2\xi\omega dz+dz^2); \mu,\xi \in \mathbb{R}.$$

where $\omega = ydx - xdy$.

$g_{\mathcal{B}\mu,\xi}$ are defined on the domain $\mathcal{B}_{\mu,\xi}$ which is, according to μ, ξ, a region of $\mathbb{R}^3$ or the whole 3-space $\mathbb{R}^3$. The metrics $g_{\mathcal{B}\mu,\xi}$ are invariant under rotations about (Oz)-axis and translations along the same axis. They generalyse this one of Heisenberg and Euclidean metrics.

The minimal surface equation on $\mathcal{B}_{\mu,\xi}$ for a graph function $z = f(x,y)$ is

$$f_{xx}(1+f_y^2-2\xi x f_y-\mu x^2)-2f_{xy}(f_xf_y+\xi(yf_y-xf_x)+\mu xy)$$
$$+f_{yy}(1+f_x^2+2\xi y f_x-\mu y^2)=0$$

where the index in f denotes partial derivation.

The affine planes $z = f(x, y) = ax + by + c$ are solutions of previous equation. The euclidean helicoid $z = f(x, y) = a\tan^{-1}(\frac{y}{x}) + b; a, b, c \in \mathbb{R}$ stay as minimal surface in $\mathcal{B}_{\mu,\xi}$ independently and regardless of μ and ξ. We classify the axially symmetric minimal surfaces in $\mathcal{B}_{\mu,\xi}$.

In last, we characterise that, the only riemannian metrics which have a constant determinant on $\mathcal{B}_{\mu,\xi}$ and which admit all the planes as minimal surfaces are Heisenberg's metrics.

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(*)The appellation *"planal metrics"* in the title is given by R. L. Bryant [6] to say that all the metrics in 3-space for which the planes are minimal surfaces.

1. Introduction

In order to understand deeply the properties and geometrical knowledge of $\mathcal{B}_{\mu,\xi}$, the study of minimal surfaces is an effective tool. We describe the principal elements of $g_{\mathcal{B}_{\mu,\xi}}$, we write the minimal surfaces equation and give some particular of them in $\mathcal{B}_{\mu,\xi}$. In the last we give a characterisation of Heisenberg's metrics.

1.1. The affine planes $z = f(x, y) = ax + by + c; a, b, c \in \mathbb{R}$ are minimal surfaces in Euclidean space $\mathbb{E}^3$ and also in Heisenberg space $\mathbb{H}_3$. There are solutions of the famous Lagrange's minimal surface equation (1760) and also solution of minimal surface equation in Heisenberg space given in [1].

Inspired by this observation, R. Lutz raised the problem: *Determine all the Riemannian metrics in regions of* $\mathbb{R}^n$ *which admit all the hyperplanes as minimal hypersurfaces.*

A first approach to answer this problem was described in [2], for the case $\mathbb{R}^3$ and their affine planes $z = f(x, y) = ax + by + c; a, b, c \in \mathbb{R}$.

In [2], we solve partially this problem and announce that the *family of Riemannian metrics,* denoted by $\mathcal{M}_3 = \{ds^2 = g_{ij}(x_1, x_2, x_3)dx_i dx_j\}$, of Lutz's problem, is 20-dimensional space and we give the particular solutions $g_{\mathcal{B}_{\mu,\xi}}$ which are only a 2-real parameters family of Riemannian metrics. These metrics appear as solutions because they have the fundamental property to be invariant under the rotations about Oz-axis and translations along the same axis. Recall that the Euclidean metric and the left invariant metrics on the Heisenberg group belong to this family. They are respectively $g_{\mathcal{B}_{0,0}}$.anb $g_{\mathcal{B}_{-\xi^2,\xi}}$.

R. L. Bryant in [6] proved that, $\mathcal{M}_3$ is effectively a 20-dimensional manifold. His proof is based on another method. He used the theory of the moving frame and the exterior differential system theory of Elie Cartan. The author in [6] did not give explicitly the metrics of $\mathcal{M}_3$. Lutz's problem and the answer to our announce in [2] was solved by Th. Hangan in [9] .

1.2. In this paper we shall study the familly $g_{\mathcal{B}\mu,\xi}$. We explicit all the basic elements of this metrics as the Riemannian curvature tensor, the Ricci tensor, the Maurer-Cartan equations, and different geometrical elements of this metrics...

We write, as in Euclidean and in Heisenberg space, the associated minimal surface equation in $\mathcal{B}_{\mu,\xi}$ for a surface as a graph of the function $z = f(x, y)$. Naturally this equation generalyses Lagrange's equation and the one in Heisenberg space given in [1] . We give some particular solutions and classify the family of all the axially symmetric minimal surfaces in $\mathcal{B}_{\mu,\xi}$. These surfaces are explicitly expressed by the elliptic integrals in Legendre form of first and second kind in terms of the parameters ξ and μ.

1.3. In [9] pp. 333, the author announced the Theorem 2: *Modulo an isometry of the euclidean space* (R^3, g_E), *where* g_E *is the euclidean metric, the riemannian metrics* $g_{H,k} = dx^2 + dy^2 + (dz + k(ydx - xdy))^2$ *are the only polynomials metrics solutions of R. Lutz's problem.*

In fact, always in [9] pp. 332, in Theorem 3, the author gave all the metrics solutions of R. Lutz's problem and specified that in 3-dimensional, the solutions are always rational. In the end of our paper we proved that the only polynomial solutions of R. Lutz's are Heisenberg's metrics stem from the $g_{\mathcal{B}\mu,\xi}$.

2. Preliminaries

2.1. Lutz's problem, in $\mathbb{R}^3$, is to find all the Riemannian metrics on an open set Ω in $\mathbb{R}^3$ such as all the traces $\mathcal{P} \cap \Omega$ of planes $\mathcal{P}$ of $\mathbb{R}^3$ on Ω will be minimal surfaces.

We take an element of $\mathcal{M}_3$ in the form $ds^2 = g_{ij}(x_1, x_2, x_3)dx_i dx_j \in \mathcal{M}_3$. We use the coordinates (x_1, x_2, x_3) instead of (x, y, z) to describe temporarily the family $\mathcal{M}_3$. We take $\mathbf{G} = (g_{ij})$ the matrix of fundamental tensor components and $\Delta_{ij} = (-1)^{i+j} m_{ij}$ the matrix in which the elements are the minors of $\mathbf{G}$, m_{ij} is the minor of g_{ij}.

In [2] , Theorem.1, the author established the partial differential equations system for which the elements of $\mathcal{M}_3$ must verify all the planes which are minimal surfaces. This system is:

$$(\mathcal{S}) \qquad 2\sum_{r=1}^{r=3} \Delta_{(ij}\Delta_{k)r,r} = \sum_{r=1}^{r=3} \Delta_{r(i}\Delta_{jk),r}.$$

where $\Delta_{ij,r} = \dfrac{\Delta_{ij}}{\partial x_r}$ and the parenthesis for express the symmetry of the index where $1 < i, j, k, r < 3$. (S) is a P.D.E system of 10 equations (see [2] p. 3078) where the Heisenberg metrics and $g_{\mathcal{B}\mu,\xi}$ are solutions.

Th. Hangan in [9], p. 329, Theorem.1 and 2, generalised the problem for all dimension n and establish the P.D.E. system satisfied directly by the components g_{ij} of the metric $ds^2 = g_{ij}dx_i dx_j$ in order that $ds^2 \in \mathcal{M}_n$, where $g_{ij} = g_{ij}(x_1, x_2, ..., x_n)$ and he solved the P.D.E. system.

$\mathcal{M}_n$ is the family of Riemannian metrics in $\mathbb{R}^n$ which admit all hyperplanes as minimal hypersurfaces and $\dim \mathcal{M}_n = n(n+1)^2(n+2)/12$. For $n = 3$, $\dim \mathcal{M}_3 = 20$. This result was the confirmation of announce of the author in [2] and the one of R. L. Bryant in [6] .

2.**2.** Let the family of the metrics

$$g_{\mathcal{B}\mu,\xi} = \frac{1}{1 - (\mu + \xi^2)(x^2 + y^2)}(dx^2 + dy^2 - \mu\omega^2 + 2\xi\omega dz + dz^2); \mu, \xi \in \mathbb{R}$$

where $\omega = ydx - xdy$.

$g_{\mathcal{B}\mu,\xi}$ are defined on the region $\mathcal{B}_{\mu,\xi}$ of the Cartesian 3-space $\mathbb{R}^3(x, y, z)$ as

$$\mathcal{B}_{\mu,\xi} = \begin{cases} \{\forall(x, y, z) \in \mathbb{R}^3 \text{ when } \mu + \xi^2 \leq 0\} \\ \{\forall(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 < (\mu + \xi^2)^{-1}, \mu + \xi^2 > 0\} \end{cases}$$

We put $r^2 = x^2 + y^2, \lambda = \mu + \xi^2, \delta = 1 - \lambda r^2$.

2.3 .1°) Fundamental tensor $g_{\mathcal{B}\mu,\xi}$ admit the associated matrix (g_{ij}) and its inverse (g^{ij}) as quadratic forms. These matrices are

$$(g_{ij}) = \frac{1}{\delta}\begin{pmatrix} 1 - \mu y^2 & \mu xy & \xi y \\ \mu xy & 1 - \mu x^2 & -\xi x \\ \xi y & -\xi x & 1 \end{pmatrix},$$

$$(g^{ij}) = \begin{pmatrix} 1 - \lambda x^2 & -\lambda xy & -\xi y \\ -\lambda xy & 1 - \lambda y^2 & \xi x \\ -\xi y & \xi x & 1 - \mu r^2 \end{pmatrix}.$$

Its determinant is $\det(g_{ij}) = 1/\delta^2$.

2°) Christoffel symbols in coordinates (x, y, z) for the Remannian metric are

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{l=3} g^{lk}(g_{jl,i} + g_{li,j} - g_{ij,l}); i,j,k = 1,2,3.$$

In coordinates we have explicitly

$$\begin{array}{lll}
\Gamma^1_{11} = \dfrac{\lambda x(1-\mu y^2)}{\delta}; & \Gamma^2_{11} = \dfrac{y(\mu-\xi^2-\mu\lambda y^2)}{\delta}; & \Gamma^3_{11} = \dfrac{2\mu\xi xy}{\delta} \\
\Gamma^1_{12} = \dfrac{y(\xi^2+\mu\lambda x^2)}{\delta}, & \Gamma^2_{12} = \dfrac{x(\xi^2+\mu\lambda y^2)}{\delta}, & \Gamma^3_{12} = \dfrac{\mu\xi(y^2-x^2)}{\delta}, \\
\Gamma^1_{13} = 0, & \Gamma^2_{13} = \dfrac{-\xi}{\delta}, & \Gamma^3_{13} = \dfrac{\mu x}{\delta}, \\
\Gamma^1_{22} = \dfrac{x(\mu-\xi^2-\mu\lambda x^2)}{\delta}, & \Gamma^2_{22} = \dfrac{\lambda y(1-\mu x^2)}{\delta}, & \Gamma^3_{22} = \dfrac{-2\mu\xi xy}{\delta}, \\
\Gamma^1_{23} = \dfrac{\xi}{\delta}, & \Gamma^2_{23} = 0, & \Gamma^3_{23} = \dfrac{\mu y}{\delta}, \\
\Gamma^1_{33} = \dfrac{-\lambda x}{\delta}, & \Gamma^2_{33} = \dfrac{-\lambda y}{\delta}, & \Gamma^3_{33} = 0.
\end{array}$$

3°) The orthonormal frame field $\mathbf{e} = (e_1, e_2, e_3)$ is

$$\begin{cases}
e_1 = \sqrt{\dfrac{\delta}{1-\lambda y^2}}\left(\dfrac{\partial}{\partial x} - \xi y\dfrac{\partial}{\partial z}\right) \\
e_2 = \dfrac{1}{\sqrt{1-\lambda y^2}}\left(-\lambda xy\dfrac{\partial}{\partial x} + (1-\lambda y^2)\dfrac{\partial}{\partial y} + \xi x\dfrac{\partial}{\partial z}\right) \\
e_3 = \sqrt{\delta}\dfrac{\partial}{\partial z}.
\end{cases}$$

The associated dual coframe is $\theta = (\theta_1, \theta_2, \theta_3)$ where

$$\begin{cases}
\theta_1 = \dfrac{1}{\sqrt{(1-\lambda y^2)\delta}}((1-\lambda y^2)dx + \lambda xydy) \\
\theta_2 = \dfrac{dy}{\sqrt{1-\lambda y^2}} \\
\theta_3 = \dfrac{1}{\sqrt{\delta}}((\xi(ydx - xdy) + dz).
\end{cases}$$

Note that the 1-form θ_3 is a contact form ($\theta_3 \wedge d\theta_3 \neq 0$) if and only if $\xi \neq 0$.

4°) The Levi-Civita connection associate D on $\mathcal{B}_{\mu,\xi}$ is described by

$$\begin{cases}
D_{e_1}e_1 = \dfrac{\lambda y}{\sqrt{1-\lambda y^2}}e_2 \\
D_{e_2}e_1 = -\xi e_3 \\
D_{e_3}e_1 = -\xi e_2 + \dfrac{\lambda x}{\sqrt{\delta(1-\lambda y^2)}}e_3
\end{cases}, \quad
\begin{cases}
D_{e_1}e_2 = -\dfrac{\lambda y}{\sqrt{1-\lambda y^2}}e_1 + \xi e_3 \\
D_{e_2}e_2 = 0 \\
D_{e_3}e_2 = \xi e_1 + \dfrac{\lambda y}{\sqrt{1-\lambda y^2}}e_3
\end{cases},$$

$$\begin{cases} D_{e_1}e_3 = -\xi e_2 \\ D_{e_2}e_3 = \xi e_1 \\ D_{e_3}e_3 = -\dfrac{\lambda x}{\sqrt{\delta(1-\lambda y^2)}}e_1 - \dfrac{\lambda y}{\sqrt{1-\lambda y^2}}e_2. \end{cases}$$

Directly, the Lie brackets are

$$\begin{cases} [e_1, e_2] = -\dfrac{\lambda y}{\sqrt{1-\lambda y^2}}e_1 + 2\xi e_3 \\ [e_1, e_3] = -\dfrac{\lambda x}{\sqrt{\delta(1-\lambda y^2)}}e_3, \\ [e_2, e_3] = -\dfrac{\lambda y}{\sqrt{1-\lambda y^2}}e_3. \end{cases}$$

5°) These metrics can be written in sum of squares form as

$$g_{\mathcal{B}\mu,\xi} = \left(\frac{1-\lambda y^2}{\sqrt{\delta(1-\lambda y^2)}}dx + \frac{\lambda xy}{\sqrt{\delta(1-\lambda y^2)}}dy\right)^2 + \left(\frac{1}{\sqrt{1-\lambda y^2}}dy\right)^2 + \\ + \left(\frac{\xi y}{\sqrt{\delta}}dx - \frac{\xi x}{\sqrt{\delta}}dy + \frac{1}{\sqrt{\delta}}dz\right)^2.$$

6°) The connection forms (ω_{ij}) relating to the coframe $\theta = (\theta_1, \theta_2, \theta_3)$ are given as $d\theta_i + \sum_j \omega_{ij} \wedge \theta_j = 0$. Explicitly these are expressed by

$$(\omega_{ij}) = \begin{pmatrix} 0 & -\xi\theta_3 + \dfrac{\lambda y}{\sqrt{1-\lambda y^2}}\theta_1 & -\xi\theta_2 + \dfrac{\lambda x}{\sqrt{\delta(1-\lambda y^2)}}\theta_3 \\ \xi\theta_3 - \dfrac{\lambda y}{\sqrt{1-\lambda y^2}}\theta_1 & 0 & \xi\theta_1 + \dfrac{\lambda y}{\sqrt{1-\lambda y^2}}\theta_3 \\ \xi\theta_2 - \dfrac{\lambda x}{\sqrt{\delta(1-\lambda y^2)}}\theta_3 & -\xi\theta_1 - \dfrac{\lambda y}{\sqrt{1-\lambda y^2}}\theta_3 & 0 \end{pmatrix}.$$

7°) We have by means of the preceding the curvature forms Ω^i_j relating to $\theta = (\theta_1, \theta_2, \theta_3)$, these are $\Omega^i_j = \frac{1}{2}R^i_{jkl}\theta_k \wedge \theta_l = d\omega_{ij} + \omega_{ih} \wedge \omega_{hj}$ where the R^i_{jkl} are the components of tensor curvature:

$$\begin{cases} \Omega_2^1 = (\mu - 2\xi^2)\,\theta_1 \wedge \theta_2 + \dfrac{2\xi\lambda x}{\sqrt{\delta(1-\lambda y^2)}}\theta_1 \wedge \theta_3 \\ \qquad\quad + \dfrac{2\xi\lambda y}{\sqrt{1-\lambda y^2}}\theta_2 \wedge \theta_3 \\ \Omega_3^1 = \dfrac{2\xi\lambda x}{\sqrt{\delta(1-\lambda y^2)}}\theta_1 \wedge \theta_2 - (\mu + \dfrac{2\lambda^2 x^2}{\delta(1-\lambda y^2)})\theta_1 \wedge \theta_3 \\ \qquad\quad - \dfrac{2\lambda^2 xy}{(1-\lambda y^2)\sqrt{\delta}}\theta_2 \wedge \theta_3 \\ \Omega_3^2 = \dfrac{2\xi\lambda y}{\sqrt{1-\lambda y^2}}\theta_1 \wedge \theta_2 - \dfrac{2\lambda^2 xy}{(1-\lambda y^2)\sqrt{\delta}}\theta_1 \wedge \theta_3 \\ \qquad\quad + (2\xi^2 + \mu - \dfrac{2\lambda}{1-\lambda y^2})\theta_2 \wedge \theta_3. \end{cases}$$

8°) The Riemannian curvature tensor and the Ricci tensor are described respectively in coordinates (x, y, z) by the formulas:

$$\begin{cases} R_{213}^1 = \dfrac{2\xi\lambda x}{\sqrt{\delta(1-\lambda y^2)}}, \; R_{223}^1 = \dfrac{2\xi\lambda y^2}{\sqrt{1-\lambda y^2}}, \\ R_{323}^1 = -\dfrac{2\lambda^2 xy}{(1-\lambda y^2)\sqrt{\delta}}, R_{313}^1 = -\mu - \dfrac{2\lambda^2 x^2}{\delta(1-\lambda y^2)}, \\ R_{212}^1 = \mu - 2\xi^2, \quad R_{323}^2 = 2\xi^2 + \mu - \dfrac{2\lambda}{1-\lambda y^2}. \end{cases}$$

and the Ricci tensor

$$\begin{cases} \Re_{11} = -2\xi^2 - \dfrac{2\lambda^2 x^2}{\delta(1-\lambda y^2)}, \; \Re_{12} = -\dfrac{2\lambda^2 xy}{(1-\lambda y^2)\sqrt{\delta}}, \; \Re_{13} = -\dfrac{2\xi\lambda y^2}{\sqrt{1-\lambda y^2}} \\ \Re_{23} = \dfrac{2\xi\lambda x}{\sqrt{\delta(1-\lambda y^2)}}, \qquad \Re_{22} = 2\mu - \dfrac{2\lambda}{1-\lambda y^2}, \qquad \Re_{33} = 2\xi^2 - \dfrac{2\lambda}{\delta}. \end{cases}$$

Hence the scalar curvature is

$$K = \frac{1}{6}\left(2\mu - \frac{4\lambda}{\delta}\right) .$$

3. Minimal surfaces equation in $\mathcal{B}_{\mu,\xi}$

3.1. Let $\mathbb{S}$ be an immersed surface in $\mathcal{B}_{\mu,\xi}$ which is given as a graph of a function $z = f(x, y)$. The position vector of $\mathbb{S}$ is expressed as a vector valued function $X(x, y) = (x, y, f(x, y))$. The first fundamental form of $\mathbb{S}$ is a Riemannian metric $\mathbf{I} = X^\star g_{\mathcal{B}_{\mu,\xi}|\mathbb{S}}$ on $\mathbb{S}$ induced by the ambiant metric $g_{\mathcal{B}_{\mu,\xi}}$ on $\mathcal{B}_{\mu,\xi}$ as

$$\mathbf{I}(dx, dy) = Edx^2 + 2Fdxdy + Gdy^2.$$

If f_x, f_y, f_{xx}, f_{xy}, f_{yy} are the partial derivatives of f with respect to x, y. We set $dz = f_x dx + f_y dy$ in $g_{\mathcal{B}\mu,\xi|\mathbb{S}}$. This gives

$$g_{\mathcal{B}\mu,\xi|\mathbb{S}} = \frac{1}{\delta}\left[\begin{array}{c}(1+f_x^2+2\xi y f_x-\mu y^2)dx^2+2(f_xf_y+\xi y f_y-\xi x f_x+\mu xy)dxdy+\\ +(1+f_y^2-2\xi x f_y-\mu x^2)dy^2\end{array}\right].$$

We have explicitly

$$\begin{cases} E = \dfrac{1}{\delta}\left[1-\lambda y^2+(f_x+\xi y)^2\right] \\ F = \dfrac{1}{\delta}\left[\lambda xy+(f_x+\xi y)(f_y-\xi x)\right] \\ G = \dfrac{1}{\delta}\left[1-\lambda x^2+(f_y-\xi x)^2\right]. \end{cases}$$

3.2. Take a vector field $\mathbf{N}$ normal to $\mathbb{S}$ in $\mathcal{B}_{\mu,\xi}$. The second fundamental form $\mathbf{II}$ derived from $\mathbf{n} = \mathbf{N}/\|\mathbf{N}\|$ is defined by the Gauss formula $D_Y X_\star Z = X_\star(\nabla_Y Z) + \mathbf{II}(Y,Z)\mathbf{n}$ for all vector fields Y and Z on $\mathbb{S}$. The Gauss formula induces a connection ∇ on $\mathbb{S}$. This connection coincides with the Levi-Civita connection of the Riemannian submanifold (2-manifold) $(\mathbb{S}, \mathbf{I})$. Note that $\mathbf{II}$ can be defined alternatively by the formula $\mathbf{II} = -g_{\mathcal{B};\mu,\xi|\mathbb{S}}(D\mathbf{n}, dX)$. Since $\mathbb{S}$ is a graph of a function $z = f(x,y)$, we can choose a unit vector field $\mathbf{n} = (n_1, n_2, n_3)$ where

$$\begin{cases} n_1 = \dfrac{\xi y + f_x - \lambda x(xf_x + yf_y)}{\delta W} \\ n_2 = \dfrac{-\xi x + f_y - \lambda y(xf_x + yf_y)}{\delta W} \\ n_3 = \dfrac{\mu r^2 - 1 + \xi(xf_y - yf_x)}{\delta W} \end{cases}$$

where W is the norm of $\mathbf{N}$

$$\|\mathbf{N}\| = W = \frac{1}{\delta}\sqrt{1+f_x^2+f_y^2-2\xi(xf_y-yf_x)-\lambda(xf_x+yf_y)^2-\mu r^2}.$$

The second fundamental form is explicitly given as

$$\mathbf{II}(dx,dy) = Ldx^2 + 2Mdxdy + Ndy^2$$

where

$$\begin{cases} L = \dfrac{1}{W\delta^2}\left[\begin{array}{c} \delta f_{xx} + 2\xi f_x f_y + 2\mu\xi xy + (\xi^2-\mu)(yf_y - xf_x) + \\ +\lambda(xf_x + yf_y)(f_x^2 + \mu y^2) \end{array}\right] \\ M = \dfrac{1}{W\delta^2}\left[\begin{array}{c} \delta f_{xy} + (\lambda f_x f_y - \lambda\mu xy)(xf_x + yf_y) - \mu(x^2 - y^2) + \\ +(\mu - \xi^2)(xf_y + yf_x) + \xi(f_y^2 - f_x^2) \end{array}\right] \\ N = \dfrac{1}{W\delta^2}\left[\begin{array}{c} \delta f_{yy} - 2\xi f_x f_y - 2\mu\xi xy - (\xi^2-\mu)(yf_y - xf_x) + \\ +\lambda(xf_x + yf_y)(f_y^2 + \mu x^2). \end{array}\right] \end{cases}$$

3.3. Let us denote the following matrice-valued functions associated with **I** and **II** respectively by:

$$\tilde{\mathbf{I}} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \tilde{\mathbf{II}} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

The solutions k_1 and k_2 of the caracteristic equation $\det(\mathbf{II} - k\mathbf{I}) = 0$ are the *principal curvatures* of $\mathbb{S}$. The average over k_1 and k_2 denoted by H is the *mean curvature* of $\mathbb{S}$. The mean curvature is computed by $H = (EN + GL - 2FM)/2(EG - F^2)$.

A surface $\mathbb{S} : z = f(x, y)$ is said to be *minimal* if $H = 0$.

3.4. The differential equation $H = 0$ for a surface $\mathbb{S}$, as a graph $z = f(x, y)$, is the *minimal surface equation* in $\mathcal{B}_{\mu,\xi}$. Insert E, F, G; L, M, N in the differential equation $H = 0$ we obtain explicitly the minimal surface equation

$$(\mathcal{E}_{\mu,\xi}) \qquad \begin{array}{c} f_{xx}(1 + f_y^2 - 2\xi xf_y - \mu x^2) - 2f_{xy}(f_x f_y + \xi(yf_y - xf_x) + \mu xy) + \\ +f_{yy}(1 + f_x^2 + 2\xi yf_x - \mu y^2) = 0 \end{array}$$

That is, from now, the minimal surface equation in $\mathcal{B}_{\mu,\xi}$.

In the plane of parameters (ξ, μ), the Heisenberg metrics are described by the parabola $\mu = -\xi^2$. This graph characterises Heisenberg's metrics of the family $g_{\mathcal{B}\mu,\xi}$. The origin of the parameters plane $(\xi, \mu) = (0, 0)$ corresponds to Lagrange's minimal surfaces equation.

3.5. We can obtain the preceding equation $(\mathcal{E}_{\mu,\xi})$ if we study the behaviour of neighbouring surfaces $(\mathbb{S}_\tau)$ as a graph described by the equation $z_\tau = f(x, y) + \tau h(x, y)$ as the process which is used by Lagrange for his variational calculation to have the minimal surface equation in the Euclidean space $\mathbb{E}^3$ We have $\mathbb{S}_0 = \mathbb{S}$, and if we denote the first fundamental form of $\mathbb{S}_0$ by $E_\circ, F_\circ, G_\circ$ the variational calculus gives

$$\frac{\partial}{\partial x}\left(\frac{f_x + \xi y - \lambda x(xf_x + yf_y)}{\delta\sqrt{E_\circ G_\circ - F_\circ^2}}\right) + \frac{\partial}{\partial y}\left(\frac{f_y - \xi x - \lambda y(xf_x + yf_y)}{\delta\sqrt{E_\circ G_\circ - F_\circ^2}}\right) = 0$$

which is the *divergence form* of $(\mathcal{E}_{\mu,\xi})$.

4. Minimal surfaces in $\mathcal{B}_{\mu,\xi}$

Minimal surfaces theory in $\mathbb{E}^3$ starts with constructing, classifing fundamental examples of minimal surfaces. We have a very beautiful particular minimal surfaces, as axially minimal surfaces, ruled surfaces, translation surfaces..., until the entire resolution of Lagrange's equation by Weierstrass (1866).

Also recall in Physics, the famous Plateau problem, Steiner problem...,these works in different subjects on the preceding questions obtained, as a reward, the Field's medal obtained by Tibor Rado and Jesse Douglas in 1936.

In this section we study elementary and fundamental examples of minimal surfaces in $\mathcal{B}_{\mu,\xi}$. As in Heisenberg space [1], we start to search for some particular solutions of $(\mathcal{E}_{\mu,\xi})$ in $\mathcal{B}_{\mu,\xi}$.

4.1. It is clear that the linear function $z = f(x,y) = ax+by+c; a,b,c \in \mathbb{R}$ is a solution of $(\mathcal{E}_{\mu,\xi})$ because it is the motivation of the search for metrics $g_{\mathcal{B}\mu,\xi}$.

4.2. Euclidean helicoids can be characterised as minimal surface in $\mathbb{E}^3$ which is a graph of a function in the form $f(x,y) = g(\frac{y}{x})$. In this subsection we search for minimal surfaces written in the form $z = f(x,y) = g(\frac{y}{x})$ and must be minimal in $\mathcal{B}_{\mu,\xi}$.

Let $\mathbb{S}$ be a surface which is graph of a function in the form $f(x,y) = g(\frac{y}{x})$. Put $u = \frac{y}{x}$ for $x \neq 0$. Then we have

$$f_x = -\frac{y}{x^2}g', f_y = \frac{1}{x}g', f_{xx} = \frac{2y}{x^3}g' + \frac{y^2}{x^4}g''$$
$$f_{xy} = -\frac{1}{x^2}g' - \frac{y}{x^3}g'', f_{yy} = \frac{1}{x^2}g''.$$

Here g' et g'' are the derivatives with respect to u. We insert these data into the minimal surface equation $(\mathcal{E}_{\mu,\xi})$ in $\mathcal{B}_{\mu,\xi}$ and multiply it by x^2. Then we obtain the classical differential equation independtly and regardless of μ and ξ :

$$(1+u^2)g'' + 2ug' = 0.$$

One can see easily that the general solution to this O. D. E. is given explicitly by $f(x,y) = g(\frac{y}{x}) = a\tan^{-1}(\frac{y}{x}) + b; a,b \in \mathbb{R}$. However we have

Proposition 4.2. *The only minimal surface in* $\mathcal{B}_{\mu,\xi}$ *which has the form* $f(x,y) = g(\frac{y}{x})$ *is the surfaces* $z = f(x,y) = a\tan^{-1}(\frac{y}{x}) + b; a, b \in \mathbb{R}$.

4.3. By analogy to the case in $\mathbb{H}_3$, where the hyperbolic paraboloid is a minimal surface, we also have in $\mathcal{B}_{\mu,\xi}$ the particular minimal surface $z = f(x,y) = \sqrt{-\mu}xy$, $\mu < 0$. which is also a hyperbolic paraboloid.

In general, if we put $f(x,y) = \xi y\left[T(x) + x\right]$, we have $f_x = \xi y(T' + 1)$, $f_y = \xi(T + x)$; $f_{xx} = \xi y T''$; $f_{xy} = \xi(T' + 1)$, $f_{yy} = 0$ where T' et T'' are the derivatives with respect to x. Now we insert these data to the minimal surface equation $(\mathcal{E}_{\mu,\xi})$ to obtain the differential equation

$$T''(\xi^2 T^2 + 1 - \lambda x^2) - 2(T' + 1)(\xi^2 T(T' + 2) + \lambda x) = 0.$$

The solutions of this one are minimal surfaces in $\mathcal{B}_{\mu,\xi}$. It is very difficult to solve explicitly this equation but for $\lambda = 0$, we find again the Heisenberg case which is in [3].

4.4. The metrics $g_{\mathcal{B}_{\mu,\xi}}$ are invariant under rotations about (Oz)-axis and translations along the same axis. Based on this fundamental property, in this subsection, we classify the axially symmetric minimal surfaces in $\mathcal{B}_{\mu,\xi}$.

Put $f(x,y) = U(r)$ and as before $r^2 = x^2 + y^2$. We have

$$f_x = \frac{x}{r}U', f_y = \frac{y}{r}U', f_{xx} = \frac{y^2}{r^3}U' + \frac{x^2}{r^2}U'',$$
$$f_{yy} = \frac{x^2}{r^3}U' + \frac{y^2}{r^2}U'', f_{xy} = \frac{-xy}{r^3}U' + \frac{xy}{r^2}U''.$$

Here U', U'' are the derivatives with respect to r. From this we have the differential equation and we obtain the following minimal surface equation:

$$r(1 - \mu r^2)U'' + U'(1 + U'^2) = 0.$$

This equation, depend only on μ, becomes an equation of the first order. Hence, we get $U' = V$. To solve this differential equation we must discuss two cases depending on μ:

1°) $\mu = 0$, the solution is $U'^2 = k^2/\left(r^2 - k^2\right)$, where k is constant. The solution is the axial symmetric surface

$$U(r) = k\cosh^{-1}(\frac{r}{k}) + c_1.$$

If $k \neq 0$, the graph of this surface is the catenoïd in $\mathbb{E}^3$ discovered as a minimal surface by Meusnier (1776).

Remark: If $k = 0$, we have $U(r) = const$, which is the graph is horizontal Euclidean plane.

2°) $\mu \neq 0$, the solution is in the form: $U'^2 = \alpha^2 \dfrac{r^2 - 1/\mu}{r^2 + \alpha^2/\mu}$ where $\alpha^2 = k^2/(1-k^2)$, k is constant and $-1 < k < 1$. If $k \neq 0$, we have two cases:

i) $\mu < 0$, r verifies $r^2 > \dfrac{\alpha^2}{-\mu}$. The solution is the axial symmetric surface

$$U(r) = \alpha^2 \int_{\alpha/\sqrt{-\mu}}^{r} \sqrt{\frac{t^2 + (1/\sqrt{-\mu})^2}{t^2 - (\alpha/\sqrt{-\mu})^2}} dt + c_2.$$

This elliptic integral can be expressed by the elliptic integrals in Legendre form of first and second kind. In fact, the integral

$$I(u) = \int_{b}^{u} \sqrt{\frac{t^2 + a^2}{t^2 - b^2}} dt$$

is given by the elliptic integrals in Legendre form of first and second kind.as $I(u) = \sqrt{a^2 + b^2} \left(F(\varepsilon, s) - E(\varepsilon, s)\right) + \dfrac{1}{\mu} \sqrt{(u^2 + a^2)(u^2 + b^2)}$ with $u > b > 0$.

ii) $\mu > 0$, r satisfies $r^2 > \dfrac{1}{\mu}$. The solution is the axial symmetric surface

$$U(r) = \alpha^2 \int_{1/\sqrt{\mu}}^{r} \sqrt{\frac{t^2 - (1/\sqrt{\mu})^2}{t^2 + \left(\alpha/\sqrt{\mu}\right)^2}} dt + c_3.$$

This integral can be expressed by the elliptic integrals in Legendre form of first and second kind. In fact, the integral

$$I(u) = \int_{b}^{u} \sqrt{\frac{t^2 - b^2}{t^2 + a^2}} dt$$

is given by $I(u) = \dfrac{1}{\mu} \sqrt{(u^2 + a^2)(u^2 - b^2)} - \sqrt{a^2 + b^2} E(\varepsilon, s)$ with $u > b > 0$.

$F(\varepsilon, s)$ and $E(\varepsilon, s)$ are the elliptic integrals in Legendre form of first and second kind. These are

$$F(\varepsilon, s) = \int_{0}^{\varepsilon} \left(1 - s^2 \sin^2 \alpha\right)^{-\frac{1}{2}} d\alpha,\ E(\varepsilon, s) = \int_{0}^{\varepsilon} \left(1 - s^2 \sin^2 \alpha\right)^{\frac{1}{2}} d\alpha.$$

The modulas ε and the variable s are given by $\varepsilon = \cos^{-1}(\frac{b}{u})$, $s = \frac{a}{\sqrt{a^2+b^2}}$.

Theorem 4.4. *The only axially symmetric minimal surfaces in $\mathcal{B}_{\mu,\xi}$ are the graphs of functions $f(x,y) = U(r) = U(\sqrt{x^2+y^2})$ with $r^2 = x^2+y^2$, where*

$1^\circ : U(r) = k\cosh^{-1}(\frac{r}{k}) + c_1,\ \mu = 0, k \neq 0.$

$2^\circ : i).\ \mu < 0, U(r) = \alpha^2 \int\limits_{\alpha/\sqrt{-\mu}}^{r} \sqrt{\frac{t^2 + (1/\sqrt{-\mu})^2}{t^2 - (\alpha/\sqrt{-\mu})^2}} dt + c_2,$

$ii).\ \mu > 0, U(r) = \alpha^2 \int\limits_{1/\sqrt{\mu}}^{r} \sqrt{\frac{t^2 - (1/\sqrt{\mu})^2}{t^2 + \left(\alpha/\sqrt{\mu}\right)^2}} dt + c_3$

with $\alpha^2 = \frac{k^2}{1-k^2}$, and k =const. and $-1 < k < 1$..

4.5. Comments about the surface in the form

$$z = f(x,y) = g(\frac{y}{x}) = a\tan^{-1}(\frac{y}{x}) + b; a, b \in \mathbb{R}.$$

1°) In $\mathcal{B}_{\mu,\xi}$ and if $\xi = \mu = 0$, the minimal surface $z = a\tan^{-1}(\frac{y}{x}) + b; a, b \in \mathbb{R}$ is a right helicoid in Euclidean space $\mathbb{E}^3$. This surface is minimal in the Heisenberg space $\mathbb{H}_3$ as well as in $\mathbb{E}^3$, see [1]. For these 2 cases $g_{\mathcal{B}0,0}$ and $g_{\mathcal{B}-\xi^2,\xi}$ we have $\mathcal{B}_{\mu,\xi} = \mathbb{R}^3$.

2°) In a previous paper [4] of ours we proved also that the surface $z = a\arctan(\frac{y}{x}) + b$; a, $b \in \mathbb{R}$ is a minimal one in the well known 3-dimensional homogeneous space called the Bianchi-Cartan-Vranceanu space which (as a particular d'Atri space denote by $\mathcal{C}$, see [5] and the references therein). We recall briefly and essentially that the following 2-parameters family of homogeneous Riemannian metrics $g_{\mathcal{C}\gamma,\eta}$

$$g_{\mathcal{C}\gamma,\eta} = \frac{dx^2+dy^2}{(1+\eta(x^2+y^2))^2} + (dz + \frac{\gamma}{2}\frac{ydx - xdy}{1+\eta(x^2+y^2)})^2; \gamma, \eta \in \mathbb{R}$$

This Riemannian metrics founded by L. Bianchi, E. Cartan and G. Vranceanu. There are defined on the space denoted

$$\mathcal{C}_{\gamma,\eta} = \begin{cases} \{\forall(x,y,z) \in \mathbb{R}^3 \text{ when } \eta \geq 0\} \\ \{\forall(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 < -1/\eta, \eta < 0\}. \end{cases}$$

which is the homogeneous 3-manifold $\mathcal{C}_{\gamma,\eta}$ equipped with the riemannian metrics $g_{\mathcal{C}\gamma,\eta}$).

The metrics $g_{\mathcal{C}\gamma,\eta}$ do not belong to the family in which all the hyperplane are minimal like this one investigated in [2] and in [9] for all γ and η. This means that $g_{\mathcal{C}_{\gamma,\eta}}$ do not belong to the family $\mathcal{M}_3$ in general but the Euclidean $g_{\mathcal{C}_{0,0}}$ and Heisenberg $g_{\mathcal{C}_{\gamma,0}}$ metrics belong to $g_{\mathcal{C}_{\gamma,\eta}}$. In these 2 cases we have $\mathcal{C}_{\gamma,\eta} = \mathbb{R}^3$.

3°) The surface $z = a\tan^{-1}(\frac{y}{x}) + b; a, b \in \mathbb{R}$ is also a *maximal surface* in the 3-dimensional Minkowsky space. This surface is a space-like surface with vanishing mean curvature. A Minkowsky space $\mathbb{L}^3$ is $\mathbb{R}^3$ equipped with the Lorentz metric $ds^2_{\mathbb{L}} = dx^2 + dy^2 - dz^2$, . i.e. $\mathbb{L}^3 = (\mathbb{R}^3, ds^2_{\mathbb{L}})$

In [11] p. 307, we have the Theorem. 4.2 : *Except for the plane, only the helicoid is a maximal surface in $\mathbb{L}^3 = (\mathbb{R}^3, ds^2_{\mathbb{L}})$ which is a minimal surface with respect the Riemannian metric $dx^2 + dy^2 + dz^2$.*

For us, we want to say that the surface $z = a\tan^{-1}(\frac{y}{x}) + b; a, b \in \mathbb{R}$ is a maximal surface which means that this one is a solution of the maximal surface equation for $z = f(x, y)$. Maximal surface's equation is (see [11])

$$(\mathcal{L}) \qquad f_{xx}(1 - f_y^2) + 2f_xf_yf_{xy} + f_{yy}(1 - f_x^2) = 0.$$

If we look for search surfaces in the the form $z = f(x, y) = g(\frac{y}{x})$ and process as in the beginning of the subsection 4.2, we obtain the same O.D.E. equation in the Euclidean case and the unique solution in the preceeding form is $z = g(\frac{y}{x}) = a\tan^{-1}(\frac{y}{x}) + b; a, b \in \mathbb{R}$.

Instead of the Theorem 4.2. p. 307 in [11] we have in our mind the

Proposition 4.5.. *The only maximal surface in $\mathbb{L}^3$ written in the form $z = f(x, y) = g(\frac{y}{x})$ is the affine helicoid $z = a\tan^{-1}(\frac{y}{x}) + b; a, b \in \mathbb{R}$.*

Proof: We process, as in **4.**2, and report the derivative $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ into $(\mathcal{L})$, we obtain easily the same differential equation for which $z = a\tan^{-1}(\frac{y}{x})+ b; a, b \in \mathbb{R}$ is the solution of $(\mathcal{L})$.

4°) In $\mathcal{C}_{\gamma,\eta}$ the metrics $g_{\mathcal{C}_{\gamma,\eta}}$ are not isomorphic between them for different parameters (γ, η). Heisenberg metrics as particular case of the preceding $g_{\mathcal{C}_{\gamma,\eta}}$ which occurs when $\eta = 0$ and $g_{\mathcal{C}_{\gamma,0}} = dx^2 + dy^2 + (dz + \frac{\gamma}{2}(ydx - xdy))^2; \gamma \in \mathbb{R}$ are, also, not isometric between them for arbitrary γ.

In metrics $g_{\mathcal{B}_{\mu,\xi}}$, the Heisenberg case happen on the parabola $\mu = -\xi^2$ in the parameters plane (ξ, μ) and we have $g_{\mathcal{B}_{-\xi^2,\xi}} = dx^2 + dy^2 + (dz + \xi(ydx - xdy))^2$ and also are not isometric between them for different $\xi \in \mathbb{R}$ and never isometric between them for arbitrary $\mu, \xi \in \mathbb{R}$.

We observe, however, that the form $z = a\tan^{-1}(\frac{y}{x}) + b; a, b \in \mathbb{R}$ is a minimal surface in $\mathbb{E}^3, \mathbb{H}_3, \mathcal{B}_{-\xi^2,\xi}$ and $\mathcal{C}_{\gamma,\eta}$ and maximal surface in $\mathbb{L}^3$ independently and regardless of the parameters as well in $\mathcal{B}_{\mu,\xi}$ as in $\mathcal{C}_{\gamma,\eta}$.

In general, it is very unusual to find the same solution simultaneously for 5 different P.D.E. The affine surface $z = a\tan^{-1}(\frac{y}{x}) + b; a, b \in \mathbb{R}$ is a very particular surface.

5. One caracterisation of Heisenberg metrics in $\mathcal{B}_{\mu,\xi}$

In [9], the author, in Theorem 3, gave all the metrics solutions of R. Lutz's problem and specifie that on 3-dimensional, the solutions are always rational. That is the case of the metrics

$$g_{\mathcal{B}\mu,\xi} = \frac{1}{1-(\mu+\xi^2)r^2}(dx^2+dy^2-\mu\omega^2+2\xi\omega dz+dz^2); \mu, \xi \in \mathbb{R}, \omega = ydx - xdy$$

These $g_{\mathcal{B}\mu,\xi}$ are defined on the domain $\mathcal{B}_{\mu,\xi}$ which is a region of $\mathbb{R}^3$ or the whole 3-space $\mathbb{R}^3$. There are invariant under rotations about (Oz)-axis and translations along the same axis and generalyse the one of Heisenberg and Euclidean metrics.

The tensor $g_{\mathcal{B}\mu,\xi}$ admit the associated matrix (g_{ij}) as a quadratic form. The matrix

$$(g_{ij}) = \frac{1}{\delta}\begin{pmatrix} 1-\mu y^2 & \mu xy & \xi y \\ \mu xy & 1-\mu x^2 & -\xi x \\ \xi y & -\xi x & 1 \end{pmatrix},$$

has (see § **2**.3.) $\det(g_{ij}) = \frac{1}{(1-(\mu+\xi^2)(x^2+y^2))^2}$. We have $\det(g_{ij}) = 1$ if and only if $(\mu+\xi^2)(x^2+y^2) = 0, \forall x, y \in \mathbb{R}$, for $\mu+\xi^2 = 0$ which corresponds to our famous Heisenberg metrics obtained from $g_{\mathcal{B}\mu,\xi}$ when $\mu = -\xi^2$.

If we put together this last calculus and Theorem 3 in [9] according to the author that all the metrics solutions of R. Lutz's problem on 3-dimensional are always rational. We have

Theorem 5.1: *The only metrics solutions of R Lutz problem in* $\mathcal{B}_{\mu,\xi}$ *such that the tensor metrics have* $\det(g_{\mathcal{B}\mu,\xi}) = 1$, *are the Heisenberg's metrics*

$$\begin{aligned} g_{\mathcal{B}-\xi^2,\xi} &= dx^2 + dy^2 + \xi^2(ydx - xdy)^2 + 2\xi(ydx - xdy)dz + dz^2 \\ &= dx^2 + dy^2 + dz + (dz + \xi(ydx - xdy))^2, \xi \in \mathbb{R}, \mathcal{B}_{-\xi^2,\xi} = \mathbb{R}^3. \end{aligned}$$

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