**Applied Mathematical Sciences, Vol. 3, 2009, no. 49, 2427 - 2436**

# The Bernstein Operational Matrix of Integration

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**Abstract**

An accurate method is proposed to solve problems such as identification, analysis and optimal control using the Bernstein orthonormal polynomials operational matrix of integration. The Bernstein polynomials are first orthogonalized, normalized and then their operational matrix of integration is obtained. An example is given to illustrate the proposed method.

**Mathematics Subject Classification:** 41A10, 49J15

**Keywords:** The Bernstein polynomials; Operational matrix; Differential equation

## 1. Introduction

Approximations by orthonormal family of functions have played a vital role in the development of physical sciences, engineering and technology in general and mathematical analysis in particular since long. In the last three decades, they have been playing an important part in the evaluation of new techniques to solve problems such as identification, analysis and optimal control. The aim of these techniques has been to obtain effective algorithms that are suitable for the digital computers. The motivation and philosophy behind this approach is that it transforms the underlying differential equation of the problem to an algebraic equation, thus simplifying the solution process of the problem to a great extent. The basic idea of this technique is as follows:

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(i) The differential equation is converted to an integral equation via multiple integration.

(ii) Subsequently, the various signals involved in the integral equation are approximated by representing them as linear combinations of the orthonormal basis functions and truncating them at optimal levels.

(iii) Finally, the integral equation is converted to an algebraic equation by introducing the operational matrix of integration of the basis functions.

The key idea of the technique depends on the following integral property of the basis vector $\varphi(t)$

$$\int_a^t \cdots \int_a^t \varphi(\sigma)\,(d\sigma)^k \approx P_{m+1}^k\,\varphi(t), \qquad (1)$$

where $\varphi(t) = [\varphi_0(t), \varphi_1(t), \ldots, \varphi_m(t)]^T$ in which the elements $\varphi_0(t), \varphi_1(t), \ldots, \varphi_m(t)$ are the basis functions, orthogonal on a certain interval $[a,b]$ and $P_{m+1}$ is the operational matrix for integration of $\varphi(t)$. Note that $P_{m+1}$ is a constant matrix of order $(m+1)\times(m+1)$.

Using the operational matrix of an orthonormal system of functions to perform integration for solving, identifying and optimizing a linear dynamic system has several advantages:

(i) The method is computer oriented, thus solving higher order differential equation becomes a matter of dimension increasing, (ii) The solution is a multi-resolution type and (iii) the solution is convergent, even though the size of increment may be large.

Until now, the operational matrix of integration has been determined for several types of orthogonal basis functions, such as the Walsh function [1-2], block- pulse function [3-4], Laguerre series [5-7], Chebyshev polynomials [8-9], Legendre polynomials [10-11] , Fourier series [12-13] and Bessel series [14]. Later Gu and Jiang [15] derived the Haar wavelets operational matrix of integration followed by Razzaghi and Yousefi [16] who gave the Legendre wavelets operational matrix of integration.

The aim of present paper is to derive the Bernstein orthonormal polynomials matrix of integration $P_{m+1}$. The matrix $P_{m+1}$ may be used to solve problems of system analysis and synthesis in a manner similar to those of the other orthogonal functions. The Bernstein polynomials are first orthonormalized and the operational matrix of integration is then derived. A numerical example is given to illustrate the efficiency of the proposed method.

## 2. The Bernstein polynomials:

A Bernstein polynomial, named after Sergei Natanovich Bernstein, is a polynomial in the Bernstein form, that is a linear combination of Bernstein basis polynomials.
The Bernstein basis polynomials of degree $n$ are defined by

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} \quad , \qquad for \;\; i = 0,1,2,\cdots,n \,. \tag{2}$$

There are $(n+1)$ $n^{th}$ degree Bernstein basis polynomials forming a basis for the linear space $V_n$ consisting of all polynomials of degree less than or equal to $n$ in **R**[x]**-**the ring of polynomials over the field **R.** For mathematical convenience, we usually set $B_{i,n} = 0$ if $i < 0$ or $i > n$.
Any polynomial $B(x)$ in **R**[x] may be written as

$$B(x) = \sum_{i=0}^{n} \beta_i \, B_{i,n}(x) \,. \tag{3}$$

Then $B(x)$ is called a polynomial in Bernstein form or Bernstein polynomial of degree $n$. The coefficients $\beta_i$ are called Bernstein or Bezier coefficients.But several mathematicians call Bernstein basis polynomials $B_{i,n}(x)$ as the Bernstein polynomials. We will follow this convention as well. These polynomials have the following properties:

$(i)$ $B_{i,n}(0) = \delta_{i0}$ and $B_{i,n}(1) = \delta_{in}$ , where $\delta$ is the Kronecker delta function.

$(ii)$ $B_{i,n}(t)$ has one root, each of multiplicity $i$ and $n-i$, at $t = 0$ and $t = 1$ respectively.

$(iii)$ $B_{i,n}(t) \geq 0$ for $t \in [0,1]$ and $B_{i,n}(1-t) = B_{n-i,n}(t)$.

$(iv)$ For $i \neq 0$, $B_{i,n}$ has a unique local maximum in $[0,1]$ at $t = i/n$ and the maximum value $i^i n^{-n} (n-i)^{n-i} \binom{n}{i}$ .

$(v)$ The Bernstein polynomials form a partition of unity i.e. $\sum_{i=0}^{n} B_{i,n}(t) = 1$.

$(vi)$ It has a degree raising property in the sense that any of the lower-degree polynomials (degree $< n$) can be expressed as a linear combinations of polynomials of degree $n$. We have,

$$B_{i,n-1}(t) = \left(\frac{n-i}{n}\right) B_{i,n}(t) + \left(\frac{i+1}{n}\right) B_{i+1,n}(t) \;.$$

$(vii)$ Let $f(x) \in C\,[0,1] -$ (the class of continuous functions on $[0,1]$), then

$B_n(f)(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) B_{i,n}(x)$ converges to $f(x)$ uniformly on $[0,1]$ as $n \to \infty$.

($viii$) Let $f(x) \in C^{(k)}[0,1]-$ (the class of $k-$ times differentiable function with $f^{(k)}$ continuous), then $\left\| B_n(f)^{(k)} \right\|_\infty \le \frac{(n)_k}{n^k} \left\| f^{(k)} \right\|_\infty$ and $\left\| f^{(k)} - B_n(f)^{(k)} \right\|_\infty \to 0$ as $k \to \infty$, where $\|.\|_\infty$ is the sup. norm and $\frac{(n)_k}{n^k} = \left(1-\frac{0}{n}\right)\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$ is an eigen value of $B_n$; the corresponding eigen function is a polynomial of degree $k$.

## 3. The orthonormal polynomials:

Using Gram- Schmidt orthonormalization process on $B_{i,n}$ and normalizing, we obtain a class of orthonormal polynomials from Bernstein polynomials. We call them orthonormal Bernstein polynomials of order $n$ and denote them by $b_{on}, b_{1n}, \cdots, b_{nn}$ .

For $n = 5$ the five orthonormal polynomials are given by

$$b_{05}(t) = \sqrt{11}(1-t)^5$$

$$b_{15}(t) = 6\left[5(1-t)^4 t - \frac{1}{2}(1-t)^5\right] \quad (4)$$

$$b_{25}(t) = \frac{18\sqrt{7}}{5}\left[10(1-t)^3 t^2 - 5(1-t)^4 t + \frac{5}{18}(1-t)^5\right]$$

$$b_{35}(t) = \frac{28}{\sqrt{5}}\left[10(1-t)^2 t^3 - 15(1-t)^3 t^2 + \frac{30}{7}(1-t)^4 t - \frac{5}{28}(1-t)^5\right]$$

$$b_{45}(t) = 7\sqrt{3}\left[5(1-t)t^4 - 20(1-t)^2 t^3 + 18(1-t)^3 t^2 - 4(1-t)^4 t + \frac{1}{7}(1-t)^5\right]$$

$$b_{55}(t) = 6\left[t^5 - \frac{25}{2}(1-t)t^4 + \frac{100}{3}(1-t)^2 t^3 - 25(1-t)^3 t^2 + 5(1-t)^4 t - \frac{1}{6}(1-t)^5\right]$$

A function $f \in L^2[0,1]$ may be written as

$$f(t) = \lim_{n \to \infty} \sum_{i=0}^{n} c_{in}\, b_{in}(t), \tag{5}$$

where, $c_{in} = \langle f, b_{in} \rangle$ and $\langle , \rangle$ is the standard inner product on $L^2[0,1]$.
If the series (5) is truncated at $n = m$, then

$$f \cong \sum_{i=0}^{m} c_{im}\, b_{im} = C^T B(t), \tag{6}$$

where,

$$C = [c_{0m}, c_{1m}, \cdots, c_{mm}]^T \tag{7}$$

and

$$B(t) = [b_{0m}(t), b_{1m}(t), \cdots, b_{mm}(t)]^T. \tag{8}$$

## 4. The operational matrix of integration.

The orthonormal Bernstein polynomials operational matrix of integration of order $(m+1)\times(m+1)$ will be derived now. To achieve this, consider the following integral

$$\int_0^t b_{im}(x)dx = \varphi_i(t), \qquad 0 \le t < 1, \qquad i = 0,1,\cdots,m.$$

$$= \sum_{j=0}^{m} c^i_{jm}\, b_{jm}(t),$$

$$= [c^i_{0m}, c^i_{1m}, \cdots, c^i_{mm}]\, B(t), \qquad \text{for } 0 \le i \le m. \tag{9}$$

Using equations (8) and (9), we obtain

$$\int_0^t B(x)\,dx = P_{m+1}\, B(t), \tag{10}$$

where the operational matrix $P_{m+1}$ of integration associated with orthonormal Bernstein polynomials is given by

$$P_{m+1} = (c^{i}_{j\,m})^{m}_{i,\ j=0} \tag{11}$$

and

$$c^{i}_{j\,m} = \left\langle \varphi_i ,\, b_{j\,m} \right\rangle . \tag{12}$$

For $m = 5$, the matrix $P_6$ is denoted by $P$ and is given as follows:

$$\mathrm{P} := \begin{pmatrix} 0.152778 & 0.288948 & 0.241533 & 0.20663 & 0.159329 & 0.092228 \\ -0.012563 & 0.125 & 0.242527 & 0.180128 & 0.146743 & 0.082341 \\ 0.002216 & -0.022048 & 0.097222 & 0.191725 & 0.116686 & 0.077868 \\ -0.000624 & 0.006211 & -0.027389 & 0.069444 & 0.134479 & 0.051021 \\ 0.000242 & -0.002406 & 0.010608 & -0.026896 & 0.041667 & 0.065296 \\ -0.00099 & 0.000992 & -0.004375 & 0.011092 & -0.017183 & 0.013889 \end{pmatrix} \tag{13}$$

## 5. Numerical example

The following example shows the computational power of the Bernstein polynomial operational matrix of integration.
Consider a linear time-varying system

$$a\,\dot{y}(t) + y(t) = u(t), \qquad with \quad y(0) = 0, \tag{14}$$

where $u(t)$ is the unit step function. The analytic solution of (14) is $y(t) = 1 - e^{-t/a}$. Gu and Jiang [15] considered this problem with $a = 0.25$ and gave an approximate solution by using Haar wavelets with four, six and ten basis functions. Paraskevopoulos et all. [12] considered the same problem with $a = 1$ and used Fourier series operational matrix of integration of orders $(11\times 11)$ and $(21\times 21)$ to obtain approximate solutions. In 2001, Razzaghi and Yousefi [16] used Legendre wavelets operational matrix of order $(6\times 6)$ to solve this problem. We obtain approximate solution of (14) using the Bernstein operational matrix of integration $P_{m+1}$ by taking $m = 4$, $5$ and compare the solutions.
Integrating (14) from $0$ to $t$, we get

$$a\,y(t) + \int_0^t y(x)\,dx = \int_0^t u(x)\,dx . \tag{15}$$

Using (6), the unknown function $y(t)$ and unit step function $u(t)$ are approximated as

$$y(t) = C^T B(t) \text{ and } u(t) = d^T B(t), \tag{16}$$

where $C = [c_{0m}, c_{1m}, \ldots, c_{mm}]^T$ is to be determined. Substituting (16) in (15), we obtain

$$a C^T B(t) + \int_0^t C^T B(x)\, dx = \int_0^t d^T B(x)\, dx. \tag{17}$$

Using (10) , we get

$$a C^T B(t) + C^T P_{m+1} B(t) = d^T P_{m+1} B(t). \tag{18}$$

As (18) holds for all $t \in [0,1)$, it reduces to

$$a C^T + C^T P_{m+1} = d^T P_{m+1}. \tag{19}$$

Taking transpose of (19), one obtains

$$(a I + P_{m+1}^T) C = P_{m+1}^T d .$$

Writing $Q = a I + P_{m+1}^T$, and $E = P_{m+1}^T d$, we get

$$Q C = E, \tag{20}$$

where $I$ is the $(m+1)\times(m+1)$ unit matrix. Eqn. (20) is a set of algebraic equations whose solution gives $c_{im}$, $0 \le i \le m.$ Solving (20), one gets

$$C = Q^{-1} E. \tag{21}$$

Finally, the solution $y(t)$ is obtained by substituting (21) into (16).For $m = 5$, the $d^T$ is given by

$$d^T = [0.552771, 0.5, 0.440959, 0.372678, 0.288675, 0.166667]. \tag{22}$$

In figures 1 and 2, graphs of the exact solution as well as those of the approximate solution $y(t) = C^T B(t)$ for $m = 4$ *and* $5$ are given taking $a = 0.25$, respectively . Figure 3 depicts the corresponding errors between the approximate solutions. From Fig.

3, it can be seen that the accuracy increases quite fast as we go from level $m = 4$ *to* $m = 5$.

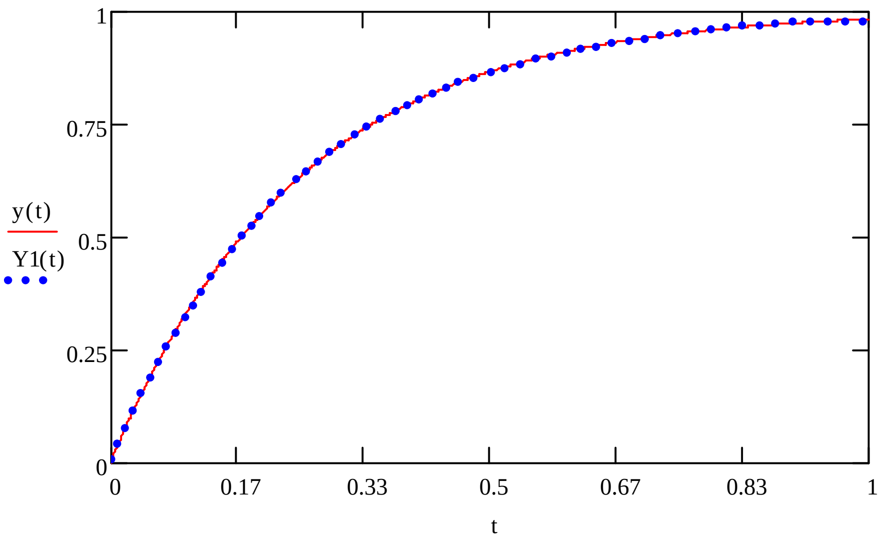


**Fig.1.** The exact solution $y(t)$ (solid line) and the approximate solution denoted by $Y1(t)$ (dotted line) truncated at level $m = 4$ .

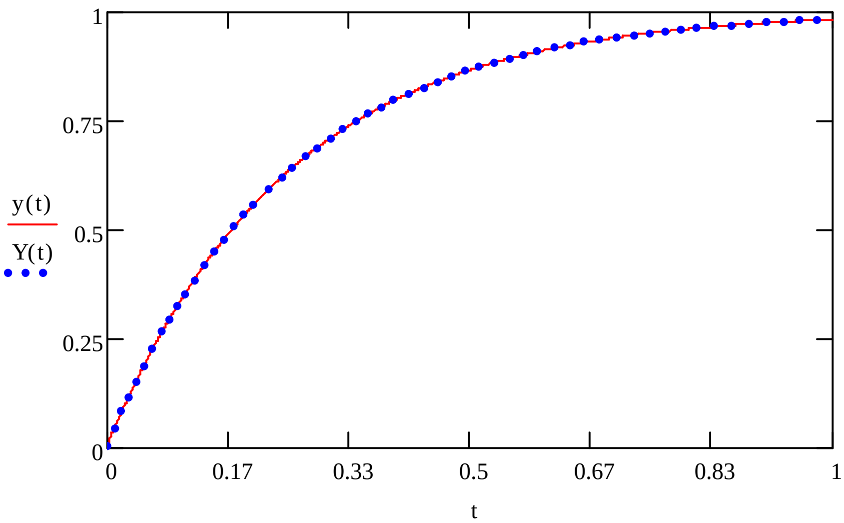


**Fig.2.** The exact solution $y(t)$ (solid line) and the approximate solution denoted by $Y(t)$ (dotted line) truncated at level $m = 5$ .

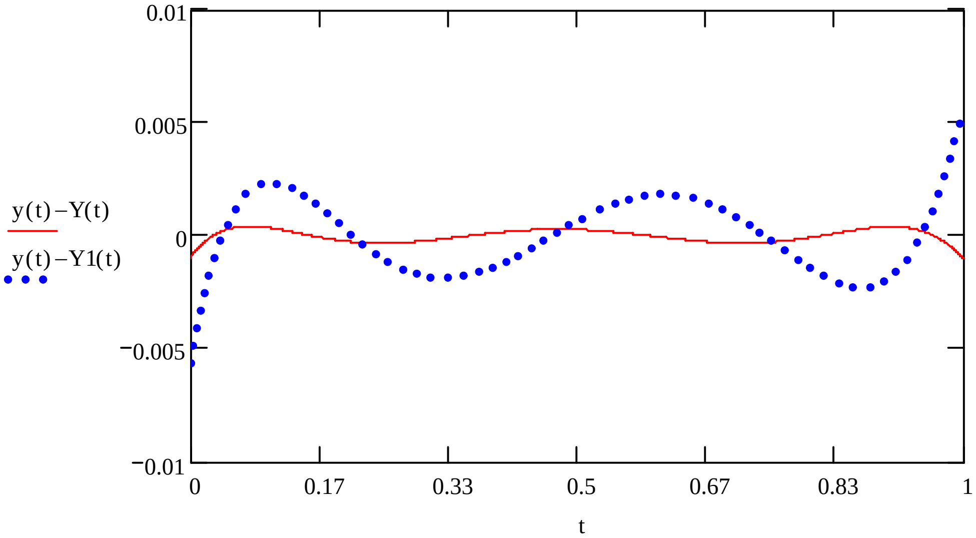


**Fig.3**. Comparison between the errors at $m = 5$ (solid line) and $m = 4$ (dotted line).

## 6. Conclusion.

The uniform approximation capabilities of Bernstein polynomials coupled with the fact that only a small number of polynomials are needed to obtain a satisfactory result makes our method very attractive. It gives better approximation compared to that of paraskevopoulos et all. [12] , Gu and Jiang [15], and Razzaghi and Yousefi [16].

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Received: February, 2009