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A Study on Homogeneous Fuzzy Semi-Markov Model

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Abstract

In this paper, we have defined homogeneous fuzzy semi-Markov process through the interval fuzzy transition possibilities between the states. A large number of results have been obtained including the following conditional possibilities: the process will be in state j after a time t given that it entered at starting time in state i; the process will survive to a time t, given that the starting state; that it will continue to remain in the starting state up to time t; that it reach state j in the next transition, if the previous state was in i and no state change occurred up to time t. We have also analyzed the steady state behavior of homogeneous fuzzy semi-Markov process through steady state behavior of fuzzy Markov chain and average time spent in each state. These approaches are demonstrated by considering web navigational model.

Mathematics Subject Classification: 04A72, 60K15

Keywords: Possibility Space, Conditional Possibility, Fuzzy Transition Possibilities, Fuzzy Semi-Markov Kernel, Fuzzy semi-Markov Model, Steady State Possibilities

1 Introduction

In this paper, the homogeneous fuzzy semi-Markov model is proposed as a useful tool for predicting the evolution of the web access during the specified period through possibilities, by assuming the state transitions as fuzzy transition possibilities on the state space. Important theoretical results and applications for classical semi-Markov models can be found in [1, 3, 4], [6 - 12], [15 - 17]. In this paper, the model is examined under the assumption of the fuzzy transition between the states through the concept of possibility of events. Since the transition between the states of a system cannot be precisely measured due to the system that is intrinsically fuzzy, the decisions are associated with fuzzy transition that can be defined as possibilities on the state space of a system. This model also predicts the steady state behavior of the system. This model has the following advantages:

1. Not only the uncertainties in the different states in which the evolution of transitions into the consideration, but also the uncertainties in the elapsed time in each state.

2. All the states are interrelated, therefore any transition is considered.

3. Fewer and less rigid working hypothesis is needed.

4. Needs only raw data obtained from observations with no strong assumption about any standard possibility functions regarding the possibility variable analyzed.

5. The conclusions are simply based on a list of all computed possibilities derived directly from raw data.

This paper is constructed as follows. Section 2 recalls the basic definitions of possibility space. In section 3, the definition of a discrete time homogeneous fuzzy semi-Markov model, convolution of two fuzzy semi-Markov kernels, Chapman-kolmogorov equation for fuzzy semi-Markov kernel and the basic equations for the interval fuzzy transition possibility are provided. Section 4 gives the solution for the interval fuzzy transition possibilities. Section 5 deals with the steady state analysis of homogeneous fuzzy semi-Markov process. Section 6 illustrates the fuzzy semi-Markov model described in section 3, 4 and 5. The conclusion is discussed in section 7. All the definitions and results are based on max - min operation and we know that the computation in the max - min operation are more robust to perturbation when compared to usual addition and multiplication operation.

2 Preliminary Notes

In this section, we recall some of the basic definitions of possibility space.

Definition 2.1. *(Possibility Space [5])*
Let Γ be the universe of discourse and Ψ be the power set of Γ. The possibility measure is a mapping $\sigma : \Psi \longrightarrow [0,1]$ such that

1. $\sigma(\Phi) = 0$; $\sigma(\Gamma) = 1$
2. $\sigma(\bigcup_i A_i)$=$\sup_i(\sigma(A_i))$

for every arbitrary collection A_i of Ψ. Then (Γ, Ψ, σ) is called as Possibility Space.

Definition 2.2. *(Conditional Possibility [5])*
Let (Γ, Ψ, σ) be a possibility space and $A, B \in \Psi$. Then the possibility of A conditional on B is defined as

$$\sigma(A/B) = \begin{cases} 1 & if \;\; \sigma(AB) = \sigma(B) \\ \sigma(AB) & if \;\; \sigma(AB) < \sigma(B) \end{cases}$$

Definition 2.3. *(Total Possibility Law [5])*
Let (Γ, Ψ, σ) be a possibility space and $\{A_i\}$ be the collection of sets such that $\bigcup A_i = \Gamma$ and $B \in \Psi$. Then ,

$$\begin{aligned} \sigma(B) &= \sup_i \sigma(A_i B) \\ &= \sup_i \min[\sigma(B/A_i), \sigma(A_i)] \end{aligned}$$

Definition 2.4. *(Possibility Variable [5])*
Let (Γ, Ψ, σ) be a possibility space and U be an arbitrary universe. A possibility variable X is a mapping from Γ to U. If a possibility variable can take on countable number of possible values(i.e.,U is countable), then it is called discrete possibility variable. If U is uncountable, then X is called continuous possibility variable.

Definition 2.5. *(Point Possibility Distribution Function [5])*
Let (Γ, Ψ, σ) be a possibility space and U be an arbitrary universe. If $X : \Gamma \longrightarrow U$ is a discrete possibility variable, then the point possibility distribution function is,

$$g(x) = \sigma(X = x), \forall x \in U.$$

3 Discrete Time Homogeneous Fuzzy Semi-Markov Model

A classical semi-Markov process [10] is defined as a sequence of two dimensional random variable $\{X_n, T_n; t \in T\}$,with the properties (1)X_n is a discrete time Markov chain taking values in a countable set S of the system and represents its state after transition n, (2) the holding times $T_{n+1}-T_n$ between two transitions are random variable, whose distribution depends on the present state and the state after the next transition and is given as

$$\begin{aligned} p[X_{n+1} = j, T_{n+1} - T_n \leq t/X_0, X_1, \ldots, X_n; T_0, T_1, \ldots, T_n] \\ = p[X_{n+1} = j, T_{n+1} - T_n \leq t/X_n] \end{aligned}$$

Since for many systems due to uncertainties and imprecision of data, the estimation of precise values of probability is very difficult. For this reason, we have used possibilities defined on possibility space and since the computation using max-min operation are more robust to perturbation when compared to usual addition and multiplication operation, we have followed max-min operation through out the paper. We now discuss about the discrete time homogeneous fuzzy semi-Markov model, whose transitions are taken as fuzzy transition possibilities on the state space.

Let E={1, 2, ... ,m} be the state space and let (Γ, Ψ, σ)be a possibility space. We define the following possibility variables:

$$J_n : \Gamma \longrightarrow E, \quad S_n : \Gamma \longrightarrow N$$

where J_n represents the state at the n-th transition and S_n represents the time of the n-th transition. The process $(J_n, S_n)_{n\in N}$ is called homogenous fuzzy Markov renewal process if

$$\begin{aligned} \sigma[J_{n+1} = j, S_{n+1} \leq t/J_0, J_1, \ldots J_n = i; S_0, S_1, \ldots S_n] \\ = \sigma[J_{n+1} = j, S_{n+1} \leq t/J_n = i] \end{aligned}$$

and for j≠i

$$\tilde{Q}_{ij}(t) = \sigma[J_{n+1} = j, S_{n+1} - S_n \leq t/J_n = i]$$

is the associated homogeneous fuzzy semi-Markov kernel $\tilde{Q}$.
These fuzzy semi-Markov kernels can be expressed in matrix form as

$$\tilde{Q} = (\tilde{Q}_{ij}(t)) = \begin{bmatrix} \tilde{Q}_{11}(t) & \tilde{Q}_{12}(t) & . & . & . & \tilde{Q}_{1m}(t) \\ \tilde{Q}_{21}(t) & \tilde{Q}_{22}(t) & . & . & . & \tilde{Q}_{2m}(t) \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ \tilde{Q}_{m1}(t) & \tilde{Q}_{m2}(t) & . & . & . & \tilde{Q}_{mm}(t) \end{bmatrix}_{mxm}$$

called fuzzy semi-Markov kernel matrix.
Thus, we have

$$\tilde{p}_{ij} = \lim_{t \longrightarrow \infty} \tilde{Q}_{ij}(t), i, j \in E, j \neq i$$

$$= \sigma[J_{n+1} = j / J_n = i]$$

represents the possibility of a system making its next transition to state j, given that it entered state i at time t and $\tilde{P} = (\tilde{p}_{ij})_{i,j}$ is the m x m fuzzy transition possibility matrix of the embedded homogeneous fuzzy Markov chain $(J_n)_{n \in N}$.

3.1 Chapman-Kolmogorov Equation for Fuzzy Semi-Markov Kernel

To derive Chapman-Kolmogorov Equation for Fuzzy Semi-Markov Kernel (FSMK), we first define the convolution of two fuzzy semi-Markov kernels.

3.1.1 Convolution of Two Fuzzy Semi-Markov Kernels

For two FSMK matrices $\tilde{Q} = (\tilde{Q}_{ij}(t))$ and $\tilde{T} = (\tilde{T}_{ij}(t))$ where $\tilde{Q}_{ij}(t),\ \tilde{T}_{ij}(t)$ are defined on E, we define its convolution as

$$\tilde{Q} * \tilde{T} = \tilde{U},$$

where $\tilde{U} = (\tilde{U}_{ij}(t))$ and

$$\tilde{U}_{ij}(t) = \max_{k \in E}\{ \max_{\tau=0,1,\dots,t} \{\min[\tilde{Q}_{ik}(\tau), \tilde{T}_{kj}(t-\tau)]\}\}$$

For our conveneience, let

$$\max_{\tau=0,1,\dots,t}\{\min[\tilde{Q}_{ik}(\tau), \tilde{T}_{kj}(t-\tau)]\} = \tilde{Q}_{ik}(\tau) \circledcirc \tilde{T}_{kj}(t-\tau)$$

and hence

$$\tilde{U}_{ij}(t) = \max_{k \in E}\{\tilde{Q}_{ik}(\tau) \circledcirc \tilde{T}_{kj}(t-\tau)\}$$

3.1.2 Chapman-Kolmogorov equation for FSMK

We now derive the Chapman-Kolmogorov equation for FSMK by induction method.
consider

$$\tilde{Q}^{(1)}_{ij}(t) = \sigma[J_1 = j, S_1 - S_0 \leq t / J_0 = i] = \tilde{Q}_{ij}(t)$$

This implies $\tilde{Q}^{(1)} = \tilde{Q}$ and $\tilde{Q}_{ij}^{(0)}(t)= \begin{cases} 0, & \text{if t} \leq 0 \\ 1, & \text{otherwise} \end{cases}$

Now consider,

$$\tilde{Q}_{ij}^{(2)}(t) = \sigma[J_2 = j, S_2 - S_1 \leq t/J_0 = i]$$

The state j can be reached to state i through some intermediate state k with duration time t in 2 steps as

$$\begin{aligned} \tilde{Q}_{ij}^{(2)}(t) &= \min[\sigma[J_2 = j, S_2 - S_1 \leq t - \tau/J_1 = k], \sigma[J_1 = k, S_1 - S_0 \leq \tau/J_0 = i]] \\ &= \min[\tilde{Q}_{kj}^{(1)}(t-\tau), \tilde{Q}_{ik}^{(1)}(\tau)] \\ &= \min[\tilde{Q}_{ik}^{(1)}(\tau), \tilde{Q}_{kj}^{(1)}(t-\tau)] \end{aligned}$$

Since these intermediate steps can take values k=1, 2, ... , m corresponding to their time durations, we have

$$\tilde{Q}_{ij}^{(2)}(t) = \max_{k\in E}\{\max_{\tau=0,1,\dots,t}\{\min[\tilde{Q}_{ik}^{(1)}(\tau), \tilde{Q}_{kj}^{(1)}(t-\tau)]\}\}$$

This implies

$$\begin{aligned} \tilde{Q}^{(2)} &= \tilde{Q}^{(1)} * \tilde{Q}^{(1)} \\ &= \tilde{Q} * \tilde{Q} \end{aligned}$$

By induction, we have

$$\tilde{Q}_{ij}^{(m+1)}(t) = \max_{k\in E}\{\max_{\tau=0,1,\dots,t}\{\min[\tilde{Q}_{ik}^{(m)}(\tau), \tilde{Q}_{kj}^{(1)}(t-\tau)]\}\}, m \geq 0.$$

This implies

$$\tilde{Q}^{(m+1)} = \tilde{Q}^{(m)} * \tilde{Q}^{(1)} = \tilde{Q}^{m} * \tilde{Q}$$

Thus in general, we have

$$\tilde{Q}^{(m+n)} = \tilde{Q}^{(m)} * \tilde{Q}^{(n)}$$

and this is called Chapman-kolmogorov equation for FSMK.

Note that

$$\tilde{Q}_{ij}^{(n)}(t) = \sigma[J_n = j, S_n - S_{n-1} \leq t/J_0 = i]$$

3.2 Evolution Equation of a Discrete Time Homogeneous Fuzzy semi-Markov Model

In this section, we derive the evolution equation of a discrete time homogeneous fuzzy semi-Markov model. The evolution equation represents the interval fuzzy transition possibility from state i to reach state j with time duration t.

We now define the conditional cumulative distribution function of the waiting time in each state, given the state subsequently occupied by,

$$\tilde{F}_{ij}(t) = \sigma[S_{n+1} - S_n \leq t/J_{n+1} = j, J_n = i]$$

Since the FSMK $\tilde{Q}_{ij}(t)$ is both characterized by a fuzzy Markov chain $(J_n)_{n\in N}$ and transition time $(S_n)_{n\in N}$ which depends on both the present state and the next state, we can rewrite $\tilde{Q}_{ij}(t)$ as

$$\begin{aligned}
\tilde{Q}_{ij}(t) &= \sigma[J_{n+1} = j, S_{n+1} - S_n \leq t/J_n = i] \\
&= \min[\sigma[S_{n+1} - S_n \leq t/J_{n+1} = j, J_n = i], \sigma[J_{n+1} = j/J_n = i]] \\
&= \min[\tilde{F}_{ij}(t), \tilde{p}_{ij}] \\
&= \min[\tilde{p}_{ij}, \tilde{F}_{ij}(t)]
\end{aligned}$$

Without loss of generality, we denote the waiting time distributions as $\tilde{f}_{ij}(t)$ and $\tilde{D} = (\tilde{f}_{ij}(t))_{i,j}$ represent the duration matrix. Let us introduce the possibility that the process will leave state i in a time t as:

$$\tilde{H}_i(t) = \sigma[S_{n+1} - S_n \leq t/J_n = i]$$

Obviously, it results that

$$\begin{aligned}
\tilde{H}_i(t) &= \max_{j\neq i} \tilde{Q}_{ij}(t) \\
&= \max_{j\neq i}\{\min[\tilde{p}_{ij}, \tilde{F}_{ij}(t)]\}, i, j \in E
\end{aligned}$$

Let us define the possibility that the process has been in state i for time duration 't' without transitioning to other state as

$$\tilde{S}_i(t) = \sigma[S_{n+1} - S_n > t/J_n = i]$$

Now it is possible to define the discrete time homogeneous fuzzy semi-Markov process Z:

$$Z = (Z_t, t \in R_0^+)$$

representing for each waiting time t, the state occupied by the process $Z_t = J_{N(t)}$, where $N(t) = \max\{n, T_n \leq t\}$. This fuzzy semi-Markov process is both characterized by a set of fuzzy transition matrices $\tilde{P}$ and a set of duration matrix $\tilde{D}$. Now we define the interval fuzzy transition possibilities in the following way:

$$\tilde{\phi}_{ij}(t) \quad = \quad \sigma[Z_t = j/Z_0 = i]$$

By taking all the possible mutually exclusive ways in which it is possible for the event to take place, we could prove that for $\forall t, t \geq 0$.

$$\tilde{\phi}_{ij}(t) \quad = \quad \max\{\min[\tilde{\delta}_{ij}, \tilde{S}_i(t)], \max_{l\in E}\{\max_{\tau=0,1,\ldots,t}\{\min[\tilde{p}_{il}, \tilde{f}_{il}(\tau), \tilde{\phi}_{lj}(t-\tau)]\}\}\}$$

where

$$\tilde{\delta}_{ij}= \begin{cases} 0, & \text{i} \neq \text{j} \\ 1, & \text{i=j} \end{cases}$$

The above equation represents the possibility of remaining in state i without any change from time t and possibility of having changed in state i and of having reached in some way to state j and of staying in this state at time t and this equation is called as the evolution equation of a discrete time homogeneous fuzzy semi-Markov model.

4 Solution for Interval Fuzzy Transition Possibilities

We now discuss the solution for interval fuzzy transition possibilities. This solution represents the future behavior of the system starting from one state and reaching to other state through some intermediate states during the specified time of duration. Let there be 'k' number of transitions to state j at time 't', given that the process started from state i. Let us also assume that these 'k' transitions occur successively from state i with time duration $t_1, t_2, \ldots$and reaches to state j for the given duration time 't'.

Let us first assume that, there occurs one transition from state i with duration time t_1 and reaches to state j. Then the evolution equation of homogeneous fuzzy semi-Markov process is written as

$$\tilde{\phi}_{ij}(t) \quad = \quad \max\{\min[\tilde{\delta}_{ij}, \tilde{S}_i(t)], \max_{t_1=0,1,\ldots,t}\{\min[\tilde{p}_{ij}, \tilde{f}_{ij}(t_1), \tilde{S}_j(t-t_1)]\}\}$$

and the evolution equation of homogeneous fuzzy semi-Markov process from state i with duration time t_1, t_2 to reach state j occurring with two transitions is written as

$$\tilde{\phi}_{ij}(t) = \max \left\{ \begin{array}{l} \min[\tilde{\delta}_{ij}, \tilde{S}_i(t)], \max_{t_1=0,1,\ldots,t}\{\min[\tilde{p}_{ij}, \tilde{f}_{ij}(t_1), \tilde{S}_j(t-t_1)]\}, \\ \max_{l\in E}\{\max_{t_1=0,1,\ldots,t}\{max_{t_2=0,1,\ldots,t-t_1} \\ \{\min[\tilde{p}_{il}, \tilde{f}_{il}(t_1), \tilde{p}_{lj}, \tilde{f}_{lj}(t_2), \tilde{S}_j(t-t_1-t_2)]\}\}\} \end{array} \right\}$$

In general, if there are 'k' transitions occur successively from state i with duration time $t_1, t_2, ...$ and reaches to state j with total duration time 't'. Then the evolution equation of homogeneous fuzzy semi-Markov process can be written as

$$\tilde{\phi}_{ij}(t) = \max\{T_0,\, T_1,\, T_2,\, T'\}$$

where
$T_0 = \min[\tilde{\delta}_{ij}, \tilde{S}_i(t)]$

$T_1 = \max_{t_1=0,1,\ldots,t}\{\min[\tilde{p}_{ij}, \tilde{f}_{ij}(t_1), \tilde{S}_j(t-t_1)]\}$

$$T_2 = \left\{ \begin{array}{l} \max_{l\in E}\{\max_{t_1=0,1,\ldots,t}\{max_{t_2=0,1,\ldots,t-t_1}\{\min[\tilde{p}_{il}, \tilde{f}_{il}(t_1), \\ \tilde{p}_{lj}, \tilde{f}_{lj}(t_2), \tilde{S}_j(t-t_1-t_2)]\}\}\} \end{array} \right\}$$

$$T_k = \left\{ \begin{array}{l} \max_{l\neq i,\, l\in E}\{\max_{n\neq l,\, n\in E}\ldots\{\max_{w\neq j,\, w\in E} \\ \{\max_{t_1=0,1,\ldots,t}\{max_{t_2=0,1,\ldots,t-t_1}\ldots\{\max_{t_k=0,1,\ldots,t-t_1\ldots-t_{k-1}} \\ \{\min[\tilde{p}_{il}, \tilde{f}_{il}(t_1), \tilde{p}_{ln}, \tilde{f}_{ln}(t_2), \ldots, \tilde{p}_{wj}, \tilde{f}_{wj}(t_k), \tilde{S}_j(t-t_1-\ldots-t_k)]\}\}\}\}\}\} \end{array} \right\}$$

and $T' = \max_{k=3,4,\ldots} T_k, for\, k \geq 3.$

These expression formalizes the fact that possibility of homogeneous fuzzy semi-Markov process is in state j with duration t, given that it has entered state i may be derived from no transition (k=0) or from exactly one transition (k=1) or from exactly two transitions (k=2) or more ($k \geq 3$).

Let us define $\tilde{\phi}^k_{ij}(t)$ by the following possibility:

$\tilde{\phi}^k_{ij}(t)=\sigma$[the process is in state j at time t, k transitions with duration $t-t_1-\ldots-t_k$ / it was in state i]

Hence finally for i, j $\in E$,

$$\tilde{\phi}_{ij}(t) \;=\; \max_{k=0,1,2,\ldots} \tilde{\phi}^k_{ij}(t)$$

And in matrix form with $\tilde{\phi}(t) = (\tilde{\phi}_{ij}(t))_{i,j}$ and $\tilde{\phi}^k(t) = (\tilde{\phi}^k_{ij}(t))_{i,j}$

$$i.e.,\ \tilde{\phi}(t) \;=\; \max_{k=0,1,2,\ldots,\infty} \tilde{\phi}^k(t)$$

5 Steady State Analysis of Homogeneous Fuzzy Semi-Markov Process

Fuzzy semi-Markov processes can be analyzed for steady state performance in the same manner as discrete time fuzzy Markov process. To do this, we need to know the steady state possibilities of the (Fuzzy semi-Markov process's) embedded fuzzy Markov chain and the mean residence time in each state or the average time spent in each state.

5.1 Steady State Possibilities of Embedded Fuzzy Markov Chain

We see that in [2], if the powers of the fuzzy transition matrix converge in T steps to a nonperiodic solution, then the associated fuzzy Markov chain is said to be aperiodic and if the fuzzy Markov chain is aperiodic then it is said to be ergodic. We should always remember that not all fuzzy Markov process will have a steady state distribution. If the fuzzy Markov process is ergodic then ergodic fuzzy Markov process always tends to a steady state.

In [13], [14] the possibilities of remaining in each state after n steps is defined using max - min operation as a row vector $\tilde{V}^n = (\tilde{V}_1^n, \tilde{V}_2^n, ..., \tilde{V}_m^n)$ where each entry $\tilde{V}_j^n$ is the maximum of minimum of each path of length n from state i to state j and the steady state possibilities or steady state distribution is defined as a row vector $\tilde{V}^\infty = (\tilde{V}_1^\infty, \tilde{V}_2^\infty, ..., \tilde{V}_m^\infty)$ where each entry $\tilde{V}_j^\infty$ is the maximum of minimum of each path from state i to state j given as

$$\tilde{V}_j^\infty = \max\{\min_i[\tilde{V}_i^\infty, \tilde{p}_{ij}]\},\ j\ =\ 1, 2, \dots, m \qquad (5.1.5.1)$$

and is obtained from the following algorithm:

1. Initialize the components $\tilde{V}_1^\infty, \tilde{V}_2^\infty, ..., \tilde{V}_m^\infty$.

2. Fix a threshold limit(based on usage) and calculate each component of the vector by the calculation $|R_i - L_i| \leq \sigma$, where R_i and L_i are RHS and LHS of the m equations given by equation 5.1.5.1. The computation is stopped if the desired σ is fulfilled.

5.2 Average Time for Each State

We know that time spent at each state j (for j=1,2, . . . ,m) is given by

$$\tilde{S}_j(t) = \sigma[S_{n+1} - S_n > t/J_n = j]$$

If the time spent at each state varies with time duration u = 0, 1, 2,...t, then the average time spent at each state j (for j=1,2, . . . ,m) is given by

$$\tilde{\theta}_j \ = \ \max_{u=0,1,\dots,t} \tilde{S}_j(u), j \in E$$

Therefore the steady state possibility of being in state j (for j=1,2, . . . ,m) for fuzzy semi-Markov process is

$$\tilde{\varphi}_j = \begin{cases} 1, & \text{if } \min(\tilde{V}_j^\infty, \tilde{\theta}_j) = \max_{j\in E}\min(\tilde{V}_j^\infty, \tilde{\theta}_j) \\ \min(\tilde{V}_j^\infty, \tilde{\theta}_j), & \text{if } \min(\tilde{V}_j^\infty, \tilde{\theta}_j) < \max_{j\in E}\min(\tilde{V}_j^\infty, \tilde{\theta}_j). \end{cases} \qquad (5.2.5.1)$$

6 Example

Consider the below web navigation model.

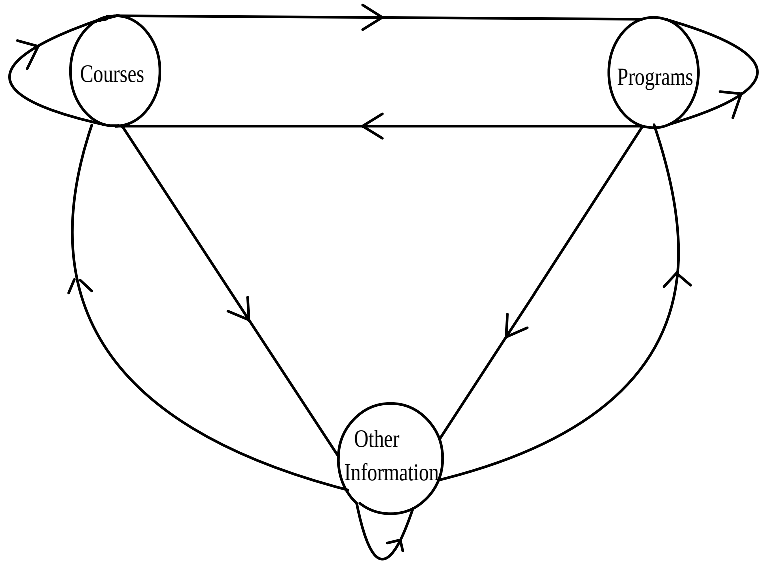

Figure 1: Web Navigational Model

The operational units Courses(C), Programs(P), Other information(OI) are the set of states with the associated connections as the transitions. With each transition we associate a possibility based on the usage for the period of 12 weeks duration. These possibilities can be obtained from various sources containing information based on actual usage patterns to track usage and failures. As the web access increases the usage information will also increase. Hence we have modeled fuzzy semi-Markov model with state space S = {C, P, OI} for the above web navigation.
The corresponding fuzzy state transition matrix is

$$\tilde{P} = \begin{array}{c} \\ C \\ P \\ OI \end{array} \begin{array}{c} \begin{array}{ccc} C & P & OI \end{array} \\ \begin{bmatrix} 0.98 & 0.98 & 0.973 \\ 0.98 & 0.776 & 0.98 \\ 0.782 & 0.99 & 0.969 \end{bmatrix} \end{array}$$

and the calculated $\tilde{D}$ for the given state space as

$$\tilde{D} = \begin{array}{c} \\ C \\ P \\ OI \end{array} \begin{array}{c} \begin{array}{ccc} C & P & OI \end{array} \\ \begin{bmatrix} 0.981 & 0.982 & 0.9 \\ 0.89 & 0.964 & 0.841 \\ 0.964 & 0.872 & 0.78 \end{bmatrix} \end{array}$$

Possibilities of staying in each state up to 12 weeks are given as follows:

$$\tilde{S}_C(12) = 0.985;\ \tilde{S}_P(12) = 0.982;\ \tilde{S}_{OI}(12) = 0.964$$

Hence the interval fuzzy transition possibility from one state to another state for the given period is given as matrix and is depicted below in Figure 2:

$$\tilde{\phi}_{ij}(12) = \begin{array}{c} \\ C \\ P \\ OI \end{array} \begin{array}{c} \begin{array}{ccc} C & P & OI \end{array} \\ \begin{bmatrix} 0.985 & 0.98 & 0.9 \\ 0.89 & 0.982 & 0.841 \\ 0.782 & 0.872 & 0.969 \end{bmatrix} \end{array}, i,\ j = C, P, OI.$$

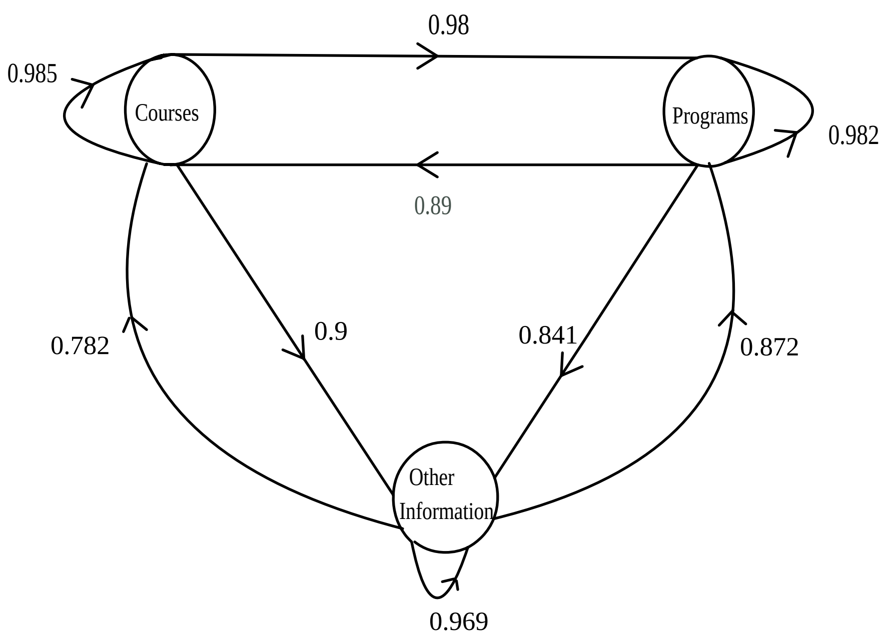

Figure 2: Interval Fuzzy Transition Possibilities

6.1 Steady State Analysis for Web Navigation Model

In section 5, we have seen that the possibility of system being in each state in the steady state as $\tilde{V}^{\infty} = (\tilde{V}_1^{\infty}, \tilde{V}_2^{\infty}, ..., \tilde{V}_m^{\infty})$. We now calculate the steady state possibilities for the navigation model discussed in the previous section. $\tilde{V}^{\infty} = (\tilde{V}_1^{\infty}, \tilde{V}_2^{\infty}, \tilde{V}_3^{\infty})$ can be determined by the equation $\tilde{V}_j^{\infty} = \max\{\min_i[\tilde{V}_i^{\infty}, \tilde{p}_{ij}]\}$, for j = C, P, OI.

By fixing the threshold limit as $\sigma = 0.05$ and following the algorithm in section 5, we can compute the steady state possibilities of the fuzzy Markov model. Thus the steady state possibilities that we have obtained are as follows (0.9, 0.85, 0.8).

The mean residence possibilities for the state space are given by

$$\tilde{\theta}_C = 0.985;\ \tilde{\theta}_P = 0.982;\ \tilde{\theta}_{OI} = 0.964$$

Based on this steady state possibilities and mean residence possibilities, the steady state possibilities of being in state j (for j=C, P, OI) of fuzzy semi-Markov process can be computed using equation 5.2.5.1.

∴ Steady state possibilities that we obtained are as follows:

$$\begin{aligned} \tilde{\varphi}_C &= 1 \\ \tilde{\varphi}_P &= 0.85 \\ \tilde{\varphi}_{OI} &= 0.8 \end{aligned}$$

From the steady state possibilities we see that $\tilde{\varphi}_{OI} = 0.8$ which indicates that the state is less frequently visited when compared to the other states and the most frequently visited state is "Course". Thus the most frequently visited states are "Course" and "Programs". Hence more focus should be given on testing links that leads to these states.

7 Conclusion

In this paper we have defined a homogeneous fuzzy semi-Markov model and presented a homogeneous fuzzy semi-Markov model approach to the dynamic evolution of web application defined by interval fuzzy transition possibilities. By means of this approach, we cannot consider only uncertainties in the possible stages of transitions, but also the uncertainties of the duration of the waiting time in each state. This method starts from the idea of evolution of interval fuzzy transition possibility through fuzzy transition possibility and duration time process and this idea allows the approach of homogeneous fuzzy semi-Markov model resulting to step transitions and the steady state possibilities of web application.

We would like to point out that this paper does not show all the potential of homogeneous fuzzy semi-Markov process environment. Indeed, by means of recurrence time process it is possible to assess the different transition possibilities as a function of the duration inside the states. Moreover, it allows to the possibility of doing a reliability analysis that considers in the future research.

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