Exponential Stability of Switched Systems
with Mixed Time Delays

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Abstract

This paper address the exponential stability for a class of switched systems with mixed time delays. Based on linear matrix inequalities and Lyapunov-Krasovskii functional approach, a geometrically switching rule for the exponential stability of the system is designed. The approach allows to compute simultaneously the two bounds that characterize the exponential stability rate of the solution. Numerical example to show the effectiveness of the proposed method is given.

1 Introduction

Hybrid system is a combination of discrete and continuous dynamical systems. These systems arise as models phenomena, which cannot be described by exclusively continuous or exclusively discrete processes [1, 8]. As an important class of hybrid systems, switched system is a family of subsystems together with rules to switch between them.

Switched systems arise in many practical models in manufacturing, communication networks, automotive engine control, chemical processes, e.g. see [1, 5, 9] and the references therein. The stability analysis of switched time-delay systems has attracted a lot of attention from many researchers [3, 4, 7, 11]. The main approach for stability analysis relies on the use of Lyapunov-Krasovskii functionals and LMI approach for constructing common Lyapunov function

and switching rules [3, 4, 6, 13]. In recent paper [4], studying a switching system composed of a finite number of linear point time-delay differential equations, it was shown that the asymptotic stability of this kind of switching systems may be achieved by using a common Lyapunov function method and minimum switching rule. Some extending results of [4] to linear switching systems with discrete and distributed time delays were given in [2]. However, the common Lyapunov functional and the switching rule were constructed base on the existence of a Hurwitz linear convex combination of non-delay system matrices. This condition is delay-independent and it was shown that the system was stable for sufficiently small delays. The problem of stability and stabilization of a class of switched neutral control systems were investigated in [12]. By using quadratic Lyapunov functions and inequalities analysis technique, a delay-dependent stability condition was formulated in term of linear matrix inequalities. However, this condition was delay-independent of the derivation of the past state.

In this paper, we study the problem of exponential stability for a class of linear switching systems with mixed time delays. By using an improved Lyapunov-Krasovskii functional, a delay-dependent conditions for the exponential stability of the systems are derived in terms of the linear matrix inequalities which allows to design switching rules and to compute simultaneously the two bounds that characterize the exponential stability rate of the solution.

The paper is organized as follows. Section 2 presents notations, definitions and technical lemmas need for the proof of the main result. Sufficient conditions for the exponential stability and numerical example to illustrate the obtained results are presented in Section 3. The paper ends with conclusions and cited references.

2   Preliminaries

The following notations will be used throughout this paper: R^n denotes the n dimensional Euclidean space with the Euclidean norm $\|.\|$ and scalar product $\langle x, y \rangle = x^T y$; $\lambda_{\max}(A)$ ($\lambda_{\min}(A)$, resp.) denotes the maximal (the minimum, resp.) of the real part of eigenvalues of A; A^T denotes the transpose of the matrix A; $Q \geq 0$ ($Q > 0$, resp.) means Q is semi-positive definite (positive definite, resp.), $A \geq B$ means $A - B \geq 0$.

Consider a switched linear system with mixed time delays of the form

$$\dot{x}(t) = A_\sigma x(t) + D_\sigma x(t-h) + E_\sigma \int_{t-r}^{t} x(s)ds,$$
$$x(t) = \phi(t), \quad t \in [-\tau, 0], \tau = \max\{h, r\}, \tag{1}$$

where $x(t) \in R^n$ is the state, $\sigma \in \bar{m} = \{1, 2, \ldots, m\}$ is piecewise constant switching signal depending on time and the system state and will be designed, $A_i, D_i, E_i, \ (i = 1, 2, \ldots, m)$ are given real matrices.

Definition 1. *Given $\alpha > 0$. The system (1) is α-exponentially stable if there exist a switching rule σ and a constant $N \geq 1$ such that every solution $x(t, \phi)$ of the system satisfies the following inequality*

$$\|x(t, \phi)\| \leq Ne^{-\alpha t}\|\phi\|, \quad t \geq 0.$$

Definition 2. *The system of matrices $\{L_i\}, i \in \bar{m} = \{1, 2, \ldots, m\}$ is said to be strictly complete if for every $x \in R^n\backslash\{0\}$ there is $i \in \bar{m}$ such that $x^T L_i x < 0$.*

Let us define
$$\Omega_i = \{x \in R^n : x^T L_i x < 0\}, i \in \bar{m}$$

It's easy to show that the system $\{L_i\}, i \in \bar{m}$ is strictly complete if and only if

$$\bigcup_{i=1}^m \Omega_i = R^n\backslash\{0\}.$$

Remark 1. *As shown in ([10]), a sufficient condition for the strictly completeness of the system $\{L_i\}$ is that there exist $\beta_i \geq 0, \sum_{i=1}^m \beta_i > 0$ such that*

$$\sum_{i=1}^m \beta_i L_i < 0.$$

If $N = 2$ then the above condition is also necessary for the strictly completeness.

The following well-known lemmas will be used in the proof of our main results.

Proposition 1. *(Matrix Cauchy inequality) For any symmetric positive definite matrix M and $x, y \in R^n$ one has*

$$\pm 2x^T y \leq x^T M x + y^T M^{-1} y.$$

Proposition 2. *For any symmetric positive definite matrix M, scalar $\gamma > 0$ and vector function $\omega : [0, \gamma] \to R^n$ such that the integrals concerned are well defined, then*

$$\left(\int_0^\gamma \omega(s)ds\right)^T M \left(\int_0^\gamma \omega(s)ds\right) \leq \gamma \int_0^\gamma \omega^T(s)M\omega(s)ds.$$

3   Main results

For given $\alpha > 0, h, r > 0$ and symmetric positive definite matrices P, Q, S, M, we denote

$$L_i = A_i^T P + P A_i + 2\alpha P + Q + rS + M, \quad i \in \bar{m},$$

$$\Omega_i = \{x \in R^n : x^T L_i x < 0\}, \quad i \in \bar{m},$$

$$\overline{\Omega}_1 = \Omega_1, \quad \overline{\Omega}_i = \Omega_i \backslash \bigcup_{j=1}^{i-1} \overline{\Omega}_j, \quad i = 2, 3, \dots, m.$$

Theorem 1. *Given $\alpha > 0$. The system (1) is α-exponentially stable if the exist symmetric positive definite matrices P, Q, S, M satisfy the following conditions:*

(i) *The system of matrices $\{L_i\}$ is strictly complete,*

(ii) $\begin{bmatrix} M & PD_i & PE_i \\ D_i^T P & e^{-2\alpha h}Q & 0 \\ E_i^T P & 0 & \dfrac{1}{r}e^{-2\alpha r}S \end{bmatrix} > 0, \quad (i = 1, 2, \dots, m).$

The switching rule is chosen as $\sigma(x(t)) = i$ whenever $x(t) \in \overline{\Omega}_i$. Moreover, all solution $x(t, \phi)$ of the system satisfies

$$\|x(t, \phi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{-\alpha t} \|\phi\|, \quad t \geq 0,$$

where $\alpha_1 = \lambda_{\min}(P), \quad \alpha_2 = \lambda_{\max}(P) + h\lambda_{\max}(Q) + \dfrac{1}{2}r^2\lambda_{\max}(S)$.

Proof. By the assumption (i) we have $\bigcup_{i=1}^{m} \Omega_i = R^n \backslash \{0\}$. It follows that

$$\bigcup_{i=1}^{m} \overline{\Omega}_i = R^n \backslash \{0\}, \quad \overline{\Omega}_i \bigcap \overline{\Omega}_j = \emptyset, i \neq j. \tag{2}$$

The switching rule is chosen as $\sigma(x(t)) = i$ whenever $x(t) \in \overline{\Omega}_i$ (this switching rule is well-defined due to (2)). So when $x(t) \in \overline{\Omega}_i$, the *ith* subsystem is activated and then we have the following subsystem

$$\dot{x}(t) = A_i x(t) + D_i x(t - h) + E_i \int_{t-r}^{t} x(s)ds.$$

Consider the following Lyapunov-Krasovskii functional

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t), \tag{3}$$

where

$$V_1(x_t) = x^T(t)Px(t),$$

$$V_2(x_t) = \int_{-h}^{0} e^{2\alpha s} x^T(t+s)Qx(t+s)ds,$$

$$V_3(x_t) = \int_{-r}^{0} \int_{s}^{0} e^{2\alpha \xi} x^T(t+\xi)Sx^T(t+\xi)d\xi ds.$$

It's easy to verify that

$$\alpha_1 \|x(t)\|^2 \le V(x_t) \le \alpha_2 \|x_t\|^2, \quad t \ge 0. \tag{4}$$

Taking derivative of $V_1(x_t)$ along trajectories of the system (1) we get

$$\dot{V}_1(x_t) = x^T(t)(A_i^T P + PA_i)x(t) + 2x^T(t)PD_i x(t-h) + 2x^T(t)PE_i \int_{t-\tau}^{t} x(s)ds.$$

Apply Proposition 1 and 2 we have

$$2x^T(t)PD_i x(t-h) \le e^{2\alpha h} x^T(t)PD_i Q^{-1} D_i^T Px(t) + e^{-2\alpha h} x^T(t-h)Qx(t-h),$$

$$2x^T(t)PE_i \int_{t-\tau}^{t} x(s)ds \le re^{2\alpha r} x^T(t)PE_i S^{-1} E_i^T Px(t)$$

$$+ \frac{1}{r} e^{-2\alpha r} \left(\int_{t-r}^{t} x(s)ds \right)^T S \left(\int_{t-r}^{t} x(s)ds \right)$$

$$\le re^{2\alpha r} x^T(t)PE_i S^{-1} E_i^T Px(t) + e^{-2\alpha r} \int_{t-r}^{t} x^T(s)Sx(s)ds.$$

Therefore,

$$\dot{V}_1(x_t) \le x^T(t) \Big[A_i^T P + PA_i \Big] x(t)$$

$$+ e^{2\alpha h} x^T(t)PD_i Q^{-1} D_i^T Px(t) + e^{-2\alpha h} x^T(t-h)Qx(t-h) \tag{5}$$

$$+ re^{2\alpha r} x^T(t)PE_i S^{-1} E_i^T Px(t) + e^{-2\alpha r} \int_{t-r}^{t} x^T(s)Sx(s)ds.$$

Next, taking derivative of V_2, V_3 along trajectories of system (1) respectively, we obtain

$$\dot{V}_2(x_t) = x^T(t)Qx(t) - e^{-2\alpha h} x^T(t-h)Qx(t-h) - 2\alpha V_2(x_t),$$

$$\dot{V}_3(x_t) = rx^T(t)Sx(t) - \int_{-r}^{0} e^{2\alpha s} x^T(t+s)Sx(t+s)ds - 2\alpha V_3(x_t) \tag{6}$$

$$\le rx^T(t)Sx(t) - e^{-2\alpha r} \int_{t-r}^{t} x^T(s)Sx(s)ds - 2\alpha V_3(x_t).$$

Combining (5), (6) we get

$$\dot{V}(x_t) + 2\alpha V(x_t) \leq x^T(t)\Big[A_i^T P + PA_i + 2\alpha P + Q + rS\Big]x(t)$$
$$+ e^{2\alpha h}x^T(t)PD_iQ^{-1}D_i^T Px(t) + re^{2\alpha r}x^T(t)PE_iS^{-1}E_i^T Px(t).$$

By Schur complement theorem, from hypothesis (ii) we have

$$M - e^{2\alpha h}PD_iQ^{-1}D_i^T P - re^{2\alpha r}PE_iS^{-1}E_i^T P > 0, \quad i \in \bar{m}.$$

Therefore,

$$\dot{V}(x_t) + 2\alpha V(x_t) \leq x^T(t)\Big[PA_i + A_i^T P + 2\alpha P + Q + rS + M\Big]x(t) \tag{7}$$
$$= x^T(t)L_ix(t), \quad t \geq 0.$$

By the completeness of the system of matrices $\{L_i\}$, for any $t \geq 0$, there exists $i \in \bar{m}$ such that $x(t) \in \overline{\Omega}_i$. Therefore, from (7) it follows that

$$\dot{V}(x_t) + 2\alpha V(x_t) \leq 0, \quad \forall t \geq 0,$$

and hence

$$V(x_t) \leq V(\phi)e^{-2\alpha t} \leq \alpha_2\|\phi\|^2 e^{-2\alpha t}, \quad t \geq 0.$$

Taking (4) into account we obtain

$$\|x(t,\phi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}}e^{-\alpha t}\|\phi\|, \quad t \geq 0,$$

which completes the proof.   □

From Remark 1 and Theorem 1 we have the following corollary.

Corollary 2. *Given $\alpha > 0$. The system (1) is α-exponentially stable if the exist a convex combination $\widehat{A}$ of A_i, symmetric positive definite matrices P, Q, S, M such that the following LMIs hold:*

i) $\widehat{A}^T P + P\widehat{A} + 2\alpha P + Q + rS + M < 0,$ (8)

ii) $\begin{bmatrix} M & PD_i & PE_i \\ D_i^T P & e^{-2\alpha h}Q & 0 \\ E_i^T P & 0 & \dfrac{1}{r}e^{-2\alpha r}S \end{bmatrix} > 0, \quad (i = 1, 2, \dots, m),$ (9)

where $\widehat{A} = \sum_{i=1}^{m}\beta_i A_i, \beta_i \geq 0, \sum_{i=1}^{m}\beta_i = 1$. The switching rule is chosen as $\sigma(x(t)) = i$ whenever $x(t) \in \overline{\Omega}_i$. Moreover, all solution $x(t,\phi)$ of the system satisfies

$$\|x(t,\phi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}}e^{-\alpha t}\|\phi\|, \quad t \geq 0.$$

Example. Consider the switching system given by

$$\dot{x}(t) = A_i x(t) + D_i x(t-h) + E_i \int_{t-r}^{t} x(s)ds, \quad i \in \bar{m} = \{1, 2\}, \tag{10}$$

where $h = 1, r = 1$ and

$$(A_1, D_1, E_1) = \left(\begin{bmatrix} -20 & 1 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \right),$$

$$(A_2, D_2, E_2) = \left(\begin{bmatrix} 4 & -1 \\ 1 & -32 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 3 & -4 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 4 \end{bmatrix} \right).$$

It is easy to check that each subsystem is unstable. For $\alpha = 0.5$, by solving LMIs (8), (9) with $\beta_1 = \beta_2 = 0.5$ we get

$$P = \begin{bmatrix} 0.6991 & -0.2583 \\ -0.2583 & 0.5142 \end{bmatrix}, \quad Q = \begin{bmatrix} 2.8282 & -2.2782 \\ -2.2782 & 3.0657 \end{bmatrix},$$

$$S = \begin{bmatrix} 1.2451 & -0.4272 \\ -0.4272 & 2.5712 \end{bmatrix}, \quad M = \begin{bmatrix} 4.1690 & -1.9356 \\ -1.9356 & 6.5331 \end{bmatrix}.$$

Then

$$L_1 = \begin{bmatrix} -17.4725 & -1.0937 \\ -1.0937 & 14.2247 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 14.0176 & 2.1479 \\ 2.1479 & -19.7109 \end{bmatrix}.$$

The switching regions are constructed by (see Figure 1)

$$\Omega_1 = \{(x_1, x_2) \in R^2 : -17.4725x_1^2 - 2.1874x_1x_2 + 14.2247x_2^2 < 0\},$$
$$\Omega_2 = \{(x_1, x_2) \in R^2 : 14.0176x_1^2 + 4.2958x_1x_2 - 19.7109x_2^2 < 0\},$$
$$\overline{\Omega}_1 = \Omega_1, \quad \overline{\Omega}_2 = \Omega_2 \backslash \overline{\Omega}_1.$$
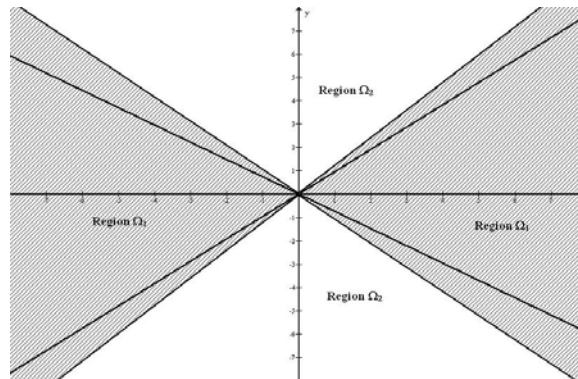

Fig.1.

Under the switching rule $\sigma(x(t)) = i$ whenever $x(t) \in \overline{\Omega}_i, i = 1, 2$, the system (10) is exponentially stable with decay rate $\alpha = 0.5$. Moreover, the solution $x(t, \phi)$ of the system satisfies

$$\|x(t, \phi)\| \le 4.7374e^{-0.5t}\|\phi\|, \quad t \ge 0.$$

4 Conclusion

This paper has proposed a switching design for the exponential stability of a class of switched systems with mixed time delays. Based on an improved Lyapunov-Krasovskii functional, the exponential stability conditions are derived in terms of linear matrix inequalities which allows to compute simultaneously the two bounds that characterize the exponential stability rate of the solution. Numerical example illustrated the obtained results is given.

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