Applied Mathematical Sciences, Vol. 3, 2009, no. 51, 2507 - 2515

New Classes Containing Generalization of Differential Operator

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Abstract

New classes containing generalization of differential operator are introduced. Characterization and other properties of these classes are studied. Moreover, Fekete-Szegö functional for these classes are obtained.

Mathematics Subject Classification: 30C45

Keywords: Differential operator; Fekete-Szegö functional, Distortion theorem

1 Introduction, Definitions and Preliminaries.

Let $\mathcal{H}$ be the class of functions analytic in U and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \tag{1.1}$$

Now we introduce a differential operator defines as follows : $\mathbf{D}_{\lambda,\delta}^{k,\alpha} : \mathcal{A} \to \mathcal{A}$ by

$$\mathbf{D}_{\lambda,\delta}^{k,\alpha} f(z) = z + \sum_{n=2}^{\infty} [n^{\alpha} + (n-1)n^{\alpha}\lambda]^k C(\delta,n) a_n z^n, \;\; k, \;\alpha \in \mathbb{N}_0, \, \lambda \geq 0, \, \delta \geq 0, \tag{1.2}$$

where

$$C(\delta,n) = \begin{pmatrix} n+\delta-1 \\ \delta \end{pmatrix} = \frac{\Gamma(n+\delta)}{\Gamma(n)\Gamma(\delta+1)}.$$

Remark 1.1. When $\alpha = 1, \lambda = 0, \delta = 0$ or $\alpha = 0, \lambda = 1, \delta = 0$ we get Sălăgean differential operator [6], $k = 0$ gives Ruscheweyh operator [5], $\alpha = 0, \delta = 0$ implies Al-Oboudi differential operator of order (k) [1], $\alpha = 1, \lambda = 0$ or $\alpha = 0, \lambda = 1$ operator (1.2) reduces to Al-Shaqsi and Darus differential operator [2] and $\alpha = 0$ poses the differential operator of order (k), which is given by the authors [3]. Note that the operator in [3] was first introduced by Al-Shaqsi and Darus [7] and further studies have been done by the same authors in [8].

Some of relations for the differential operator (1.2) are discussed in the next lemma.

Lemma 1.1. Let $f \in \mathcal{A}$. Then

$$(i)\ \mathbf{D}^{0,\alpha}_{\lambda,0} f(z) = f(z),$$
$$(ii)\ \mathbf{D}^{1,1}_{0,0} f(z) = zf'(z).$$

In the following definitions, new classes of analytic functions containing the differential operator (1.2) are introduced:

Definition 1.1. Let $f(z) \in \mathcal{A}$. Then $f(z) \in S^{k,\alpha}_{\lambda,\delta}(\mu)$ if and only if

$$\Re\Big\{\frac{z[\mathbf{D}^{k,\alpha}_{\lambda,\delta} f(z)]'}{\mathbf{D}^{k,\alpha}_{\lambda,\delta} f(z)}\Big\} > \mu, \ \ 0 \le \mu < 1, \ \ z \in U.$$

Definition 1.2. Let $f(z) \in \mathcal{A}$. Then $f(z) \in C^{k,\alpha}_{\lambda,\delta}(\mu)$ if and only if

$$\Re\Big\{\frac{[z(\mathbf{D}^{k,\alpha}_{\lambda,\delta} f(z))']'}{(\mathbf{D}^{k,\alpha}_{\lambda,\delta} f(z))'}\Big\} > \mu, \ \ 0 \le \mu < 1, \ \ z \in U.$$

The article is organized as follows: In section 2, we study the characterization and distortion theorems, and other properties of these classes. In section 3, we obtain sharp upper bound of $|a_2|$ and of the Fekete-Szegö functional $|a_3 - \nu a_2^2|$ for the classes $S^{k,\alpha}_{\lambda,\delta}(\mu)$ and $C^{k,\alpha}_{\lambda,\delta}(\mu)$. For this purpose we need the following result:

Lemma 1.2.[4] Let $p \in \mathcal{P}$, that is, p be analytic in U, be given by $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and $\Re\{p(z)\} > 0$ for $z \in U$. Then

$$|p_2 - \frac{p_1^2}{2}| \le 2 - \frac{|p_1|^2}{2}$$

and $|p_n| \le 2$ for all $n \in N$.

2 General properties of $\mathbf{D}_{\lambda,\delta}^{k,\alpha}$

In this section we study the characterization properties and distortion theorems for the function $f(z) \in \mathcal{A}$ to belong to the classes $S_{\lambda,\delta}^{k,\alpha}(\mu)$ and $C_{\lambda,\delta}^{k,\alpha}(\mu)$ by obtaining the coefficient bounds.

Theorem 2.1. Let $f(z) \in \mathcal{A}$. If

$$\sum_{n=2}^{\infty}(n-\mu)[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)|a_n| \leq 1-\mu, \quad 0 \leq \mu < 1, \tag{2.3}$$

then $f(z) \in S_{\lambda,\delta}^{k,\alpha}(\mu)$. The result (2.3) is sharp.

Proof. Suppose that (2.3) holds. Since

$$\begin{aligned} 1-\mu &\geq \sum_{n=2}^{\infty}(n-\mu)[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)|a_n| \\ &\geq \sum_{n=2}^{\infty}\mu[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)|a_n| - \sum_{n=2}^{\infty}n[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)|a_n| \end{aligned}$$

then this implies that

$$\frac{1+\sum_{n=2}^{\infty}n[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)|a_n|}{1+\sum_{n=2}^{\infty}[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)|a_n|} > \mu,$$

hence

$$\Re\Big\{\frac{z[\mathbf{D}_{\lambda,\delta}^{k,\alpha}f(z)]'}{\mathbf{D}_{\lambda,\delta}^{k,\alpha}f(z)}\Big\} > \mu.$$

We also note that the assertion (2.3) is sharp and the extremal function is given by

$$f(z) = z + \sum_{n=2}^{\infty}\frac{(1-\mu)}{(n-\mu)[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)}z^{n}.$$

Corollary 2.1. Let the hypotheses of Theorem 2.1 satisfy. Then

$$|a_n| \leq \frac{(1-\mu)}{(n-\mu)[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)}, \ \forall n \geq 2. \tag{2.4}$$

Corollary 2.2. Let the hypotheses of Theorem 2.1 be satisfied. Then for $\delta = \mu = k = 0$

$$|a_n| \leq \frac{1}{n}, \ \forall n \geq 2. \tag{2.5}$$

In the same way we can verify the following results:

Theorem 2.2. Let $f(z) \in \mathcal{A}$. If

$$\sum_{n=2}^{\infty} n(n-\mu)[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k} C(\delta,n)|a_n| \le 1-\mu, \quad 0 \le \mu < 1, \tag{2.6}$$

then $f(z) \in C^{k,\alpha}_{\lambda,\delta}(\mu)$. The result (2.6) is sharp.

Corollary 2.3. Let the hypotheses of Theorem 2.2 be satisfied. Then

$$|a_n| \le \frac{(1-\mu)}{n(n-\mu)[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k} C(\delta,n)}, \ \ \forall\, n \ge 2. \tag{2.7}$$

Also we have the following inclusion results:

Theorem 2.3. Let $0 \le \mu_1 \le \mu_2 < 1$. Then $S^{k,\alpha}_{\lambda,\delta}(\mu_1) \supseteq S^{k,\alpha}_{\lambda,\delta}(\mu_2)$.

Proof. By Theorem 2.1.

Theorem 2.4. Let $0 \le \mu_1 \le \mu_2 < 1$. Then $C^{k,\alpha}_{\lambda,\delta}(\mu_1) \supseteq C^{k,\alpha}_{\lambda,\delta}(\mu_2)$.

Proof. By Theorem 2.2.

Theorem 2.5. Let $0 \le \lambda_1 \le \lambda_2$. Then $S^{k,\alpha}_{\lambda_1,\delta}(\mu) \subseteq S^{k,\alpha}_{\lambda_2,\delta}(\mu)$.

Proof. By Theorem 2.1.

Theorem 2.6. Let $0 \le \lambda_1 \le \lambda_2$. Then $C^{k,\alpha}_{\lambda_1,\delta}(\mu) \subseteq C^{k,\alpha}_{\lambda_2,\delta}(\mu)$.

Proof. By Theorem 2.2.

We introduce the following distortion theorems.

Theorem 2.7. Let the hypotheses of Theorem 2.1 be satisfied. Then for $z \in U$ and $0 \le \mu < 1$

$$|\mathbf{D}^{k,\alpha}_{\lambda,\delta} f(z)| \ge |z| - \frac{1-\mu}{2-\mu}$$

and

$$|\mathbf{D}^{k,\alpha}_{\lambda,\delta} f(z)| \le |z| + \frac{1-\mu}{2-\mu}.$$

Proof. By using Theorem 2.1, one can verify that

$$(2-\mu)\sum_{n=2}^{\infty}[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k} C(\delta,n)|a_n| \le \sum_{n=2}^{\infty}(n-\mu)[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k} C(\delta,n)|a_n| \le 1-\mu$$

then

$$\sum_{n=2}^{\infty}[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)|a_n| \leq \frac{1-\mu}{2-\mu}.$$

Thus we obtain

$$\begin{aligned}|\mathbf{D}_{\lambda,\delta}^{k,\alpha}f(z)| &\leq |z| + \sum_{n=2}^{\infty}[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)|a_n||z|^n \\ &\leq |z| + \sum_{n=2}^{\infty}[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)|a_n||z|^2 \\ &\leq |z| + [\frac{1-\mu}{2-\mu}]|z|^2\end{aligned}$$

The other assertion can be proved as follows:

$$\begin{aligned}|\mathbf{D}_{\lambda,\delta}^{k,\alpha}f(z)| &= |z + \sum_{n=2}^{\infty}[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)a_n z^n| \\ &\geq |z - \sum_{n=2}^{\infty}(n-\mu)[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)a_n z^n| \\ &\geq |z| - \sum_{n=2}^{\infty}[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)|a_n||z|^n \\ &\geq |z| - \sum_{n=2}^{\infty}[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)|a_n||z|^2 \\ &\geq |z| - [\frac{1-\mu}{2-\mu}]|z|^2\end{aligned}$$

This completes the proof.

In the same way we can get the following result.

Theorem 2.8. Let the hypotheses of Theorem 2.2 be satisfied. Then for $z \in U$ and $0 \leq \mu < 1$

$$|\mathbf{D}_{\lambda,\delta}^{k,\alpha}f(z)| \geq |z| - \frac{(1-\mu)}{2(2-\mu)}|z|^2$$

and

$$|\mathbf{D}_{\lambda,\delta}^{k,\alpha}f(z)| \leq |z| + \frac{(1-\mu)}{2(2-\mu)}|z|^2.$$

Also, we have the following distortion results

Theorem 2.9. Let the hypotheses of Theorem 2.1 be satisfied. Then

$$|f(z)| \geq |z| - \frac{(1-\mu)\Gamma(\delta+1)}{(2-\mu)\Gamma(\delta+2)[2^{\alpha}(1+\lambda)]^{k}}|z|^2$$

and

$$|f(z)| \leq |z| + \frac{(1-\mu)\Gamma(\delta+1)}{(2-\mu)\Gamma(\delta+2)[2^{\alpha}(1+\lambda)]^{k}}|z|^2.$$

Proof. In virtue of Theorem 2.1, we have

$$(2-\mu)[2^{\alpha}(1+\lambda)]^{k}\frac{\Gamma(\delta+2)}{\Gamma(\delta+1)}\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}(n-\mu)[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)|a_n| \leq (1-\mu)$$

then

$$\sum_{n=2}^{\infty}|a_n| \leq \frac{(1-\mu)\Gamma(\delta+1)}{(2-\mu)\Gamma(\delta+2)[2^{\alpha}(1+\lambda)]^{k}}.$$

Thus we obtain

$$\begin{aligned}|f(z)| &= |z+\sum_{n=2}^{\infty}a_n z^n| \\ &\leq |z| + \sum_{n=2}^{\infty}|a_n||z|^2 \\ &\leq |z| + \frac{(1-\mu)\Gamma(\delta+1)}{(2-\mu)\Gamma(\delta+2)[2^{\alpha}(1+\lambda)]^{k}}|z|^2\end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned}|f(z)| &\geq |z-\sum_{n=2}^{\infty}a_n z^n| \\ &\geq |z| - \sum_{n=2}^{\infty}|a_n||z|^2 \\ &\geq |z| - \frac{(1-\mu)\Gamma(\delta+1)}{(2-\mu)\Gamma(\delta+2)[2^{\alpha}(1+\lambda)]^{k}}|z|^2.\end{aligned}$$

This completes the proof.

In the same way we can get the following results.

Theorem 2.10. Let the hypotheses of Theorem 2.2 be satisfied. Then $(n-\mu)[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n) \geq 1$ and $0 \leq \mu < 1$ poses

$$|f(z)| \geq |z| - \frac{(1-\mu)\Gamma(\delta+1)}{2(2-\mu)\Gamma(\delta+2)[2^{\alpha}(1+\lambda)]^{k}}|z|^2$$

and

$$|f(z)| \leq |z| + \frac{(1-\mu)\Gamma(\delta+1)}{2(2-\mu)\Gamma(\delta+2)[2^{\alpha}(1+\lambda)]^{k}}|z|^2.$$

3 Fekete-Szegö for the classes $S^{k,\alpha}_{\lambda,\delta}(\mu)\, and\, C^{k,\alpha}_{\lambda,\delta}(\mu)$

In this section we determine the sharp upper bound for $|a_2|$ for the classes $S^{k,\alpha}_{\lambda,\delta}(\mu)$ and $C^{k,\alpha}_{\lambda,\delta}(\mu)$. Moreover, we calculate the Fekete-Szegö $|a_3-\nu a_2^2|$ functional for the classes aforementioned.

Theorem 3.1. Let the hypotheses of Theorem 2.1 be satisfied. Then

$$|a_2| \le \frac{2(1-\mu)}{(2^\alpha(1+\lambda))^k}\frac{\Gamma(1+\delta)}{\Gamma(2+\delta)}$$

and for all $\nu \in \mathbb{C}$ the following bound is sharp

$$|a_3-\nu a_2^2| \le 2[\frac{(1+\delta)}{(3+\delta)}\frac{(1-\mu)}{(3^\alpha(1+2\lambda))^k}]max\Big\{1, |1+2(1-\mu)[1-\frac{\nu\Gamma(3+\delta)(3^\alpha(1+2\lambda))^k\Gamma(1+\delta)}{\big(\Gamma(2+\delta)(2^\alpha(1+\lambda))^k\big)^2}]|\Big\}.$$

Proof. Since $f \in S^{k,\alpha}_{\lambda,\delta}(\mu)$ then the condition

$$\Re\Big\{\frac{z[\mathbf{D}^{k,\alpha}_{\lambda,\delta}f(z)]'}{\mathbf{D}^{k,\alpha}_{\lambda,\delta}f(z)}\Big\} > \mu, \quad 0 \le \mu < 1, \ \ z \in U$$

is equivalent to

$$z[\mathbf{D}^{k,\alpha}_{\lambda,\delta}f(z)]' = \mathbf{D}^{k,\alpha}_{\lambda,\delta}f(z)[(1-\mu)p(z)+\mu], \ \ z \in U,$$

for some $p \in \mathcal{P}$. Equating coefficients we obtain $a_2 = A(1-\mu)p_1$, $a_3 = B[(1-\mu)^2p_1^2 + (1-\mu)p_2]$ where $A := \frac{\Gamma(1+\delta)}{\Gamma(2+\delta)(2^\alpha(1+\lambda))^k}$, $B := \frac{\Gamma(1+\delta)}{\Gamma(3+\delta)[3^\alpha(1+2\lambda)]^k}$ and further, for $C := D\{\frac{1}{2} + (1-\mu) - \frac{\nu A^2(1-\mu)}{B}\}$ where $D := B(1-\mu)$ and by using Lemma 1.2 we have $|a_3-\nu a_2^2| \le H(x) = 2D + (C-\frac{D}{2})x^2, \ x := |p_1| \le 2$. Consequently, we receive

$$|a_3-\nu a_2^2| \le \begin{cases} H(0) = 2D, & C \le \frac{D}{2} \\ \\ H(2) = 4C, & C > \frac{D}{2}. \end{cases}$$

Equality is attained for functions given by

$$\frac{z[\mathbf{D}^{k,\alpha}_{\lambda,\delta}f(z)]'}{\mathbf{D}^{k,\alpha}_{\lambda,\delta}f(z)} = \frac{1+z^2(1-2\mu)}{1-z^2}$$

and

$$\frac{z[\mathbf{D}^{k,\alpha}_{\lambda,\delta}f(z)]'}{\mathbf{D}^{k,\alpha}_{\lambda,\delta}f(z)} = \frac{1+z(1-2\mu)}{1-z}$$

respectively.

For $\mu = 0$ we receive the following corollary.

Corollary 3.1. Let the assumptions of Theorem 3.1 hold. Then for $\mu = 0$

$$|a_2| \leq \frac{2\Gamma(1+\delta)}{\Gamma(2+\delta)(2^\alpha(1+\lambda))^k}$$

and

$$|a_3 - \nu a_2^2| \leq 2[\frac{(1+\delta)}{(3+\delta)(3^\alpha(1+2\lambda))^k}]max\Big\{1, |1+2[1-\frac{\nu\Gamma(3+\delta)(3^\alpha(1+2\lambda))^k\Gamma(1+\delta)}{\big(\Gamma(2+\delta)(2^\alpha(1+\lambda))^k\big)^2}]|\Big\}.$$

In the similar manner we can prove the following result.

Theorem 3.2. Let the hypotheses of Theorem 2.2 be satisfied. Then

$$|a_2| \leq \frac{(1-\mu)}{(2^\alpha(1+\lambda))^k}\frac{\Gamma(1+\delta)}{\Gamma(2+\delta)}$$

and for all $\nu \in \mathbb{C}$ the following bound is sharp

$$|a_3 - \nu a_2^2| \leq \frac{2}{3}[\frac{(1+\delta)}{(3+\delta)}\frac{(1-\mu)}{(3^\alpha(1+2\lambda))^k}]max\Big\{1, |1+(1-\mu)[2-\frac{3\nu\Gamma(3+\delta)(3^\alpha(1+2\lambda))^k\Gamma(1+\delta)}{2\big(\Gamma(2+\delta)(2^\alpha(1+\lambda))^k\big)^2}]|\Big\}.$$

For $\mu = 0$ we receive the following corollary.

Corollary 3.2. Let the assumptions of Theorem 3.2 hold. Then for $\mu = 0$

$$|a_2| \leq \frac{\Gamma(1+\delta)}{\Gamma(2+\delta)(2^\alpha(1+\lambda))^k}$$

and

$$|a_3 - \nu a_2^2| \leq \frac{2}{3}[\frac{(1+\delta)}{(3+\delta)(3^\alpha(1+2\lambda))^k}]max\Big\{1, |1+[2-\frac{3\nu\Gamma(3+\delta)(3^\alpha(1+2\lambda))^k\Gamma(1+\delta)}{2\big(\Gamma(2+\delta)(2^\alpha(1+\lambda))^k\big)^2}]|\Big\}.$$

Acknowledgement: The authors were supported in part by ScienceFund: 04-01-02-SF0425, MOSTI, Malaysia.

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Received: March, 2009