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Haar Wavelet Method for Solving Cahn-Allen Equation

G. Hariharan

Department of Mathematics, SASTRA University
Tirumalaisamudram-613402
Thanjavur, Tamilnadu, India
hariharan@maths.sastra.edu

K. Kannan

Department of Mathematics, SASTRA University
Tirumalaisamudram-613402
Thanjavur, Tamilnadu, India
kannan@maths.sastra.edu

Abstract

In this paper, we develop an accurate and efficient Haar wavelet method for well-known Cahn-Allen equation. The proposed scheme can be used to a wide class of nonlinear equations. The power of this manageable method is confirmed. Moreover the use of Haar wavelets is found to be accurate, simple, fast, flexible, convenient, small computation costs and computationally attractive.

Keywords: Nonlinear parabolic equations; Cahn–Allen equation; Haar wavelet method

1. Introduction

Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, solid state physics, optical fibers, biology, fluid dynamics and chemical kinetics. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very essential in nonlinear wave equations. A variety of powerful methods, such as inverse scattering method [1,23], bilinear transformation [15], the tanh–sech method [21,24,28], extended tanh method [8,9,25], sine–cosine method [3,26], homogeneous balance method [10] and Exp-function method [4,14] were used to develop nonlinear dispersive and dissipative problems. Tascan and Bekir [22] have established the travelling wave solutions of the Cahn-Allen equation by using first integral method. Recently, Haar wavelets have been applied extensively for signal processing in communications and physics research, and have proved to be a wonderful mathematical tool. In solving ordinary differential equations by using Haar wavelet related method, Chen and Hsiao [7] had derived an operational matrix of integration based on Haar wavelet. Lepik [18,19,20] had solved higher order as well as nonlinear ODEs and some nonlinear evolution equations by Haar wavelet method. Hariharan et al. [13] have introduced a Haar wavelet method for solving Fisher's equation.

We introduce a Haar wavelet method for solving the Cahn-Allen equation with the initial and boundary conditions, which will exhibit several advantageous features:

i) Very high accuracy fast transformation and possibility of implementation of fast algorithms compared with other known methods.
ii) The simplicity and small computation costs, resulting from the sparsity of the transform matrices and the small number of significant wavelet coefficients.

The method is also very convenient for solving the boundary value problems, since the boundary conditions are taken care of automatically.

Beginning from 1980's, wavelets have been used for solution of partial differential equations (PDE). The good features of this approach are possibility to detect singularities, irregular structure and transient phenomena exhibited by the analyzed equations. Most of the wavelet algorithms can handle exactly periodic boundary conditions. The wavelet algorithms for solving PDE are based on the Galerkin techniques or on the collocation method.

Evidently all attempts to simplify the wavelet solutions for PDE are welcome. One possibility for this is to make use of the Haar wavelet family. Haar wavelets (which are Daubechies of order 1) consists of piecewise constant functions and are therefore the simplest orthonormal wavelets with a compact support. A drawback of the Haar wavelets is their discontinuity. Since the derivatives do not exist in the breaking points it is not possible to apply the Haar wavelets for solving PDE directly. There are two possibilities for getting out of this situation. One way is to regularize the Haar wavelets with interpolating splines (e.g. B-splines or Deslaurier-Dabuc interpolating wavelets). This approach has been applied by Cattani [6], but the regularization process considerably complicates the solution and the main advantage of the Haar wavelets-the simplicity gets to some extent lost. The other way is to make use of the integral method, which was proposed by Chen and Hsiao [7]. There are discussions by other researchers [5,12,16,17].

The paper is organized the following way. For completeness sake the Haar wavelet method is presented in Section 2. Function approximation is presented in Section 3. The method of solution of the Cahn-Allen equation is proposed in Section 4. Concluding remarks are given in Section 5.

2. Haar wavelets

Haar functions have been used from 1910 when they were introduced by the Hungarian mathematician Alfred Haar [11]. Haar wavelets are the simplest wavelets among various types of wavelets. They are step functions (piecewise constant functions) on the real line that can take only three values. Haar wavelets, like the well-known Walsh functions (Rao 1983), form an orthogonal and complete set of functions representing discretized functions and piecewise constant functions. A function is said to be piecewise constant if it is locally constant in connected regions.

The Haar transform is one of the earliest examples of what is known now as a compact, dyadic, orthonormal wavelet transform. The Haar function, being an odd rectangular pulse pair, is the simplest and oldest orthonormal wavelet with compact support. In the mean time, several definitions of the Haar functions and various generalizations have been published and used. They were intended to adopt this concept to some practical applications as well as to extend its in applications to different classes of signals. Haar functions appear very attractive in many applications as for example, image coding, edge extraction, and binary logic design.

After discretizing the differential equations in a conventional way like the finite difference approximation, wavelets can be used for algebraic manipulations in the system of equations obtained which lead to better condition number of the resulting system.

The previous work in system analysis via Haar wavelets was led by Chen and Hsiao [7], who first derived a Haar operational matrix for the integrals of the Haar function vector and put the application for the Haar analysis into the dynamical systems. Then, the pioneer work in state analysis of linear time delayed systems via Haar wavelets was laid down by Hsiao, who first proposed a Haar product matrix and a coefficient matrix. Hsiao and Wang [16] proposed a key idea to transform the time-varying function and its product with states into a Haar product matrix.

The orthogonal set of Haar function $h_i(t)$ is shown in Fig.1. This is a group of square waves with magnitudes of ± 1 in certain intervals and zeros elsewhere.

For applications of the Haar transform in logic design, efficient ways of calculating the Haar spectrum from reduced forms of Boolean functions are needed.

The Haar wavelet family for $t \in [0,1]$ is defined as follows.

$$h_i(t) = \begin{cases} 1\,, & for \quad t \in \left[\dfrac{k}{m}\,, \dfrac{k+0.5}{m} \right) \\ -1\,, & for \quad t \in \left[\dfrac{k+0.5}{m}, \dfrac{k+1}{m} \right) \\ 0\,, & elsewhere \end{cases} \tag{1}$$

Integer $m = 2^j$ $(j = 0,1,2...J)$ indicates the level of the wavelet; $k = 0,1,2,.....m-1$ is the translation parameter. Maximal level of resolution is J. The index i is calculated according the formula $i = m + k + 1$; in the case of minimal values. $m = 1, k = 0$ we have $i = 2$, the maximal value of i is $i = 2M = 2^{J+1}$. It is assumed that the value $i = 1$ corresponds to the scaling function for which $h_1 \equiv 1 \;\; in\; [0,1]$. Let us define the collocation points $t_l = (l - 0.5)/2M$, $(l = 1,2....2M)$ and discretise the Haar function $h_i(t)$; in this way we get the coefficient matrix $H(i,l) = (h_i(t_l))$, which has the dimension $2M \times 2M$.

The operational matrix of integration P, which is a 2M square matrix, is defined by the equation

$$(PH)_{il} = \int_0^{t_l} h_i(t)\,dt \tag{2}$$

$$(QH)_{il} = \int_0^{t_l} dt \int_0^{t} h_i(t)\,dt \tag{3}$$

The elements of the matrices H, P and Q can be evaluated according to (1), (2) and (3).

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P_2 = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad P_4 = \frac{1}{16}\begin{bmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$P_8 = \frac{1}{64}\begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Chen and Hsiao [7] showed that the following matrix equation for calculating the matrix P of order m holds

$$P_{(m)} = \frac{1}{2m}\begin{pmatrix} 2mP_{(m/2)} & -H_{(m/2)} \\ H^{-1}_{(m/2)} & O \end{pmatrix}$$

where O is a null matrix of order $\frac{m}{2}\times\frac{m}{2}$,

$$\mathrm{H_{mXm}} \triangleq \left[h_m(t_0) \quad h_m(t_1) \quad ------h_m(t_{m-1}) \quad \right] \tag{4}$$

and $\frac{i}{m} \le t < i + \frac{1}{m}$ and $H^{-1}_{mxm} = \frac{1}{m} H^{T}_{mxm} \; diag(r)$

It should be noted that calculations for $P_{(m)}$ and $H_{(m)}$ must be carried out only once; after that they will be applicable for solving whatever differential equations. Since H and H^{-1} contain many zeros, this phenomenon makes the Haar transform must faster than the Fourier transform, and it is even faster than the Walsh transform. This is one of the reason for rapid convergence of the Haar wavelet series.

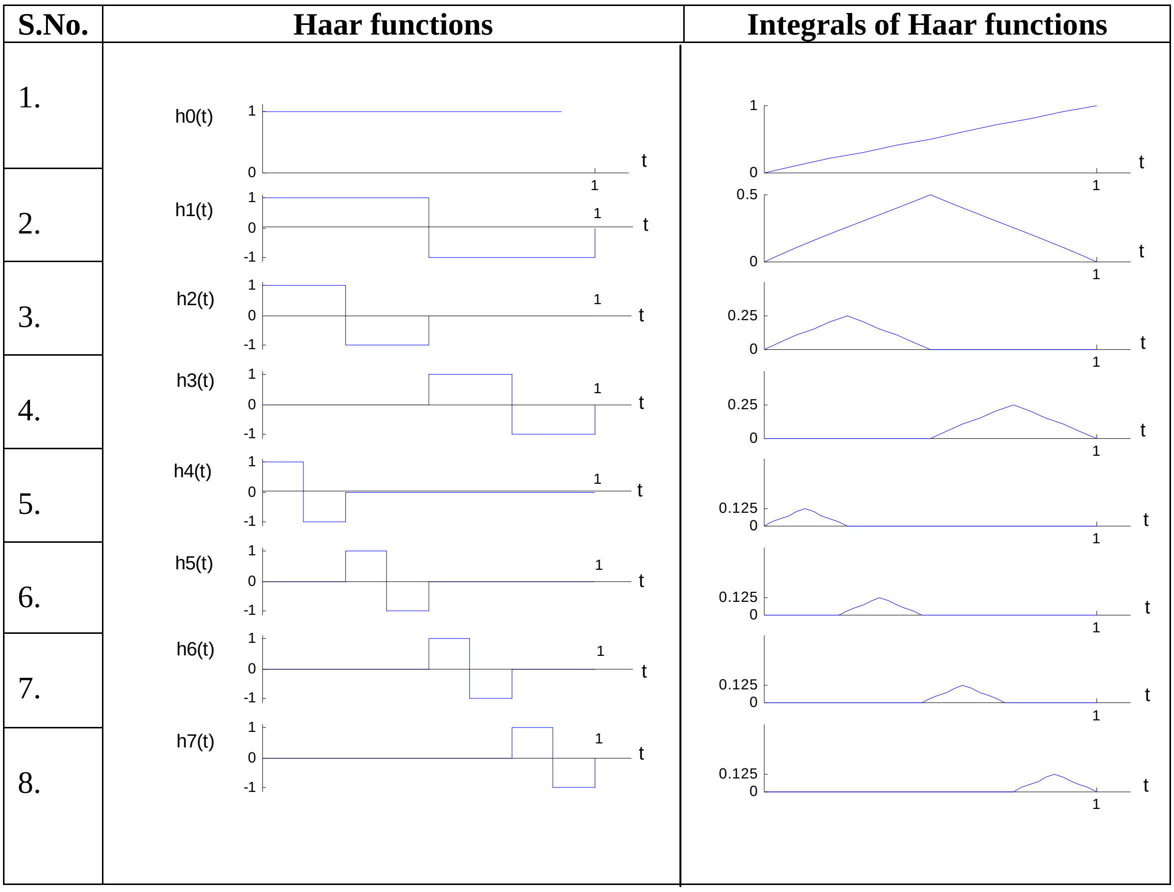

Fig. 1. First eight Haar functions

3. Function approximation

Any function $y(x) \in L^2[0,1)$ can be decomposed as

$$y(x) = \sum_{n=0}^{\infty} c_n h_n(x) \tag{5}$$

where the coefficients c_n are determined by

$$c_n = 2^j \int_0^1 y(x) h_n(x) dx \tag{6}$$

Where $n = 2^j + k,\ j \geq 0,\ 0 \leq k < 2^j$. Specially $c_0 = \int_0^1 y(x)dx.$

The series expansion of $y(x)$ contains an infinite terms. If $y(x)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then $y(x)$ will be terminated at finite terms, that is

$$y(x)=\sum_{n=0}^{m-1} c_n h_n(x)=c_{(m)}^T h_{(m)}(x) \tag{7}$$

Where the coefficients $c_{(m)}^T$ and the Haar function vector $h_{(m)}(x)$ are defined as

$$c_{(m)}^T=[c_0, c_1, ..., c_{m-1}]$$

and $h_{(m)}(x)=[h_0(x), h_1(x), ..., h_{m-1}(x)]^T$ where 'T' means transpose and $m=2^j$.

4. The method of solution of the Cahn-Allen equation

We study the nonlinear parabolic PDE given by

$$u_t=u_{xx}-u^n+u. \tag{8}$$

with the initial condition $u(x,0)=f(x), \quad 0\le x\le 1$

and the boundary conditions $u(0,t)=g_0(t),\ u(1,t)=g_1(t), \quad 0<t\le T$

For $n=3$ Eq. (8) becomes Cahn-Allen equation [2]. It arises in many scientific applications such as mathematical biology, quantum mechanics and plasma physics. It is well known that wave phenomena of plasma media and fluid dynamics are modeled by kink shaped and tanh solution or bell shaped sech solutions [27].

Let us divide the interval (0,1] into N equal parts of length $\Delta t=(0,1]/N$ and denote $t_s=(s-1)\Delta t,\ s=1,2,.....N$. We assume that $\dot{u}''(x,t)$ can be expanded interms of Haar wavelets as formula

$$\dot{u}''(x,t)=\sum_{n=0}^{m-1} c_s(n) h_n(x)=c_{(m)}^T h_{(m)}(x) \tag{9}$$

where . and ' means differentiation with respect to t and x respectively, the row vector $c_{(m)}^T$ is constant in the subinterval $t\in(t_s, t_{s+1}]$

Integrating formula (9) with respect to t from t_s to t and twice with respect to x from 0 to x, we obtain

$$u''(x,t)=(t-t_s)c_{(m)}^T h_{(m)}(x)+u''(x,t_s) \tag{10}$$

$$\begin{aligned} u(x,t)=(t-t_s)c_{(m)}^T Q_{(m)} h_{(m)}(x)+u(x,t_s)-u(0,t_s) \\ +x[u'(0,t)-u'(0,t_s)]+u(0,t) \end{aligned} \tag{11}$$

$$\dot{u}(x,t)=c_{(m)}^T Q_{(m)} h_{(m)}(x)+x\dot{u}'(0,t)+\dot{u}(0,t) \tag{12}$$

By the boundary conditions, we obtain

$$u(0,t_s)=g_0(t_s), \qquad u(1,t_s)=g_1(t_s)$$

$$\dot{u}(0,t)=g_0{}'(t), \qquad \dot{u}(1,t)=g_1{}'(t)$$

Putting $x=1$ in formulae (11) and (12), we have

$$u'(0,t)-u'(0,t_s)=-(t-t_s)c_{(m)}^T Q_{(m)} h_{(m)}(x)+g_1(t)-g_0(t)-g_1(t_s)+g_0(t_s) \tag{13}$$

$$\dot{u}'(0,t)=g_1'(t)-c_{(m)}^T Q_{(m)} h_{(m)}(x)-g_0{}'(t) \tag{14}$$

Substituting formulae (13) and (14) into formulae (10)-(12), and discretizising the results by assuming $x \to x_l$, $t \to t_{s+1}$ we obtain

$$u''(x_l,t_{s+1})=(t_{s+1}-t_s)c_{(m)}^T h_{(m)}\left(x_l\right)+u''(x_l,t_s) \tag{15}$$

$$\begin{aligned} u(x_l,t_{s+1}) &= (t_{s+1}-t_s)c_{(m)}^T Q_{(m)} h_{(m)}(x_l)+u(x_l,t_s)-g_0(t_s)+g_0(t_{s+1}) \\ &\quad + x_l[-(t_{s+1}-t_s)c_{(m)}^T P_{(m)} f+g_l(t_{s+1})-g_0(t_{s+1})-g_1(t_s)+g_0(t_s)] \end{aligned} \tag{16}$$

$$\dot{u}(x_l,t_{s+1})=c_{(m)}^T Q_{(m)} h_{(m)}(x)+g_0'(t_{s+1})+x_l[-c_{(m)}^T P_{(m)} f+g_1'(t_{s+1})-g_0'(t_{s+1})] \tag{17}$$

Where the vector f is defined as

$$f=[1, \underbrace{0,......,0}_{(m-1)\ elements}]^T$$

In the following the scheme

$$\dot{u}(x_l,t_{s+1})=u''(x_l,t_{s+1})-u^3(x_l,t_{s+1})+u(x_l,t_{s+1}) \tag{18}$$

which leads us from the time layer t_s to t_{s+1} is used.

Substituting equations (15)-(17) into the equation (18), we gain

$$\begin{aligned} & c_{(m)}^T Q_{(m)} h_{(m)}(x_l)+x_l[-c_{(m)}^T P_{(m)} f+g_1'(t_{s+1})-g_0'(t_{s+1})]+g_0'(t_{s+1}) \\ & =u''(x_l,t_{s+1})-u^3(x_l,t_{s+1})+u(x_l,t_{s+1}) \end{aligned} \tag{19}$$

From formula (19) the wavelet coefficients $c_{(m)}^T$ can be successively calculated.

Computer simulation was carried out in the cases $m=32$ and $m=64$, the computed results were compared with the exact solution, more accurate results can be obtained by using a larger m.

The exact solution of Eq. (8) in a closed form is given by

$$u\left(x,t\right)=\frac{1}{1+e^{-\frac{\sqrt{2}}{2}\left(x+\frac{3\sqrt{2}}{2}t\right)+c_0}}, \quad \text{where } c_0 \text{ is integration constant.}$$

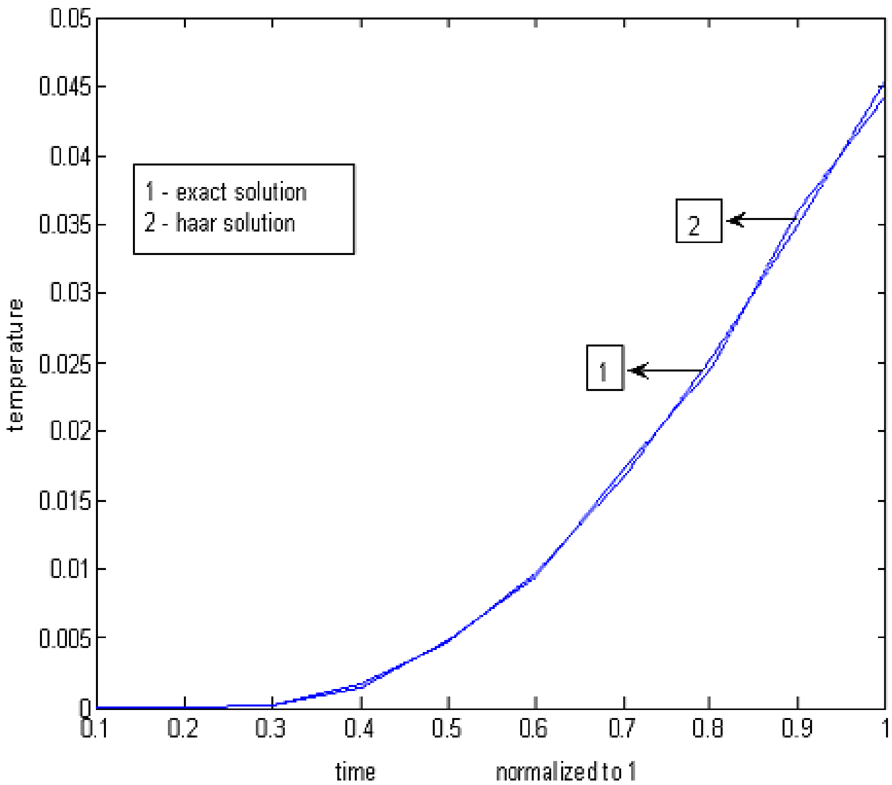

Fig.2 Comparison between exact and Haar solution of the Cahn-Allen equation x = 10 and k = 12.5

Our results can be compared to Wazwaz's results [27]

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a personal computer System Vostro 1400 Processor x86 Family 6 Model 15 Stepping 13 Genuine Intel ~1596 Mhz.

5. Conclusion

The theoretical elegance of the Haar wavelet approach can be appreciated from the simple mathematical relations and their compact derivations and proofs. It has been well demonstrated that in applying the nice properties of Haar wavelets, the differential equations can be solved conveniently and accurately by using Haar wavelet method systematically. In comparison with existing numerical schemes used to solve the nonlinear parabolic equations, the scheme in this paper is an improvement over other methods in terms of accuracy. It is worth mentioning that

Haar solution provides excellent results even for small values of m ($m=16$). For larger values of m (i.e., $m=32$, $m=64$, $m=128$ and $m=256$), we can obtain the results closer to the real values. The main goal of this work is to apply the Haar wavelet method to the well-known Cahn-Allen equation that appears in many scientific applications. The work also confirmed the power of the Haar wavelet method in handling nonlinear equations in general. This method can be easily extended to find the solution of all other non-linear parabolic equations. Another benefit of our method is that the scheme presented here, with some modifications, seems to be easily extended to solve model equations including more mechanical, physical or biophysical effects, such as nonlinear convection, reaction, linear diffusion and dispersion.

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