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Improving Efficiency Scores of Inefficient Units with Restricted Primary Resources

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Abstract

An important issue in Data Envelopment Analysis (DEA) is finding the efficiency scores of Decision Making Units (DMUs). For improving the efficiency scores of inefficient units, it is important to consider the primary resources fixed and restricted. In this paper, we propose a model to improve the efficiency scores of inefficient units with restricted primary resources. We also suggest the perturbation on primary resources in such a way that the efficiency scores of inefficient units improved and the efficient DMUs remain unchanged.

Keywords: data envelopment analysis, reallocation, primary resources, perturbation.

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1. Introduction

Data envelopment analysis (DEA), originally introduced by Charnes et al (1978), nonparametric method for evolution the efficiency scores of decision making units (DMUs). By considering different technology assumptions, a series of DEA models, such as CCR, BCC models, have been proposed in the literature. Recently Emrouznejad et al (2008) publisher a survey on the DEA models and application. Reallocation is an important application of DEA, first introduced by Golany and Tamir (1995) propose an output oriented resource allocation model in which there are constraints on the total input consumption. Since they affect on the aggregated efficiencies. In this respect, many researchers have developed new DEA models for reallocation problem. For example, Cook and Kress (1999) a DEA approach for cost allocation problems which are based on two principles: invariance and pareto-minimality. Their approach cannot be used directly to determine a cost allocation among the DMUs. Cook and Zhu (2005) extend the Cook and Kress approach, and provide a practical approach to the cost allocation problem. Jahanshahloo et al (2004) proposes an approach in which without solving linear programming problems only using simple formula, the equitable allocation is achieved. Their method is based on invariance principle of Cook and Kress. Beasley (2003) introduced a non-linear resource allocation model based on the ratio form that aimed at jointly computing inputs and outputs for each DMU for the next period with the objective of maximizing the average efficiency. Recently Chen et al (2007) uses the relative efficiency of one assignment relative to others instead of measuring the cost or profit.

Existing restriction in reallocation DEA models detect some efficient units as inefficient ones. In this paper we consider the problem of imposing restrictions on the reallocation DEA models. We assume reallocation flows as the perturbed information on the primary resources among the units. This means that where an input of an inefficient DMU moves on as the input of efficient units. Our purpose, it is to solve this problem and improving the efficiency scores of inefficient DMUs, with the condition in which the efficiency scores of efficient DMUs are remain unchanged, and also the efficiency scores of inefficient units increased as much as possible.

The rest of paper is organized as fallows: Section 2 reviews the necessary preliminaries. Section 3 introduces our proposed model. Then in section 4 we illustrate the proposed model with a numerical example. The conclusion remarks and further research are given in Section 5.

2. Preliminaries

Assume that we have n DMUs, where DMU_j $(j = 1,...,n)$ consumes $x_j = (x_{1j},...,x_{mj})^t$ as inputs and produces $\mathbf{y}_j = (y_{1j},...,y_{sj})^t$ as outputs. For evaluation of DMU_k under constant return to scale technology the following linear programming model, CCR model, should be solved.

$$
\begin{aligned}
\theta_k^* &= \max\ \mathbf{u}\mathbf{y}_k \\
&\quad s.t. \\
&\qquad \mathbf{u}\mathbf{y}_j - \mathbf{v}\mathbf{x}_j \le 0 \quad j = 1,\dots,n \\
&\qquad\qquad \mathbf{v}\mathbf{x}_k = 1 \\
&\qquad \mathbf{u} \ge \mathbf{0},\ \mathbf{v} \ge \mathbf{0}
\end{aligned}
\tag{1}
$$

Where, $\mathbf{u} = (u_1,\dots,u_s)$ and $\mathbf{v} = (v_1,\dots,v_m)$ are defined as the vectors of weights for outputs and inputs, respectively.

We partition all DMUs into two groups. We denote efficient units as the first and second group contains inefficient units, indicated by E and $\hat{E}$ respectively. We define p as cardinal of E and q as cardinal of $\hat{E}$ and we define aggregate efficiency scores as follows: $\overline{\theta} = \theta_1^* + \theta_2^* + \dots + \theta_n^*$

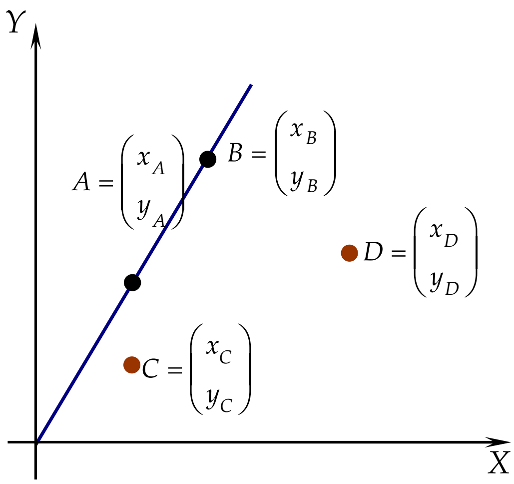

Figure 1: Before perturbation over the input

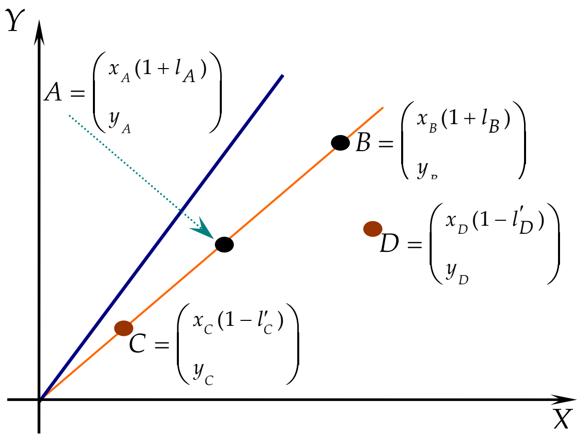

Figure 2: After perturbation on input

Figure 1 indicates a two-dimensional production possibility set (PPS) consisting four DMUs, A, B, C and D. As it is shown **A** and **B** are efficient units, so *E*= {A, B}, and **C** and **D** are inefficient units, so $\hat{E}$= {C, D}. Figure 2 shows the motivation of this study. As the figure indicates we are looking for a maximum increasing of the input of the units **A** and **B** as follows: $x_j(1+l_j), j = A, B,$ and maximum decreasing of the input of units **C** and **D** as $x_j(1-l'_j), j = C, D,$ without changing the output side, in such a way that the efficient scores of units **A** and **B** remains unchanged and the efficiency scores of the other units are improved, if it is possible. With this perturbation, one can improve the efficiency values of some inefficient unit without worsening the scores of efficient units.

3. Methodology

In this section, we determine a new scale of changes that increases a given input of all DMUs in E and decreases the input of all DMUs in $\hat{E}$. Our purpose is to find the maximum perturbation for which all of the efficient DMUs remain unchanged. We propose a model that combines p CCR models by introducing the following constraints:

$$\begin{aligned} &\sum_{r=1}^{s} u_r y_{rk} - \sum_{i=1,i\neq o}^{m} (v_i x_{ik} + v_o (x_{ok}(1+l_{ok}))) \leq 0 \qquad \forall\, k \in E \\ &\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1,i\neq o}^{m} (v_i x_{ij} + v_o (x_{oj}(1-l'_{oj}))) \leq 0 \qquad \forall j \in \hat{E} \end{aligned} \tag{2}$$

Where $p = |E| \geq 1$ and o denotes the index of under investigation input. In addition, we insert the constraint $\sum_{k\in E}\sum_{r=1}^{s} u_r y_{rk} = p$ for all efficient units, this preserves efficient DMU_k ($k \in E$) unchanged.

We also use impose the constraint $\sum_{k=1}^{p} l_k - \sum_{j=1}^{q} l_j{}' = 0$ for considering the restricted primary resource.

Furthermore assume the constraint $x_{\theta_j} - l_j{}' \geq \alpha_j$ is introduced to that supply requiring input of DMU_j where α_j is determine by decision maker.

Considering all of the suggested constraints we have:

$$\begin{aligned} Max \quad & \{l_1,\dots,l_p, l'_1,\dots,l'_q\} \\ s.t \quad & V_t X_t + v_{to} l_t = 1 && \forall\, t \in E \\ & U_t Y_k - \sum_{i=1,i\neq o}^{m} (v_{ti} x_{ik} + v_{to}(x_{ok}(1+l_{ok}))) \leq 0, && \forall\, k, t \in E \\ & U_t Y_j - \sum_{i=1,i\neq o}^{m} (v_{ti} x_{ij} + v_{to}(x_{oj}(1-l'_{oj}))) \leq 0, && \forall j \in \hat{E} \;\&\; \forall\, t \in E \\ & \sum_{t\in E}\sum_{r=1}^{s} u_{pr} y_{pr} = p \\ & \sum_{k=1}^{p} l_k - \sum_{j=1}^{q} l'_j = 0 \\ & x_{oj} - l'_j \geq \alpha_j, && \forall j \in \hat{E} \\ & l_k, l'_j \geq 0, && \forall k \in E \,\&\, \forall j \in \hat{E} \\ & U_t \geq 0, V_t \geq 0, && \forall t \in E. \end{aligned} \tag{3}$$

Where $U_t = (u_{t1},\dots,u_{ts})$
$V_t = (v_{t1},\dots,v_{tm})$

Obviously, model (3) is a multi objective programming (MOP) and the model is nonlinear. To linearize, we define, Where $\overline{v}_{ti} = \frac{v_{ti}}{v_{to}}$ & $\overline{u}_{tr} = \frac{u_{tr}}{v_{to}}$ & $\overline{v}_{to} = \frac{1}{v_{to}}$ $(i = 1,...,m, i \neq o$ & $r = 1,...,s$ & $\forall t \in E)$

In addition, for transform model (3) to an LP model, we change objective function of model (3) to maximization of Z = $\sum_{k=1}^{p} l_k + \sum_{j=1}^{q} l_j'$. So, we have the following LP model:

$$
\begin{array}{lll}
Max & Z = \sum_{k=1}^{p} l_k + \sum_{j=1}^{q} l_j' & \\
s.t & \overline{V}_t X_t + l_t = \overline{v}_{to} & \forall t \in E \\
& \overline{U}_t Y_k - \sum_{i=1, i \neq o}^{m} (\overline{v}_{ti} x_{ik} + (x_{ok}(1 + l_{ok}))) \leq 0, & \forall k, t \in E \\
& \overline{U}_t Y_j - \sum_{i=1, i \neq o}^{m} (\overline{v}_{ti} x_{ij} + (x_{oj}(1 - l_{oj}'))) \leq 0, & \forall j \in \hat{E} \;\&\; \forall t \in E \\
& \sum_{t \in E} \sum_{r=1}^{s} \overline{u}_{tr} y_{tr} = \overline{v}_{to} & \\
& \sum_{k=1}^{p} l_k - \sum_{j=1}^{q} l_j' = 0 & \\
& x_{oj} - l_j' \geq \alpha_j & \forall j \in \hat{E} \\
& l_k, l_j' \geq 0, & \forall k \in E \;\&\; \forall j \in \hat{E} \\
& \overline{U}_t \geq 0, \overline{V}_t \geq 0 & \forall t \in E.
\end{array}
\tag{4}
$$

Where, $L^* = (l_1^*,...,l_p^*, l_1'^*,...,l_q'^*)$ is an optimal solution model (4), $l_j'^*$ is the value reduced in the input o^{th} of inefficient DMU_j, and l_k^* is the added value to the o^{th} input of efficient DMU_k ($j \in \hat{E}, k \in E$). Clearly $p + q = n$.

Note that if the primary resources of all DMUs are variable, we consider the constraint $\sum_{k=1}^{p} l_k - \sum_{j=1}^{q} l_j' = M$.

Now we show the following theorem.

Theorem 1. Model (4) is feasible.

Proof. The proof is clear, as that $L = (0,...,0)$ is a feasible solution, of the model. In the following theorem we assume the efficient units remained unchanged.

Theorem 2. The maximum perturbation on a specified input for improving the efficiency scores of inefficient DMUs (holding the efficient units) is L^*.

Proof. On the contrary, assume $\overline{L}^* = (\overline{l}_1^*,...,\overline{l}_p^*, \overline{l}_1'^*,...,\overline{l}_q'^*)$ denote the maximum perturbation, for which

$$\overline{L}^* \geq L^*, \overline{L}^* \neq L \tag{5}$$

Clearly $\overline{L}^*$ is a feasible solution of model (4) and inequalities given in (5) are a contradiction.

4. Illustrative example

In this section we demonstrate the proposed model (4) using 49 schools, a real data set originally give in Charnes et al. (1981). The data consists five inputs and three outputs. The inputs measure the education level of mother, the highest occupation level of a family member, the frequency of parental visits to the school, parental involvement with their children, and the number of teachers at a given site. The outputs are regarding to reading and math ability as well as self-esteem.
Now, we apply model (4) and obtain following results CCR model (1), we conclude that $|E| = 17 \& |\hat{E}| = 32$, and the original aggregate scores is $\bar{\theta} = 46.35$.

The results are summarized in Table 1.

Table 1: Result of new model for schools data

η_1 [1]	η_2	η_3	η_4	η_5
47.27	47.7	47.8	48.98	48.6

To see the results of applying model (4), the above table shows that perturbation on the second input for all DMUs increases the aggregate efficiency scores of all units from 46.35 to 47.7. Similar discussion is true for the remaining columns of Table 1. Also we propose the perturbation on the fourth input is better to be implemented, as it is the maximum perturbation model (4) among the inputs using.

5. Conclusion

In this paper, we have investigated the aggregate efficiency scores of all DMUs and proposed a new DEA model for perturbation on a specified input. Also we have shown that this perturbation is the maximum perturbation on a specified input for improving the efficiency scores of inefficient DMUs (holding the efficient units). As a further research, we offer to extend our proposed model that the modified model could deal with all component of inputs vector. In real world situations, some reallocation of resources is possible but it usually is restricted, especially in the short run. Thus, it might be impossible to move employees across the country, and hiring new employees (as well as firing old employees) might be very costly or even impossible due to shortage of qualified labor (or existing

[1] $\eta_j =$ Aggregate efficiency scores after perturbation on the j^{th} input

contracts), therefore, we should mention that our proposed model In this paper could be use reallocation of resource cost for all DMUs.

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