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Quality Index of Balanced Digraphs

Zbigniew R. Bogdanowicz

Armament Research, Development and Engineering Center
Building 95, Picatinny
New Jersey 07806, USA
E-mail: zbigniew.bogdanowicz@conus.army.mil

Abstract

Let G be a multidigraph with vertices $V(G)$ and arcs $A(G)$. G is *balanced* if $d_G^+(v) = d_G^-(v)$ for every $v \in V(G)$. Let r_i, r_j be the number of cycles in G that include $a_i, a_j \in A(G)$ respectively, and $r_i \geq r_j$. *Local quality index* of G is defined as the largest ratio $\frac{r_i}{r_j}$ for opposite arcs a_i, a_j. *Global quality index* of G is defined as the largest ratio $\frac{r_i}{r_j}$ for distinct arcs a_i, a_j. In this work we prove that for balanced multidigraphs only a local quality index can have a finite upper bound, and it exists when we are not limited to considering only simple cycles. In addition, we prove that $\frac{3}{2}$ is the best upper bound in such case and it is not attainable by a finite balanced multidigraph.

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1 Introduction

A digraph G is *balanced* if $d_G^+(v) = d_G^-(v)$ for every vertex v of G. Throughout this paper by digraph G we mean a finite multidigraph. Furthermore, if there are parallel arcs then we consider them distinguishable. Some fundamental properties of balanced digraphs are well known. For example, every connected balanced digraph has an Eulerian cycle [2], and some properties related to that have been explored. Studying the properties of balanced digraphs and the results obtained in this work might have a direct impact on the better understanding of *spin networks* [4], which are at the core of the theory of *quantum gravity* [5] - an alternative to *string theory* [3].

In this paper we consider simple as well as general cycles. That is, a general cycle is allowed to visit an arc at most once (same as a simple cycle) but it's allowed to visit a vertex multiple times (as opposed to a simple cycle)

[1]. Cycles $C_1, C_2, ..., C_n$ are pairwise distinct if there are no two cycles C_i, C_j defined by the same arc sequence.

Let G be a balanced digraph defined on vertices $V(G)$ and arcs $A(G)$. Let r_i, r_j be the number of cycles in G that include $a_i, a_j \in A(G)$ respectively, and $r_i \geq r_j$. *Local quality index* of G is defined as the largest ratio $\frac{r_i}{r_j}$ in G for opposite arcs a_i, a_j in G. *Global quality index* of G is defined as the largest ratio $\frac{r_i}{r_j}$ in G for distinct arcs a_i, a_j in G. A local or global quality index can be considered as a measure of how balanced G is. That is, the closer the value of either a local or global quality index is to 1 the more balanced we can say is G.

In this work we only consider the upper bound for local and global indices because it's easy to see that the lower bound in both cases equals 1 when G is a simple bidirectional cycle. Let $\phi(G), \Phi(G)$ be the local and global quality indices respectively when only simple cycles are considered in G. Otherwise, the local and global indices are denoted by $\psi(G), \Psi(G)$ respectively. In the next two sections we will consider simple cycles separately from general cycles. In Section 2 for simple cycles we show that we cannot find a finite $k \in R^+$ such that for any balanced G either $\phi(G) < k$ or $\Phi(G) < k$. Otherwise, in Section 3 for balanced G we prove that $\psi(G) < \frac{3}{2}$ and we show that there is no finite $k \in R^+$ that satisfies $\Psi(G) < k$. However, we conjecture that for the connected balanced digraphs a finite upper bound k exists (i.e., $\Psi(G) < k$), and we give support for that conjecture with Theorem 3.7.

2 Quality Index for Simple Cycles

In this section we show that for simple cycles in balanced G there is no finite upper bound for either $\phi(G)$ or $\Phi(G)$. To prove the next theorem we introduce a balanced digraph G defined for finite integer i and five vertices, which is illustrated in Figure 1.

Theorem 2.1 *Let $k \in R^+$ be a finite number. Then there exists balanced digraph G such that $\phi(G) > k$.*

Proof. Suppose that a finite upper bound $k \in R^+$ exists. Consider balanced G and opposite arcs a_1, a_2 from Figure 1. Let r_1, r_2 represent the number of cycles in G containing a_1, a_2 respectively. Clearly, there is only one simple cycle $a_1a_2 = a_2a_1$ that includes a_1 and a_2. There is also exactly one simple cycle that includes a_2 and excludes a_1, which is $a_2a_3a_4$. So, $r_2 = 2$. On the other hand, the number of cycles which include a_1 and exclude a_2 of the form $a_1a_5a_{2j}...a_7$ for $i \geq j \geq 3$ tends to infinity as $i \to \infty$. So, $\frac{r_1}{r_2} \to \infty$, which means that there exists a sufficiently large i for which $\phi(G) > k$ - a contradiction. $\square$

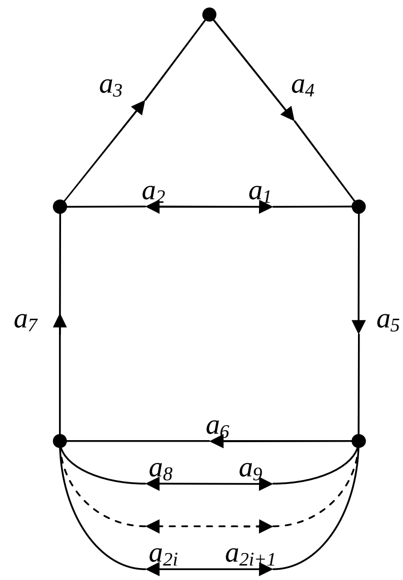

Figure 1: Balanced G on 5 vertices and $i \geq 4$

Corollary 2.2 *Let $k \in R^+$ be a finite number. Then there exists balanced digraph G such that $\Phi(G) > k$.*

Proof. Since opposite arcs are distint, then by definition of $\phi(G), \Phi(G)$ and Theorem 2.1 $\Phi(G) \geq \phi(G) > k$. □

Based on the above results the natural and intriguing problem arises to determine the families of balanced digraphs for which there exists an upper bound for either $\phi(G)$ or $\Phi(G)$.

3 Quality Index for General Cycles

We first focus on the local quality index. It turns out that for general cycles in a balanced digraph G the finite upper bound for $\psi(G)$ exists. To derive such a bound we first prove the following simple lemma.

Lemma 3.1 *Let G be a balanced digraph. If $\psi(G) > 1$ then G contains a cycle that includes a_i and excludes a_j for some pair of opposite arcs $a_i, a_j \in A(G)$.*

Proof. If $\psi(G) > 1$ then there exists at least one pair of opposite arcs in G. Suppose two opposite arcs $a_1, a_2 \in A(G)$ maximize $\frac{r_1}{r_2}$ in respect to other opposite arcs, where r_1 represents the number of cycles with a_1, r_2 represents the number of cycles with a_2, and $r_1 \geq r_2$. If there is no cycle in G that includes a_1 and excludes a_2 then $r_1 = r_2$. So, by definition $\psi(G) = 1$ - a contradiction. □

Theorem 3.2 *Let G be a balanced digraph with at least one pair of opposite arcs. Then $\psi(G) < \frac{3}{2}$.*

Proof. Without loss of generality assume that two opposite arcs $a_1, a_2 \in A(G)$ maximize $\frac{r_1}{r_2}$ in respect to other opposite arcs, where r_1 represents the number of cycles with a_1, r_2 represents the number of cycles with a_2, and $r_1 \geq r_2$. Assume $a_1 = (u, v)$. Let x be the number of cycles in G that includes a_1 and excludes a_2. If $\psi(G) = 1$ then we are done - so we can assume $\psi(G) > 1$. Then by Lemma 3.1 $x \geq 1$. Let S_1 be a set of all cycles in G such that each cycle in S_1 contains a_1 and excludes a_2. Let S_2 be a set of all cycles in G such that each cycle in S_2 contains arc sequence a_1a_2. Let S_3 be a set of all cycles in G such that each cycle in S_3 contains arc sequence a_2a_1. Clearly, $S_2 \cap S_3 = \{a_1a_2\} = \{a_2a_1\}$. Consider path $P_1(G) = v...u$ that uniquely identifies cycle $P_1(G)a_1 \in S_1$. Let H be a digraph obtained be removal of a_1, a_2 from G. So, H is a balanced digraph and $P_1(G) = P_1(H)$ is valid in H. Let $d^+_{P_1(H)}(x), d^-_{P_1(H)}(x)$ be indegree and outdegree of vertex x induced by path $P_1(H)$. We can construct from $P_1(H)$ a cycle in H, $C(H)$, as follows. Let $u = u_1$. Since H is balanced and $d^+_{P_1(H)}(u) = d^-_{P_1(H)}(u) + 1$ then there exists $P_2(H) = P_1(H)(u_1, u_2)$ if arc (u_1, v) is not available, where $u_2 \neq v$. So, $d^+_{P_1(G)}(u_2) = d^-_{P_1(G)}(u_2) + 1$ is satisfied in such a case. Hence, if $d^+_{P_i(G)}(u_i) = d^-_{P_i(G)}(u_i) + 1$ and arc (u_i, v) is not available then we can construct $P_{i+1}(H) = P_i(H)(u_i, u_{i+1})$, where $i \geq 2$ and $d^+_{P_{i+1}(H)}(u_{i+1}) = d^-_{P_{i+1}(H)}(u_{i+1}) + 1$. So, there must exist at least one path $P_g(H)$ such that $u_g = v$. We choose $P_g(H)$ that minimizes g. Adding a_1a_2 to our constructed cycle $C(H) = P_1(H)P_g(H)$ induces $C_1(G) = P_1(G)a_1a_2P_g(G) \in S_2$, and adding a_2a_1 to $C(H) = P_1(H)P_g(H)$ induces $C_2(G) = P_1(G)P_g(G)a_2a_1 \in S_3$. $C_1(G)$ uniquely identifies $P_1(G)$ by discarding the arcs associated with the shortest subpath $a_1a_2...a_k$, where $a_k = (x, v)$ and x is any vertex in $V(G)$. Also $C_2(G)$ uniquely identifies $P_1(G)$ by discarding the arcs associated with the shortest subpath $a_k...a_2a_1$, where $a_k = (u, x)$ and x is any vertex in $V(G)$. This implies that $|S_2| \geq |S_1|$ and $|S_3| \geq |S_1|$. Furthermore, because $S_2 \cap S_3 = \{a_1a_2\} = \{a_2a_1\}$ and cycles a_1a_2, a_2a_1 were not induced by augmentation of $C(H)$ then $|S_2| \geq |S_1| + 1$ and $|S_3| \geq |S_1| + 1$.

Consider now $\psi(G) = \frac{r_1}{r_2}$. Because $S_2 \cap S_3 = \{a_1a_2\}$, $|S_2| \geq |S_1| + 1$ and $|S_3| \geq |S_1| + 1$ then $r_1 \geq |S_1| + (|S_1| + 1)) + (|S_1| + 1) - 1$. So, we can express $r_1 = x + (x + 1) + (x + 1) - 1 + y = 3x + y + 1$ for $x > 0$ and some $y \geq 0$. Here y accounts for all cycles in G that contain non consecutive arcs a_1, a_2. In addition, $r_2 \geq (|S_1| + 1)) + (|S_1| + 1) - 1$ that implies $r_2 \geq (x + 1) + (x + 1) - 1 + y = 2x + y + 1$ for established x, y based on r_1. Consequently, $\psi(G) \leq (3x+y+1)/(2x+y+1) = g(x, y)$. We assume $\psi(G) = (3x + y + 1)/(2x + y + 1)$ - worst case (i.e., largest possible). By assuming

$x, y \in R^+$ we obtain $\partial g(x,y)/\partial y < 0$, which means that $g(x,y)$ is maximized for $y = 0$. So, $g(x,y)$ cannot be larger than $g(x) = (3x+1)/(2x+1)$ for fixed $x > 0, y \geq 0$. We now assume $\psi(G) = (3x+1)/(2x+1) = g(x)$ and obtain $d(g(x))/dx > 0$. This means that

$$\psi(G) < \lim_{x\to\infty} \frac{3x+1}{2x+1} = \frac{3}{2},$$

which completes the proof. □

Theorem 3.3 *Let $k \in R^+$ be given and satisfy $k < \frac{3}{2}$. Then there exists balanced digraph G such that $k < \psi(G) < \frac{3}{2}$.*

Proof. By Theorem 3.2 $\psi(G) < \frac{3}{2}$. So, for balanced digraph G and given $k < \frac{3}{2}$ either $k < \psi(G) < \frac{3}{2}$ or $\psi(G) \leq k < \frac{3}{2}$ must be satisfied. Suppose that for every balanced G we obtain $\psi(G) \leq k < \frac{3}{2}$. Consider G and opposite arcs a_1, a_2 from Figure 1. Let S_1 be a set of all cycles in G that contain a_1 and exclude a_2. Let S_2 be a set of all cycles in G that contain arc sequence a_1a_2. Let S_3 be a set of all cycles in G that contain arc sequence a_2a_1. Suppose that cycle C in G includes a_1 and a_2. Then it's easy to see that C must be either $C = ...a_1a_2...$ or $C = ...a_2a_1...$, so a_1 and a_2 must be two consecutive arcs in the sequence. Also, $S_2 \cap S_3 = \{a_1a_2\} = \{a_2a_1\}$.

Clearly, there is exactly one cycle $a_2a_3a_4$ in G that includes a_2 and excludes a_1. Every cycle C_1 in S_1 is of the form $a_1a_5a_{2j}...a_7$. Every cycle C_2 in S_2 is of the form $a_1a_2a_3a_4a_5a_{2j}...a_7$. So, C_2 is uniquely identified from C_1 by replacing a_1 with $a_1a_2a_3a_4$. Conversely, C_1 is uniquely identified from C_2 by replacing $a_1a_2a_3a_4$ with a_1. Every cycle C_3 in S_3 is of the form $a_5a_{2j}...a_7a_3a_4a_2a_1$. So, C_3 is uniquely identified from C_1 by replacing a_1 with $a_3a_4a_2a_1$. Conversely, C_1 is uniquely identified from C_2 by replacing $a_3a_4a_2a_1$ with a_1. Let $|S_1| = x$ and let r_1, r_2 represent the number of cycles containing a_1, a_2 respectively. Then we have $r_1 = |S_1| + |S_2| + |S_3| - 1 = |S_1| + (|S_1| + 1) + (|S_1| + 1) - 1 = 3x + 1$. For r_2 we have $r_2 = 1 + |S_2| + |S_3| - 1 = 1 + (|S_1| + 1) + (|S_1| + 1) - 1 = 2x + 2$. We note that x increases as i increases. Define $g(x) = (3x+1)/(2x+2)$. Since $d(g(x))/dx > 0$ and $k < \frac{3}{2}$ then there exists sufficiently large i such that $g(x) > k$, which means that also $\psi(G) > k$ - a contradiction. □

So, by Theorems 3.2, 3.3 we conclude

Corollary 3.4 *Ratio $\frac{3}{2}$ is the best upper bound for $\psi(G)$ and this bound is not attainable by a finite balanced digraph.*

Proof. By Theorems 3.2, 3.3 ratio $\frac{3}{2}$ is the lowest upper bound for $\psi(G)$, and by Theorem 3.2 $\psi(G) = \frac{3}{2}$ is not possible. □

We finally focus on the global quality index $\Psi(G)$.

Theorem 3.5 *Let $k \in R^+$ be a finite number. Then there exists balanced digraph G such that $\Psi(G) > k$.*

Proof. Suppose that finite $k \in R^+$ exists that satisfies $\Psi(G) < k$ for every balanced G. Consider balanced H that consists of two components A and B. Let $A = G$, where G is from Figure 1. Let B be cycle $C_c = b_1 b_2 ... b_n$ on n vertices, where $b_j \in A(H)$ for $n \geq j \geq 1$. Let r_1, r_2 represent the number of cycles in H containing a_1, b_1 respectively. Since $r_2 = 1$ then $\frac{r_1}{r_2} = r_1$. The number of cycles in H containing a_1 increases as i increases. So, there exists H for sufficiently large i that satisfies $\Psi(H) > k$ - a contradiction. □

By Theorem 3.5 there is no finite upper bound for $\Psi(G)$. This doesn't seem to be the case for the connected balanced digraphs, and we state the following:

Conjecture 3.6 *There exists finite $k \in R^+$ such that for any connected balanced digraph G we have $\Psi(G) < k$.*

If the above conjecture is true than the challenge lies in determining for connected balanced G the upper bound for $\Psi(G)$. Our next result proves that such an upper bound exceeds $\frac{3}{2}$, which is the best upper bound for $\psi(G)$. The proof is based on the digraph illustrated in Figure 2.

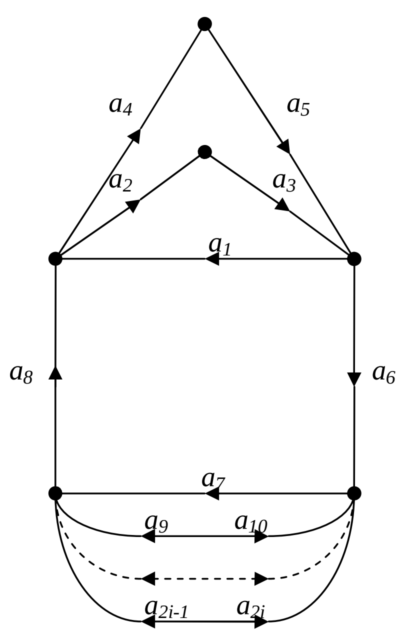

Figure 2: Balanced G on 6 vertices and $i \geq 5$

Theorem 3.7 *Let $k \in R^+$ be given and satisfy $k < 2$. Then there exists connected balanced digraph G such that $k < \Psi(G) < 2$.*

Proof. Suppose that for given $k < 2$ there is no connected balanced digraph G that satisfies $k < \Psi(G) < 2$. Consider G from Figure 2. Let r_6, r_1 represent the number of cycles in G containing a_6, a_1 respectively. Consider path $P_j = a_6 a_{2j-1} ... a_8$ in G for $i \geq j \geq 4$. Let x be the total number of such paths. Then in G there are: x cycles of type $P_j a_2 a_3$, x cycles of type $P_j a_4 a_5$, x cycles of type $P_j a_2 a_3 a_1 a_4 a_5$, and x cycles of type $P_j a_4 a_5 a_1 a_2 a_3$, which constitute all cycles with P_j. Since a_6 only belongs to cycles that include P_j then $r_6 = 4x$. We note that there are only two cycles $a_1 a_2 a_3$, $a_1 a_4 a_5$ that include a_1 and exclude P_j. So, $r_1 = 2x + 2$. Define $g(x) = \frac{r_6}{r_1} = 4x/(2x+2) = 2x/(x+1)$. Clearly, x tends to infinity as $i \to \infty$. Since $d(g(x))/dx > 0$ and $k < 2$ then there exists sufficiently large i such that $2 > g(x) > k$, which means that also $2 > \Psi(G) > k$ - a contradiction. □

Finally, if one can prove that Conjecture 3.6 holds for $k = 2$ then by Theorem 3.7 we could conclude that for the connected balanced digraph G we have 2 as the best upper bound for $\Psi(G)$. This would also mean that the number of cycles in G with a given arc is strictly less than the double number of cycles with any given arc.

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