Applied Mathematical Sciences, Vol. 3, 2009, no. 54, 2681 - 2694

Efficient Techniques for the Diffusion Equation Subject to the Specification of Energy

M. A. Rehman

Department of Mathematics, GC University
Lahore 54000, Pakistan
aziz3037@yahoo.com

M. S. A. Taj

Department of Mathematics
COMSATS Institute of Information Technology
Defence Road, Off Raiwind Road, Lahore, Pakistan
shahadat259@yahoo.com

Abstract

An efficient numerical technique is developed for the solution of the diffusion equation: $u_t = u_{xx} + s(x,t), 0 < x < X, 0 < t \leq T$, subject to $u(x,0) = f(x), 0 < x < X, u(1,t) = g(t), 0 < t \leq T$ and the specification of energy $\int_0^b u(x,t)dx = M(t), 0 < b < X, 0 < t \leq T$.

Mathematics Subject Classification: 2000: 65M06, 65N40

Keywords: Diffusion equation, nonlocal boundary condition, specification of energy, third-order numerical methods, method of lines

1 INTRODUCTION

Many physical phenomena are modelled by nonclassical boundary value problems with non-local boundary conditions.

In this paper we have considered the diffusion equation with a non-local condition : the so called energy specification. This is a linear constant having the form $\int_0^b u(x,t)dx = E(t)$ where b is a constant and $E(t)$ is the given function. Along with a one dimensional parabolic equation, this condition is quite different from the classical boundary condition. Non-local boundary value problems have certainly been one of the fastest growing areas in various application fields. Science and industry are both responsible for this growth in the last two decades.

Non-local boundary value problems can be classified according to what is being unknown: (i) initial conditions or (ii) boundary conditions [2, 3]. The present work focuses on the first group of these non-local boundary problems.

In the last decade, the development of the numerical techniques for the solution of the non-local boundary value problems has been an important research topic in many branches of science and engineering. Particularly thermoelasticity has been the subject of some resent works [6, 7].

Certain problems arising in the thermodynamics, heat conduction, plasma physics can be reduced to the nonlocal problems with integral condition.

Mathematical formation of this kind also arises in the transport of reactive and passive contaminates in aquifer, an area of active interdisciplinary research of mathematicians. We refer the reader to [4, 5] for the derivative of the mathematical models and for the precise hypothesis and analysis.

The authors of [11] have given an example from metrology. This example is the model for the evaluation of the temperature distribution of air near the ground during the calm clear nights.

One very common characteristic of all these models is that they all express a conservation of certain quantity (mass, momentum, heat, etc.) in any moment for any sub domain. This in many applications is the most desirable feature of the approximation method when it comes to the solution of the corresponding initial boundary value problem.

Much attention has been paid in the literature to development, analysis and implementation of accurate methods for the numerical solution of time dependent partial differential equation with non-local boundary condition.

This paper considers the problem of obtaining numerical approximation to the concentration $u = u(x,t)$ which satisfies the partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + s(x,t), \;\; 0 < x < X, \;\; 0 < t \leq T \tag{1}$$

subject to the given initial condition

$$u(x,0) = f(x), \;\; 0 < x < X \tag{2}$$

the boundary condition

$$u(1,t) = g(t), \;\; 0 < t \leq T \tag{3}$$

and the non-local boundary condition

$$\int_0^b u(x,t)dx = M(t), \;\; 0 < t \leq T, \;\; 0 < b < X \tag{4}$$

where f, g, M and s are known functions and are assumed to be sufficiently smooth to produce a smooth classical solution of u.

A number of sequential numerical procedures have been suggested in the literature for the solution of this problem: see, for instant, [1] and [8].

In the present paper the method of lines semi discretization approach will be used to transform the model partial differential equation (PDE) into a system of first-order, linear, ordinary differential equations (ODEs), the solution of which satisfies a certain recurrence relation involving matrix exponential terms. A suitable rational approximation will be used to approximate such exponentials leading to an L_0-stable algorithm which may be parallelized through the partial fraction splitting technique.

2 DISCRETIZATION AND RECURRENCE RELATION

Dividing the interval $[0, X]$ into $N+1$ subintervals each of width h, so that $(N+1)h = X$, and the time variable t into time steps each of length l gives a rectangular mesh of points with co-ordinates $(x_m, t_n) = (mh, nl)$ where $(m = 0, 1, 2, ..., N+1$ and $n = 0, 1, 2, ...)$ covering the region $R = [o < x < X]*[t > 0]$ and its boundary ∂R consisting of lines $x = 0, x = X$ and $t = 0$.

The space derivative in (1) may be approximated to the third-order accuracy at some general point (x, t) of the mesh by the expression:

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{1}{12h^2}\{11u(x-h,t) - 20u(x,t) + 6u(x+h,t) + 4u(x+2h,t) \\ &- u(x+3h,t)\} + \frac{h^3}{12}\frac{\partial^5 u(x,t)}{\partial x^5} + O(h^4) \quad as \quad h \to 0 \end{aligned} \tag{5}$$

It is worth noting that the equation (5) is valid only for $(x, t) = (x_m, t_n)$ with $m = 1, 2, 3, ..., N-2$. To attain the same accuracy at the points (x_i, t_n) for $i = N-1$ and N, special formulae must be developed which approximate $\frac{\partial^2 u(x,t)}{\partial x^2}$ not only to third order but also with dominant error term $\frac{h^3}{12}\frac{\partial^5 u(x,t)}{\partial x^5}$ for $x = x_{N-1}, x_N$ and $t = t_n$. Such approximations will be:

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{1}{12h^2}\{u(x-3h,t) - 6u(x-2h,t) + 26u(x-h,t) - 40u(x,t) \\ &+ 21u(x+h,t) - 2u(x+2h,t)\} + \frac{h^3}{12}\frac{\partial^5 u(x,t)}{\partial x^5} \\ &+ O(h^4) \quad as \quad h \to 0 \end{aligned} \tag{6}$$

and

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{12h^2}\{2u(x-4h,t) - 11u(x-3h,t) + 24u(x-2h,t) - 14u(x-h,t)$$

$$
\begin{aligned}
&- \quad 10u(x,t) + 9u(x+h,t)\} + \frac{h^3}{12}\frac{\partial^5 u(x,t)}{\partial x^5} \\
&+ \quad O(h^4) \quad as \quad h \to 0
\end{aligned} \qquad (7)
$$

at the mesh points (x_{N-1}, t_n) and (x_N, t_n) respectively.

3 TREATMENT OF THE NON-LOCAL BOUNDARY CONDITION

The integral in (4) may be approximated by using a quadrature rule such as Simpson's rule as used by [10] to give:

$$
\int_0^b u(x,t)dx \approx \frac{h^*}{3}[u(0,t)+4\sum_{i=1}^{\frac{J}{2}} u(2i-1,t)+2\sum_{i=1}^{\frac{J}{2}-1} u(2i,t)+u(J,t)]+O(h^4) \quad (8)
$$

in which $h^* = \frac{b}{J}$.

Applying (1) with (5), (6) and (7) to all the interior mesh points of the grid at time level $t = t_n$ produces a system of ordinary differential equations of the form:

$$
\frac{d\mathbf{U}(t)}{dt} = A\mathbf{U}(t) + v(t), t > 0 \qquad (9)
$$

with

$$
\mathbf{U}(0) = f \qquad (10)
$$

in which $\mathbf{U}(t) = [u_1(t), u_2(t), ..., u_N(t)]^T$ and $f = [f(x_1), f(x_2), ..., f(x_N)]^T$ T denoting the transpose and matrix A is of order N which is given by:

$$
A = \frac{1}{12h^2}\begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \cdots & \alpha_J & \bigcirc \\
11 & -20 & 6 & 4 & -1 & & & \\
 & 11 & -20 & 6 & 4 & -1 & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & 11 & -20 & 6 & 4 & -1 \\
 & & & & 11 & -20 & 6 & 4 \\
 & & & 1 & -6 & 26 & -40 & 21 \\
\bigcirc & & & 2 & -11 & 24 & -14 & -10
\end{bmatrix}
$$

where $\alpha_1 = -64, \alpha_2 = -16, \alpha_3 = -40, \alpha_4 = -23$

and $\alpha_i = \begin{cases} -44 & \text{for i=5(2)J-1} \\ -22 & \text{for i=6(2)J-2} \\ -11 & \text{for i=J.} \end{cases}$

and $v(t) = (33/12h^2)[M(t), 0, 0, ..., -g(t) - 2g(t) + 9g(t)]^T$ where T denoting transpose.
The solution of the system (9) and (10) satisfies the recurrence relation:

$$\mathbf{U}(t) = exp(lA)f + \int_0^t exp[(t-s)A]v(s)ds, t \geq 0 \tag{11}$$

which satisfies the recurrence relation:

$$\mathbf{U}(t+l) = exp(lA)\mathbf{U}(t) + \int_t^{t+l} exp[(t+l-s)A]v(s)ds, t = 0, l, 2l, ... \tag{12}$$

To approximate the matrix exponential in (12) we use the rational approximation [12] for real scalar θ which is of the form:

$$E_M(\theta) = \frac{\sum_{K=0}^{M-1} b_K \theta^K}{\sum_{K=0}^{M} a_K (-\theta)^K} \tag{13}$$

where M is a positive integer and $a_0 = 1, a_M, b_{M-1} \neq 0$ and $a_K \geq 0$ for all $K = 1, 2, 3, ..., M$. Matching $E_M(\theta)$ with the first M+1 terms of the Maclaurin's expansion of $exp(\theta)$ leads to the following relations in the parameters:

$$a_M = (-1)^{M-1} \sum_{K=0}^{M-1} (-1)^K \frac{a_K}{(M-K)!} \tag{14}$$

and

$$b_K = \sum_{i=0}^{K} (-1)^i \frac{a_i}{(K-i)!}, K = 0, 1, 2, ..., M-1 \tag{15}$$

The magnitude of the coefficient of the error term is:

$$\mu_M = \sum_{K=0}^{M-1} \frac{(M-K)(-1)^{K+1} a_K}{(M-K+1)!} \tag{16}$$

Two particular cases of (13) are $E_1(\theta)$ which is the (1,0) Pade approximation and $E_2(\theta)$, which is discussed in [14]. In the present paper we are concerned with $E_3(\theta)$, so, for $M = 3$ we have:

$$E_3(\theta) = \frac{b_0 + b_1\theta + b_2\theta^2}{a_0 - a_1\theta + a_2\theta^2 - a_3\theta^3} = \frac{p(\theta)}{q(\theta)} \tag{17}$$

where the coefficients a_i and b_i are real and $a_0 = 1, b_0 = 1, b_1 = 1 - a_1, b_2 = \frac{1}{2} - a_1 + a_2$ and $a_3 = \frac{1}{6} - \frac{a_1}{2} + a_2$. So we have:

$$exp(lA) = G^{-1}(I + (1-a_1)lA + (\frac{1}{2} - a_1 + a_2)l^2A^2) \tag{18}$$

where

$$G = I - a_1 lA + a_2 l^2 A^2 - (\frac{1}{6} - \frac{1}{2}a_1 + a_2) l^3 A^3 \tag{19}$$

Note that denominator of $E_3(\theta)$ has distinct real zeros [13] provided that $a_2^2 - 3a_1a_3 > 0$.

We have chosen $a_1 = (\frac{65431}{50000})$ and $a_2 = (\frac{171151}{300000})$ gives $a_3 = (\frac{4143}{50000})$, $b_1 = -(\frac{15431}{50000})$, $b_2 = -(\frac{14287}{60000})$. It can be shown that using these values, L-stability is granted [15].

The integral in (12) may be approximated by a quadrature formula of the form :

$$\int_t^{t+l} exp[(t+l-s)A]v(s)ds = W_1 v(s_1) + W_2 v(s_2) + W_3 v(s_3) \tag{20}$$

where $s_1 \neq s_2 \neq s_3$ and W_1, W_2 and W_3 are matrices.
It can be shown that:

(i) <u>when $v(s) = [1, 1, 1, ..., 1]^T$</u>

$$W_1 + W_2 + W_3 = M_1 \tag{21}$$

where $M_1 = A^{-1}(exp(lA) - I)$.

(ii) <u>when $v(s) = [s, s, s, ..., s]^T$</u>

$$s_1 W_1 + s_2 W_2 + s_3 W_3 = M_2 \tag{22}$$

where $M_2 = A^{-1}[texp(lA) - (t+l)I + A^{-1}(exp(lA) - I)]$ and

(iii) <u>when $v(s) = [s^2, s^2, s^2, ..., s^2]^T$</u>

$$s_1^2 W_1 + s_2^2 W_2 + s_3^2 W_3 = M_3 \tag{23}$$

where

$$M_3 = A^{-1}[t^2 exp(lA) - (t+l)^2 I + 2A^{-1}\{texp(lA) - (t+l)$$

$$+A^{-1}(exp(lA) - I)\}]$$

Taking $s_1 = t$, $s_2 = t + \frac{l}{2}$, $s_3 = t + l$ and then solving (21), (22) and (23) simultaneously gives:

$$W_1 = \frac{2}{l^2}\{(t^2 + \frac{3}{2}lt + \frac{l^2}{2})M_1 - (2t + \frac{3}{2}l)M_2 + M_3\} \tag{24}$$

$$W_2 = \frac{-4}{l^2}\{(t^2 + lt)M_1 - (2t + l)M_2 + M_3\} \quad (25)$$

$$W_3 = \frac{2}{l^2}\{(t^2 + \frac{l}{2}t)M_1 - (2t + \frac{l}{2})M_2 + M_3\} \quad (26)$$

It can easily be shown that:

$$W_1 = \frac{2}{l^2}(A^{-1})^3\{(\frac{1}{2}l^2A^2 - \frac{3}{2}lA + 2I)exp(lA) - (\frac{1}{2}lA + 2I)\} \quad (27)$$

$$W_2 = \frac{-4}{l^2}(A^{-1})^3\{(2I - lA)exp(lA) - (2I + lA)\} \quad (28)$$

$$W_3 = \frac{2}{l^2}(A^{-1})^3\{(2I - \frac{1}{2}lA)exp(lA) - (2I + \frac{3}{2}lA + \frac{1}{2}l^2A^2)\} \quad (29)$$

Now replacing $exp(lA)$ by (18) and (19) in (27)-(29):

$$W_1 = \frac{l}{6}\{I + (4 - 9a_1 + 12a_2)lA\}G^{-1} \quad (30)$$

$$W_2 = \frac{2l}{3}\{I - (1 - 3a_1 + 6a_2)lA\}G^{-1} \quad (31)$$

$$W_3 = \frac{l}{6}\{I + (3 - 9a_1 + 12a_2)lA + (1 - 3a_1 + 6a_2)l^2A^2\}G^{-1} \quad (32)$$

Hence (12) can be written as:

$$\mathbf{U}(t + l) = exp(lA)\mathbf{U}(t) + W_1v(t) + W_2v(t + \frac{l}{2}) + W_3v(t + l) \quad (33)$$

where W_1, W_2 and W_3 are given by (30)-(32).

4 DEVELOPMENT OF ALGORITHM

Assuming that a_1, a_2 and a_3 satisfy the condition given in [13] to produce real zeros for $q(\theta)$ and $r_i (i = 1, 2, 3)$ are distinct real zeros of the denominator of (17) then:

$$G = (I - \frac{l}{r_1}A)(I - \frac{l}{r_2}A)(I - \frac{l}{r_3}A) \quad (34)$$

and the approximation to the matrix exponential function may be written in partial fraction form as:

$$exp(lA) = \sum_{j=1}^{3} P_j(I - \frac{l}{r_j}A)^{-1} \quad (35)$$

where $P_j (j = 1, 2, 3)$, the partial-fraction coefficients of $E_3(\theta)$, are defined by:

$$p_j = \frac{1 + (1 - a_1)r_j + (\frac{1}{2} - a_1 + a_2)r_j^2}{\prod_{\substack{i=1 \\ i \neq j}}^{3} (1 - \frac{r_j}{r_i})}, j = 1, 2, 3 \quad (36)$$

Hence

$$exp(lA)\mathbf{U}(t) = [p_1(I - \frac{l}{r_1}A)^{-1} + p_2(I - \frac{l}{r_2}A)^{-1} + p_3(I - \frac{l}{r_3}A)^{-1}]\mathbf{U}(t) \quad (37)$$

The implementation of the method using a parallel architecture with three processors is based on the partial decomposition [9]. of
$exp(lA)\mathbf{U}(t)$,$W_1v(t)$, $W_2v(t + \frac{l}{2})$ and $W_3v(t + l)$ in (33).
Hence

$$\begin{aligned} \mathbf{U}(t+l) &= A_1^{-1}[p_1\mathbf{U}(t) + \frac{l}{6}(p_4v(t) + 4p_7v(t + \frac{l}{2}) + p_{10}v(t + l))] \\ &+ A_2^{-1}[p_2\mathbf{U}(t) + \frac{l}{6}(p_5v(t) + 4p_8v(t + \frac{l}{2}) + p_{11}v(t + l)) \\ &+ A_3^{-1}[p_3\mathbf{U}(t) + \frac{l}{6}(p_6v(t) + 4p_9v(t + \frac{l}{2}) + p_{12}v(t + l))] \quad (38) \end{aligned}$$

where $A_i = I - \frac{l}{r_i}A, i = 1, 2, 3.$

$$p_{3+j} = \frac{1 - (4 - 9a_1 + 12a_2)r_j}{\prod_{\substack{i=1 \\ i \neq j}}^{3} (1 - \frac{r_j}{r_i})}, j = 1, 2, 3. \quad (39)$$

$$p_{6+j} = \frac{1 - (1 - 3a_1 + 6a_2)r_j}{\prod_{\substack{i=1 \\ i \neq j}}^{3} (1 - \frac{r_j}{r_i})}, j = 1, 2, 3. \quad (40)$$

$$p_{9+j} = \frac{1 + (3 - 9a_1 + 12a_2)r_j + (1 - 3a_1 + 6a_2)r_j^2}{\prod_{\substack{i=1 \\ i \neq j}}^{3} (1 - \frac{r_j}{r_i})}, j = 1, 2, 3. \quad (41)$$

so,

$$\mathbf{U}(t + l) = y_1 + y_2 + y_3 \quad (42)$$

in which y_1, y_2 and y_3 are respectively the solutions of the systems:

$$A_1y_1 = p_1\mathbf{U}(t) + \frac{l}{6}[p_4v(t) + 4p_7v(t + \frac{l}{2}) + p_{10}v(t + l)] \quad (43)$$

$$A_2y_2 = p_2\mathbf{U}(t) + \frac{l}{6}[p_5v(t) + 4p_8v(t+\frac{l}{2}) + p_{11}v(t+l)] \tag{44}$$

$$A_3y_3 = p_3\mathbf{U}(t) + \frac{l}{6}[p_6v(t) + 4p_9v(t+\frac{l}{2}) + p_{12}v(t+l)] \tag{45}$$

5 THE PARALLEL ALGORITHM

Equations (43)-(45) have great importance in the parallel environment since they can be used to solve the corresponding linear algebraic systems on processors operating concurrently. $\mathbf{U}(t+l)$ in (33), the solution vector at time $t = (n+1)l$, may obtained via the parallel algorithm using three different processors in the following form:

Processor 1

Step 1: Input l, r_1, $\mathbf{U}_0$, A
Step 2: Compute p_1, p_4, p_7, p_{10} and $I - \frac{l}{r_1}A$
Step 3: Decompose $I - \frac{l}{r_1}A = L_1U_1$
Step 4: Evaluate $v(t)$, $v(t+\frac{l}{2})$ and $v(t+l)$
Step 5: Using $W_1(t) = \frac{l}{6}[p_4v(t) + 4p_7v(t+\frac{l}{2}) + p_{10}v(t+l)]$
Step 6: Solve $L_1U_1y_1(t) = p_1\mathbf{U}(t) + W_1(t)$

Processor 2

Step 1: Input l, r_2, $\mathbf{U}_0$, A
Step 2: Compute p_2, p_5, p_8, p_{11} and $I - \frac{l}{r_2}A$
Step 3: Decompose $I - \frac{l}{r_2}A = L_2U_2$
Step 4: Evaluate $v(t), v(t+\frac{l}{2})$ and $v(t+l)$
Step 5: Using $W_2(t) = \frac{l}{6}[p_5v(t) + 4p_8v(t+\frac{l}{2}) + p_{11}v(t+l)]$
Step 6: Solve $L_2U_2y_2(t) = p_2\mathbf{U}(t) + W_2(t)$

Processor 3

Step 1: Input l, r_3, $\mathbf{U}_0$, A
Step 2: Compute p_3, p_6, p_9, p_{12} and $I - \frac{l}{r_3}A$
Step 3: Decompose $I - \frac{l}{r_3}A = L_3U_3$
Step 4: Evaluate $v(t), v(t+\frac{l}{2})$ and $v(t+l)$

Step 5: Using $W_3(t) = \frac{l}{6}[p_6v(t) + 4p_9v(t + \frac{l}{2}) + p_{12}v(t + l)]$
Step 6: Solve $L_3U_3y_3(t) = p_3\mathbf{U}(t) + W_3(t)$
Then solve

$$\mathbf{U}(t + l) = y_1(t) + y_2(t) + y_3(t)$$

In implementing these algorithms, processor (1) generates the decomposition of $I - \frac{l}{r_1}A$, and processor (2) generate the decomposition of $I - \frac{l}{r_2}A$ while the processor (3) generate the decomposition of $I - \frac{l}{r_3}$ A simultaneously.

6 NUMERICAL EXPERIMENTS

The numerical method described in this paper was applied to two problems from the literature.

EXAMPLE (1)

Consider (1),(2),(3) and (4) with

$$f(x) = exp(x), 0 < x < 1,$$

$$g(t) = \frac{e}{1 + t^2}, 0 < t < 1,$$

$$b = 0.3,$$

$$M(t) = \frac{e^{0.3} - 1}{1 + t^2}, 0 < t \leq 1$$

$$s(x,t) = \frac{-(1 + t)^2 exp(x)}{(1 + t^2)^2}, 0 < t \leq 1, 0 < x < 1$$

and with theoretical solution

$$u(x,t) = \frac{exp(x)}{1 + t^2}$$

The results of $u(x,t)$ with $h = 0.01, 0.005, 0.0025, 0.001$, and $l = 0.01, 0.005, 0.0025, 0.001$, using the scheme discussed in this paper, are shown in Table(I) and are compared with the results obtained by using the implicit finite-difference scheme of [1] and the pade scheme of [8].

In Table(1), we present the relative error, $\frac{|\mathbf{U}_{approx} - \mathbf{U}_{exact}|}{|\mathbf{U}_{exact}|}$, for our formula and the methods of [1] and [8].

The results obtained using the new scheme developed in this paper are highly accurate than those of [1] and [8]. It is also noted that the new scheme will require less CPU time. It is therefore clear that as far as efficiency is concerned, the scheme introduced in this paper is the best candidate for the model problem. This technique can also coded very efficiently on the super as well as on the parallel computers.

EXAMPLE (2)

Consider (1),(2),(3) and (4) with

$$f(x) = sin(\pi x), 0 < x < 1,$$

$$g(t) = 0, 0 < t < 1,$$

$$b = 0.75,$$

$$M(t) = \frac{2+\sqrt{2}}{2\pi} exp(-\pi^2 t), 0 < t \leq 1,$$

$$s(x,t) = 0, 0 < t \leq 1, 0 < x < 1$$

and with theoretical solution:

$$u(x,t) = exp(-\pi^2 t) sin(\pi x).$$

The results for our example (2) are given in the Table(2). Calculations were performed for different values of $h = 0.01$, 0.005, 0.0025 ,0.001 and $l = 0.01$, 0.005 ,0.0025 ,0.001.

The results show that the scheme developed in this paper gave superior results than that of [1] and [8]. It is worth mentioning that the use of only real arithmetic especially in multi-space dimension can yield with large saving of CPU time used.

7 CONCLUSIONS

In this paper, an algorithm was applied to the one-dimensional diffusion equation with a non-local replacing one standard boundary value condition. The exact solution of this system of first-order ODEs satisfies a recurrence relation involving the matrix exponential function. This function is approximated by a rational function possessing real and distinct poles, which consequently readily admits the partial fraction expansion, thereby allowing the distribution of the work in solving the corresponding linear algebraic systems on three concurrent processors.

The method developed does not require the use of complex arithmetic and need only real arithmetic in its implementation. This technique works very well for the one dimensional diffusion with integral condition. The new scheme was

Table 1: Maximum errors for Example (1) at $t = 1$

h	Numerical Methods	l=0.01	l=0.005	l=0.0025	l=0.001
0.01	The implicit scheme	8.0×10^{-04}	2.0×10^{-04}	6.0×10^{-04}	2.0×10^{-04}
	The pade scheme	6.0×10^{-05}	1.0×10^{-05}	5.0×10^{-05}	1.0×10^{-05}
	The new scheme	1.0×10^{-08}	7.0×10^{-09}	9.0×10^{-09}	1.0×10^{-08}
0.005	The implicit scheme	7.0×10^{-04}	2.0×10^{-04}	5.0×10^{-04}	3.0×10^{-04}
	The pade scheme	5.0×10^{-05}	2.0×10^{-05}	4.0×10^{-05}	3.0×10^{-05}
	The new scheme	2.0×10^{-08}	9.0×10^{-10}	2.0×10^{-09}	3.0×10^{-09}
0.0025	The implicit scheme	6.0×10^{-04}	3.0×10^{-04}	5.0×10^{-04}	3.0×10^{-04}
	The pade scheme	4.0×10^{-05}	2.0×10^{-05}	3.0×10^{-05}	2.0×10^{-05}
	The new scheme	2.0×10^{-08}	9.0×10^{-10}	2.0×10^{-09}	3.0×10^{-09}
0.001	The implicit scheme	3.0×10^{-04}	5.0×10^{-04}	7.0×10^{-05}	9.0×10^{-05}
	The pade scheme	1.0×10^{-05}	3.0×10^{-05}	4.0×10^{-06}	5.0×10^{-06}
	The new scheme	1.0×10^{-08}	3.0×10^{-09}	3.0×10^{-09}	4.0×10^{-09}

Table 2: Maximum errors for Example (2) at t=1

h	Numerical Methods	l=0.01	l=0.005	l=0.0025	l=0.001
0.01	The implicit scheme	9.0×10^{-04}	3.0×10^{-04}	2.0×10^{-04}	1.0×10^{-04}
	The pade scheme	6.0×10^{-05}	2.0×10^{-05}	1.0×10^{-05}	1.0×10^{-05}
	The new scheme	7.0×10^{-07}	7.0×10^{-07}	7.0×10^{-07}	7.0×10^{-07}
0.005	The implicit scheme	8.0×10^{-04}	4.0×10^{-04}	5.0×10^{-04}	4.0×10^{-04}
	The pade scheme	5.0×10^{-05}	2.0×10^{-05}	2.0×10^{-05}	3.0×10^{-05}
	The new scheme	1.0×10^{-08}	2.0×10^{-09}	3.0×10^{-10}	3.0×10^{-11}
0.0025	The implicit scheme	7.0×10^{-04}	5.0×10^{-04}	4.0×10^{-04}	2.0×10^{-04}
	The pade scheme	6.0×10^{-05}	2.0×10^{-05}	3.0×10^{-05}	1.0×10^{-05}
	The new scheme	1.0×10^{-08}	2.0×10^{-09}	3.0×10^{-10}	3.0×10^{-11}
0.001	The implicit scheme	3.0×10^{-04}	2.0×10^{-04}	3.0×10^{-05}	1.0×10^{-05}
	The pade scheme	1.0×10^{-05}	3.0×10^{-05}	5.0×10^{-06}	2.0×10^{-06}
	The new scheme	1.0×10^{-08}	2.0×10^{-09}	4.0×10^{-10}	4.0×10^{-11}

found to be more accurate in comparison with the two existing schemes [1] and [8] in the literature and may be implemented in parallel using a machine with three processors concurrently.

ACKNOWLEDGEMENTS: The authors are grateful for the support of Higher Education Commission of Pakistan.

References

[1] Cannon, J.R. and Van der Hock, J. An implicit finite schemefor the diffusion equation subject to the specification of mass in a portion of domain, *Numerical solutions of partial differential equations* (Parkville,1981), North-Holland Publishing, Amsterdum, 1982, pp. 527-539.

[2] Chabrowski, J. On nonlocal problems for parabolic equations, *Nagaya Math. J.* 93(1984)109-131.

[3] Choi, Y. S. Chan, K. Y. A parabolic equation with nonlocal boundary conditions arising from electrochemistry, Nonlinear Anal. *Theory Methods Appl.* 18(4)(1992)317-331.

[4] Cushman, J.H. and Ginn, T. R. Nonlocal dispersion in porous media with continously evolving scales of heterogeneity, *Transp. porous Median* 13(1993)123-138.

[5] Cushman, J.H. and Deng, F. Nonlocal reactive trasport with physical and chemical heterogeneity: localization error, *Water Resources Res.* 31(1995)2219-2237.

[6] Day, W. A. Existence of a proprty of solutions of the heat equation subjct to linear thermoelasticity and other theories, *Quart. Appl. Math.* 40(1982)319-330.

[7] Day, W.A. Parabolic equations and thermodynamics, *Quart. Appl. Math.* 50(1992)523-533.

[8] Deghan, M. On the numerical solution of diffusion equation with a non-local boundary condition, *Math. Prob. in Engg.* 2(2003)81-92.

[9] Gourlay, A. R. and Morris, J. Li. The extrapolation of first order methods for parabolic partial differential equations. II, *SIAM J. Numer, Anal.* 17(1980)641-655.

[10] Gumel, A.B. On the numerical solution of the diffusion equation subject to the specification of mass. J. Austral. Math. Soc. Ser. B 40(1999)475-483.

[11] Murthy, A. S. V. and Verwer, J. G. Solving parabolic integro-differential equations by an explicit integration method, *J. Comput. Appl. Math.* 39(1992)121-132.

[12] Taj, M. S. A. PhD. Thesis, Brunel University Uxbridge, Middlesex, England (1995).

[13] Taj, M. S. A. and Twizell, E. H. A family of third-order parallel splitting methods for prabolic partial differential equations, *Intern. J. Comput. Math.* 67(1998)411-433.

[14] Twizell, E. H., Gumel, A. B. and Arigu, M. A. Second-order, L_0-stable methods for the heat equation with time-dependent boundary conditions, *Adv. Comput. Math.* 6(1996)333-352.

[15] Voss, D. A. and Khaliq, A. Q. M. Time stepping algorithms for semidiscretized linear parabolic PDEs based on rational approximants with distinct real poles, *Adv. Comput.* Math. 6(1996)353-363.

Received: April, 2009