Applied Mathematical Sciences, Vol. 4, 2010, no. 1, 41 - 50

Algebraic and Probabilistic Retirement Models

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Abstract

This paper aims to discuss retirement from the mathematician's viewpoint. In this world of today, are individuals going to have enough retirement savings and income to survive to a certain age or death. The paper discusses the various approaches, which are mostly used to compute how to invest in preparation of retirement days. The approaches discussed in this paper include future lifetimes, average future lifetimes, and probability theory perspective [4]. The discussion is more focused on comparative study where the three approaches are compared in order to ascertain the one best representative of the whole aspects of retirement. It extends to discuss how to determine how much is sufficient to invest depending on appropriate approach.

Mathematics Subject Classification: 60B10, 60B15

Keywords: Future Lifetimes, Average Future Lifetimes, Risk Pooling, Multiple Decrement Theory and Probability Theory Perspective

1. Introduction

The days of the stereotypical rocking chair retirement are long gone. Retirements now can come in any year. Some people happily abandon the working world at early ages while others continue to work in some form until the day they die. Because retirement years can be spent in so many different ways, making concrete financial plans is complicated, demanding a careful look at many choices and variables. Nonetheless, breaking down the issues and understanding how the math works are important first steps in creating rationale, achievable financial strategies [3].

Retirement is a transitional stage of life that can be a pleasant experience for some, yet traumatic for others. Subjectively, retirement is a self-defined notion that can mean different things to different people. For some, retirement may mean reducing the amount of work hours per week from full-time to part-time status, while for others it may mean working on a voluntary basis. Objectively, retirement can be defined simply as disengagement from business or public life. From an economic perspective, retirement is a time when one is no longer gainfully employed and receives a retirement pension benefit.

2. Methodology

Usually different individuals have different goals for yearly income after retirement. For simplicity suppose that an individual wishes to receive an income of \$k after taxes at the start of each year for life, starting at the time of retirement.

From this, its evident that for the retiree to receive yearly payments of \$k, he needs k times as much as needed for a stream of \$1 yearly payments, thus lets limit ourselves to discussing \$1 payments. The installments required in investments of course depends on the early rate of return on the investments after taxes, and so on whether the retiree wants the \$1 payments to improve overtime to take effect of inflation. Taking r as our after-tax yearly rate of return, then a \$1 investment at the start of a year would grow to $\$(1+r)$ at the end of the year. However, taking into the account the yearly inflation rate the retiree expects, each end-of-the-year dollar would buy only $1/(1+g)$ times as much as at the start. This produces a real growth factor given by the following equation,

$(1+r)/(1+g)=1+L$ Where, L is the real yearly rate of return after expected taxes and inflation and is given by, $L=(r-g)/(1+g)$

To determine the amount the retirees need to have invested in order to provide \$1 at the beginning of every year of life, starting at the moment of retirement, we will illustrate the general mathematical model as given below.

3. Future Lifetimes

To calculate the above-mentioned amount, it solely depends on how long the retiree lives after the retirement. Let us denote the total number of years the retiree lives after retirement by, *t*. Then it implies that if death occurs during the first year, $t = 0$; however the retiree gets an amount equivalent to one payment i.e. initial payment on the day of retirement. Generalizing, the retiree will always get $t+1$ payments depending on the value of *t*, as the payments are done on the retirement anniversaries.

However, *t* is unknown. Taking a conservative approach, we assume that the retiree would have invested sufficient money to generate the $1 per year irrespective of the size of *t* that is to mean to infinity. This will need an investment of $\$(1+L)/L$. At the beginning of the first year, the first payment of $1 will decrease this investment to,

$$\left[\$(1+L)/L\right] - 1 = \$1/L$$

This reduced amount will now grow at the annual interest rate to back to the original amount at beginning of the second year allowing payments forever. This is shown by the following expression.

$$\$(1+L)/L$$

For example, assume $L = 4\%$. This implies that the initial investment will be given by,

$$\$(1+0.04)/0.04 = \$26.$$

In addition, after the first payment of $1, the amount reduces to,

$$\$1/0.04 = \$25$$

This grows to $\$25 + \$(25 \times 0.04) = \$26,$ beginning the whole process again the following year.

Taking a non-conservative approach that is the preferred since no one lives forever, we need only sufficient investment for some specific number of payments, that is $t+1$. However, this expression is unknown. In order to understand this approach, we will base the discussion in some collected data in order to have an understanding of various values of *t* that typically occur in real life. For this paper, we will use data from the society of actuaries [1].

4. Average Future Lifetime

From the data [1], it can be shown that if you observe the number of total future years lived by each of 50, 000 typical 65-year-old retirees, and then you calculate the average of these future times, the average will likely to be between 20.83 and 20.99 future years. In our discussion, we pick a compromise of 20.9 years as the value for the average person.

Now let us start by tackling a simple retirement plan that gives no regular yearly payments, but just a single $1,000,000 payment to any retiree who survives to age 87. Since *t* averages 20.9 years, an average retiree would die between 85 and 86 years, is given by,

$65+20.9=85.9$ years.

In other words is to say that, the retiree will not qualify for the payment at age 87.

Investing just sufficient, i.e. zero investing; to pay the benefits for a retiree who lives the average number of years cannot be a good approach. To illustrate this, let us start with 50,000 retirees. In reality, some will outlive the average and collect their $1,000,000 at age 87 years, which a $0 initial investment could not provide. Suppose now the investment had began with $D for such a retiree. This would have grown to,

$\$D(1+I)^{22}$

by that time. Therefore, D would have to satisfy $D(1+L)^{22}=1{,}000{,}000$ in order to generate the payment.

This implies that $D=1{,}000{,}000v^{22}$, where v = $1/(1+L)$. This value $ D represents the present value of the 22 years later $1000, 000. For example if we assume our previous interest, *L=4%*, then v and D respectively will be,

$$v = 1/(1+0.04) = 0.96153 \tag{1}$$

$D=1000000\times(0.96153)^{22}=\$421{,}875$.

From the actuarial data [4], it is projected that about 25,866 of the 50,000 initial retirees will survive to age 87. Thus on average an investment of

$$(25{,}866)(\$1000000)v^{22} \big/ 50{,}000$$

is required for each initial retiree so as to give the $1000,000 to those who live to age 87. At an interest rate of 4%, this comes to around $218, 245 for every initial retiree. This is from,

$$(25{,}866)(\$1000000)(0.96153)^{22} \big/ 50{,}000 = \$218{,}244$$

This gives an average amount of $218,245, which is different from the $0 needed for the average retiree. Hence, this approach is not the best.

Now applying this average age retirement model analyze the single amount of pension payment to be provided in our previous retirement payment of $1 every year. Now it implies that we take 65 years as our start point and let denote the number of retirees at this age by, Y_{65}. That is to say, Y_{65} = 50,000 form the actuarial data.

Now we can estimate the number of retirees who will be alive *t* years to receive a $1 payment using the expression

$$Y_{(65+t)}, \text{ for } t = 0, 1, 2\ldots$$

The investment needed at age 65 that would accumulate to sufficient funds which can pay $1 to each of the $Y_{(65+t)}$ survivors *t* years later is given by,

$\$P = Y_{(65+t)}v^{t}$, where *P* = Present value of money required at (65+t) years. From this we can calculate the initial investment needed to fund all these payments for the lifetime of all the initial retirees using the following expression,

$$Y(65+0)v^{0} + Y(65+1)v^{1} + Y(65+2)v^{2} + \ldots\ldots\ldots\ldots$$

Dividing this amount by the number of initial retirees (Y_{65}), we get the average amount required per the initial retirees. This is what is usually called the actuarial present value (*APV*) [2] and is given by,

$$APV = \left(Y(65+0) \big/ Y_{65}\right)v^{0} + \left(Y(65+1) \big/ Y_{65}\right)v^{1} + \left(Y(65+2) \big/ Y_{65}\right)v^{2} + \ldots\ldots\ldots \quad (2)$$

Let us take an example to understand well. For the 50,000 retirees discussed previously, the first few years' survivors from the data [1] are,

Y_{65}=*50,000*,
Y_{66}=*49,543*,
Y_{67}=*49,360*,
Y_{68}=*48,483*,

had a maximum of four payments been promised to survivors, instead of life long payments, the average required would be given by,

$$[50{,}000 + (49543 \times 0.96153) + (49{,}360 \times 0.96153^2) + (48{,}483 \times 0.96153^3)] \div 50{,}000 = \$3.72$$

Substituting in the equation (2) and substituting the value of *v* from equation (1). For the case of life-long payments, the average investment turns out to be $14.25.

Comparing this with the $26 we had calculated is needed to quarantine $1 payments forever instead of life. For latter case, an individual retiree who had invested only $14.25 at retirement would exhaust the investment if lives much beyond the average age for his cohort.

5. Probability Theory Perspective

Suppose now that the future lifetime, X of each of a wide number Y_0 of newborns is assumed to have the same probability distribution for every newborn. This does not imply that every newborn's future lifetime is the same but it implies that they all have the same chances of behavior (the probability that a newborn dies in some particular age range is the same for all of the newborns). This random behavior is described mathematically by the cumulative distribution function,

$$F(x) = \Pr[X \le x].$$

The probability that X is less than or equal to *x* or the newborn dies by age *x*. To apply this random behavior to retirement situations, the function is converted to a survival function given by,

$$S(x) = 1 - F(x) = \Pr[X > x],$$

the probability that the newborn survives beyond age *x*. The expected number, Y_x , of survivors to age *x* among the Y_0 newborns, will then be the fraction *S(x)* of the original Y_0 newborns is given by the equation,

$$Y_X = S(x) \times Y_0$$

The values Y_x we used previously therefore describe the distribution the same way *F(x)* does, since *F(x)* can be calculated from Y_x by,

$$F(x) = 1 - S(x) = 1 - \left(Y_x \big/ Y_0 \right).$$

Now refer back to our 65-year-old-retirees. Suppose we want to compute the probability that a 65-year-old survives a minimum of another *t* years. Let us denote it by P_t for simplicity. This means that this person is a former newborn who has survived 65 years, and then this probability function is the probability that a newborn survives 65+*t* years given that she has already survived 65 years. That is,

$$P_t = S(65+t) \big/ S(65) = \left(\frac{Y_{(65+t)} / Y_0}{Y_{65} / Y_0} \right) = Y_{(65+t)} \big/ Y_{65}.$$

This illustrates that if the total number of people who survive to age 65+*t* is divided by the who survive to age 65, the result is the fraction of 65-year-olds who survive *t* years.

The quotient, $Y_{(65+t)} \big/ Y_{65}$, which is equivalent to P_t appeared in the equation [2] of actuarial present value (APV). That equation for APV is therefore total sum of terms of the form $P_t v^t$ …, the factor v^t gives the present value at age 65 of $1 at age 65+*t*. In other words, the probability that the payment will in fact be made, so it generates the expected value of the present value of that payment. And the APV is the sum of such terms, one for each payment that might be made. Therefore, APV is actually the expected value of the present value of payments made so long as the retiree survives.

The true present value of payments made for life may be quite different from the expected present value of those payments. The payments will be much smaller if the retiree dies soon and much larger if the retiree lives longer. The true present value is given by the following equation,

$$1 + v + v^2 + ... + v^t = \left(1 - v^{(t+1)}\right) \big/ (1 - v).$$

Referring back to data we used in average age approach, this true present value will exceed the $14.25 if the term (*t+1*) is at least 21 that is if the retiree lives at least 20 years. The probability of surviving 20 years turns out to be approximately 0.6 [1]. This is computed from,

$P_t = {Y_{85}}/{Y_{65}}$, with t = 20.

The inference from this computation is that about 60% of the retirees who begin with the APV (the average amount required for a lifetime of payments) will run out of money prior to dying. Higher confidence in having sufficient funds needs a higher initial investment. For instance, at 99% confidence level, the initial investment required can be computed to be approximately $20.58.

6. Risk Pooling

This can be defined as a way in which retirees protect themselves from running out of money by investing at lower costs. Even though the investment to give lifelong payments to a retiree vary depending on the individuals future lifetime, when the group is large the variations try to average out. That is those living a short time check the retirees who survive a long time requiring large initial investments. Large corporate pension schemes and insurance companies give the opportunity for people to pool their risks.

For a large group of retirees, whether the total initial fund is adequate to give lifelong payments to all retirees depends on the sum over all retirees of the present value of the payments to each retiree. If presented in a normal distribution curve, the curve contracts as the number of representatives (retirees) increases indicating that the values are largely concentrated near the average.

For instance, suppose that a large group of *N*, 65 year olds deposit the amount $\$P_N$ into an investment generating interest, *L=4%*. At 99% confidence level, the total investment required to generate lifelong payments to all *N* retirees from each person is given by,

$$P_N = 14.25 + \left(10.34 / N^{o.5}\right),$$

which decreases to the $14.25 as N increases. Assuming now N=100 retirees, $P_{100} = 15.29$ whereas for N=10,000, $P_{10000} = 14.35$. These two are comparable with the 20.58 required for a single individual.

7. Multiple Decrement Theory

In this approach, transitions into the measured state are not allowed. The other models discussed assume that only death is the cause of the retirement. In this model, it assumes that retirement can be because of death, disability, retirement, and withdrawal to another occupation or out of the labor force.

Let $q_x^{(j)}$ be the probability of decrement due to cause *j* for a person age *x* between the ages *x* and *x+1*. This causes cane be represented by the following expressions,

q_x^1 Probability of death

q_x^2 Probability of disability

q_x^3 Probability of retirement

q_x^4 Probability of withdrawal

Since they are mutually exclusive, the probabilities are additive, thus
$q_x^{(t)} = q_x^{(1)} + q_x^{(2)} + q_x^{(3)} + q_x^{(4)}$, where *t*=all causes

and

$p_x^{(t)} = 1 - q_x^{(t)}$, where $p_x^{(t)}$ gives probability of remaining in the initial state throught the interval. Once the causes are all determined, decrements into state *j* as $d_x^{(j)} = l_x^{(t)} q_x^{(j)}$ from an initial number of individuals in the initially active state, $l_x^{(t)}$ and proceed to develop a multiple decrement table.

For example, suppose we have $q_x^{(1)\prime}$ coming from an associated mortality. We need to convert this rate (the rate at which the initial population would diminish, if the mortality were the only factor causing decrements), to the corresponding $q_x^{(1)}$probability. Starting with l_x individuals alive at *x*, and if we know that $d_x^{(2)} = q_x^{(2)} l_x$ will be disabled at *x+1*, $d_x^{(3)} = q_x^{(3)} l_x$ will retire and $d_x^{(4)} = q_x^{(4)} l_x$ will withdraw at *x+1*. Now assuming that these decrements proceed linearly between *x* and *x+1*, at age *x + ½* there will be $l_x - 0.5\left(d_x^{(2)} + d_x^{(3)} + d_x^{(4)}\right)$ remaining, and it is this number, and not l_x, which is on average exposed to death. Thus $q_x^{(1)\prime}$ times this number or $q_x^{(1)\prime}\left\{1 - 0.5\left(q_x^{(2)} + q_x^{(3)} + q_x^{(4)}\right)\right\} l_x = q_x^{(1)} l_x$ will die. Consequently, we have, equating coefficients, and dividing out l_x, $q_x^{(1)} q_x^{(1)\prime}\left\{1 - 0.5\left(q_x^{(2)} + q_x^{(3)} + q_x^{(4)}\right)\right\} l_x$ as a simple approximation.

8. Conclusion

From the above discussions, it is evident that the multiple decrement theory is the right approach in addressing the issue of retirement. This is because it both considers retirement at all ages and all possible causes, which may result to the retirement. However, it is more complex since it requires sound knowledge of mathematics and is tedious.

On the probability theory approach, we think it is a representative as it is simple and considers all retirement ages from the newborn until death. However, most of its results and computations are only assumptions and as such require intuitiveness when making such assumptions.

The poorest of all the approaches is the average age retirement approach since it assumes a start of zero dollars.

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Received: August, 2009