

# A Note on Integral Transforms and Partial Differential Equations

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**Abstract.** In this study, we apply double integral transforms to solve partial differential equation namely double Laplace and Sumudu transforms, in particular the wave and poisson's equations were solved by double Sumudu transform and the same result can be obtained by double Laplace transform.

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## 1. INTRODUCTION

The topic of partial differential equations is very important subject yet there is no general method to solve all the PDEs. The behavior of the solutions very much depend essentially on the classification of PDEs therefore the problem of classification for partial differential equations is very natural and well known since the classification governs the sufficient number and the type of the conditions in order to determine whether the problem is well posed and has a unique solution.

It is also well known that some of second-order linear partial differential equations can be classified as Parabolic, Hyperbolic or Elliptic however if a PDE has coefficients which are not constant, it is rather a mixed type. In many applications of partial differential equations the coefficients are not constant in fact they are a function

of two or more independent variables and possible dependent variables. Therefore the analysis to describe the solution may not hold globally for equations with variable coefficients that we have for the equations having constant coefficients.

On the other side there are some very useful physical problems where its type can be changed. One of the best known example is for the transonic flow, where the equation is in the form of

$$\left(1 - \frac{u^2}{c^2}\right) \phi_{xx} - \frac{2uv}{c^2} \phi_{xy} + \left(1 - \frac{v^2}{c^2}\right) \phi_{yy} + f(\phi) = 0$$

where  $u$  and  $v$  are the velocity components and  $c$  is a constant, see [1].

Similarly, partial differential equations with variable coefficients are also used in finance, for example, the arbitrage-free value  $C$  of many derivatives

$$\frac{\partial C}{\partial \tau} + s^2 \frac{\sigma^2(s, \tau)}{2} \frac{\partial^2 C}{\partial s^2} + b(s, \tau) \frac{\partial C}{\partial s} - r(s, \tau) C = 0$$

with three variable coefficients  $\sigma(s, \tau)$ ,  $b(s, \tau)$  and  $r(s, \tau)$ . In fact this partial differential equation holds whenever  $C$  is twice differentiable with respect to  $s$  and once with respect to  $\tau$ , see [8].

However, in the literature there was no systematic way to generate a partial differential equations by using the equations with constant coefficients, the most of the partial differential equations with variable coefficients depend on nature of particular problems.

Recently, A. Kılıçman and H. Eltayeb in [5], introduced a new method producing a partial differential equation by using the PDEs with constant coefficients and classification of partial differential equations having polynomial coefficients. Later the same authors extended this setting in [7] to the finite product of convolution of hyperbolic and elliptic PDEs where the authors considered the positive coefficients of polynomials.

In this study we are going to solve PDEs with variable boundary conditions by using double integral transform methods: double Laplace transform and double Sumudu transform.

First of all we give the following definition and during this study we use the following convolution notation: double convolution between two continuous functions  $F(x, y)$

and  $G(x, y)$  given by

$$(1.1) \quad F_1(x, y) * * F_2(x, y) = \int_0^y \int_0^x F_1(x - \theta_1, y - \theta_2) F_2(\theta_1, \theta_2) d\theta_1 d\theta_2$$

for further details and properties of the double convolutions and derivatives we refer to [3]. And also the definition of double Laplace and Sumudu transforms given by

$$(1.2) \quad L_x L_t [f(x, s)] = F(p, s) = \int_0^\infty e^{-px} \int_0^\infty e^{-st} f(x, t) dt dx$$

where  $x, t > 0$  and  $p, s$  complex value

and

$$F(v, u) = S_2 [f(t, x); (v, u)] = \frac{1}{uv} \int_0^\infty \int_0^\infty e^{-(\frac{t}{v} + \frac{x}{u})} f(t, x) dt dx$$

where  $x, t > 0$  and  $u, v$  complex value

respectively. More details see [4] and [6]. Now since we are going to apply the integral transform methods to PDE, first we must know the integral transform of partial derivatives as follows: double Laplace transform of the first order partial derivative with respect to  $x$  given by

$$L_x L_t \left[ \frac{\partial f(x, t)}{\partial x} \right] = pF(p, s) - F(0, s).$$

Also the double Laplace transform for second partial derivative with respect to  $x$  is

$$L_{xx} \left[ \frac{\partial^2 f(x, t)}{\partial^2 x} \right] = p^2 F(p, s) - pF(0, s) - \frac{\partial F(0, s)}{\partial x},$$

the double Laplace transform for second partial derivative with respect to  $t$  similarly as above given by

$$L_{tt} \left[ \frac{\partial^2 f(x, t)}{\partial^2 t} \right] = s^2 F(p, s) - sF(p, 0) - \frac{\partial F(p, 0)}{\partial t}.$$

In a similar manner the double Laplace transform of a mixed partial derivative can be deduced from single Laplace transform as

$$L_x L_t \left[ \frac{\partial^2 f(x, t)}{\partial x \partial t} \right] = psF(p, s) - pF(p, 0) - sF(0, s) - F(0, 0).$$

Similarly, the double Sumudu transform for second partial derivative with respect to  $x$  given by

$$(1.3) \quad S_2 \left[ \frac{\partial^2 f(t, x)}{\partial x^2}; (v, u) \right] = \frac{1}{u^2} F(v, u) - \frac{1}{u^2} F(v, 0) - \frac{1}{u} \frac{\partial f(v, 0)}{\partial x}.$$

Finally, the double Sumudu transform of  $\frac{\partial^2 f(t,x)}{\partial t^2}$ , given by

$$(1.4) \quad S \left[ \frac{\partial^2 f(t,x)}{\partial t^2}; (v, u) \right] = \frac{1}{v^2} F(v, u) - \frac{1}{v^2} F(0, u) - \frac{1}{v} \frac{\partial f(0, u)}{\partial t}.$$

It is well known that in order to obtain the solution of partial differential equations by integral transform methods we need the following two steps:

- Firstly, we transform the partial differential equations to algebraic equations by using the double integral transform methods
- Secondly, on using the double inverse transform to get the solution of PDEs.

Now, let us solve the linear second order partial differential equations by using the two transforms as follows:

$$(1.5) \quad au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g_1(x; y) ** g_2(x; y)$$

under boundary conditions

$$\begin{aligned} u(x, 0) &= f_1(x) * f_2(x), \quad u(0, y) = w_1(y) * w_2(y) \\ u_y(x, 0) &= \frac{d}{dx} (f_1(x) * f_2(x)), \quad u_x(0, y) = \frac{d}{dy} (w_1(y) * w_2(y)) \quad \text{and} \quad u(0, 0) = 0 \end{aligned}$$

where the symbol \*\* means double convolution [2] and  $a, b, c, d, e$  and  $f$  are constant coefficient,  $g_1(x, y)$  is the exponential function where as  $g_2(x, y)$  is a polynomial. In this study we consider that (1.5) has a solution on using the both transforms; further the inverse double Laplace and Sumudu transforms exists. In the next we discuss the solution of above equation by using double Sumudu transform, but here we use the matrix forum, before we deal with this transform we give some concept as follows, let  $P(x, y) = \sum_{k=0}^n \frac{a_k}{x^k} + \sum_{l=0}^m \frac{b_l}{y^l}$  be a function which can be represented by the following matrix product:

$$\begin{aligned} M_P(x, y) &= \begin{pmatrix} \frac{1}{x} & \frac{1}{x^2} & \frac{1}{x^3} & \dots & \frac{1}{x^n} \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \dots & \dots & a_n \\ a_2 & a_3 & \dots & \dots & a_n & 0 \\ a_3 & \dots & \dots & a_n & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & 0 & \dots & \dots & \dots & 0 \end{pmatrix} \\ &+ \begin{pmatrix} \frac{1}{y} & \frac{1}{y^2} & \frac{1}{y^3} & \dots & \frac{1}{y^m} \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \dots & \dots & b_m \\ b_2 & b_3 & \dots & \dots & b_m & 0 \\ b_3 & \dots & \dots & b_m & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_m & 0 & \dots & \dots & \dots & 0 \end{pmatrix} \end{aligned}$$

then we see that  $M_P(x, y)u = M_P(x)u + M_P(y)u$  where  $M_P(x)$ , and  $M_P(y)$  define a linear mappings in an obvious way. Now we shall write vectors  $u$  in  $\mathbb{C}^n \times \mathbb{C}^m$  as rows vectors or columns vectors interchangeably, whichever are convenient although, when  $M_P(x, y)u$  is to be computed and of course the vector  $u$  is written as a column vector

$$(1.6) \quad M_P(x, y)u = \sum_{i=1}^n \frac{1}{x^i} \sum_{k=0}^{n-i} a_{i+k} u_k + \sum_{j=1}^m \frac{1}{y^j} \sum_{l=0}^{m-j} b_{j+l} u_l$$

for any  $u_k = (u_0(y), u_2(y) \dots u_{n-1}(y))$  and  $u_l = (u_0(x), u_2(x) \dots u_{n-1}(x)) \in \mathbb{C}^n \times \mathbb{C}^m$ .

In particular case, we apply the double Sumudu transform for a linear second order partial differential equation with constant coefficient.

Let  $g(x, y)$  be a continuous on  $(0, \infty) \times (0, \infty)$ , zero on  $(-\infty, 0) \times (-\infty, 0)$  then it is locally integrable and Laplace transformable. Let  $u(x, y)$  be differentiable on  $(0, \infty) \times (0, \infty)$ , and satisfy

$$(1.7) \quad au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y)$$

under the boundary conditions

$$\begin{aligned} u(x, 0) &= f_1(x) * f_2(x), \quad u(0, y) = w_1(y) * w_2(y) \\ u_y(x, 0) &= \frac{d}{dx} (f_1(x) * f_2(x)), \quad u_x(0, y) = \frac{d}{dy} (w_1(y) * w_2(y)) \text{ and } u(0, 0) = 0 \end{aligned}$$

then the transform of this equation lead us to the

$$\begin{aligned} A(\alpha, \beta) S_x S_y [u(x, y)](\alpha, \beta) &= S_x S_y [g(x, y)](\alpha, \beta) + \begin{pmatrix} 1 & 1 \\ \alpha & \alpha^2 \end{pmatrix} \begin{pmatrix} d & a \\ a & 0 \end{pmatrix} \begin{pmatrix} U(0, \beta) \\ U_x(0, \beta) \end{pmatrix} \\ &+ \begin{pmatrix} 1 & 1 \\ \beta & \beta^2 \end{pmatrix} \begin{pmatrix} e & c \\ c & 0 \end{pmatrix} \begin{pmatrix} U(\alpha, 0) \\ U_y(\alpha, 0) \end{pmatrix} + 0 \end{aligned}$$

where  $A(p, q) = \frac{a}{\alpha^2} + b\alpha\beta + \frac{c}{\beta^2} + d\alpha + e\beta + f$  and Sumudu transform of single convolution and derivative of single convolution are given by

$$S_x [f_1(x) * f_2(x)] = \alpha F_1(\alpha) F_2(\alpha), \quad S_y [w_1(y) * w_2(y)] = \beta W_1(\beta) W_2(\beta)$$

and

$$\begin{aligned} S_x \left[ \frac{d}{dx} (f_1(x) * f_2(x)) \right] &= \alpha [F_1(\alpha) - F_1(0)] F_2(\alpha), \\ S_y \left[ \frac{d}{dy} (w_1(y) * w_2(y)) \right] &= \beta [W_1(\beta) - W_1(0)] W_2(\beta), \end{aligned}$$

respectively.

**Proposition 1.** *Let  $f(x, t)$  be  $n, m$  times differentiable on  $(0, \infty) \times (0, \infty)$  and  $f(x, t) = 0$  for  $t, x < 0$ . Then for any polynomial  $P$  of degree  $(m, n)$  then double Sumudu transform*

$$S_x S_t [P(D)f(x, y)](\alpha, \beta) = P(\alpha, \beta) S_x S_t [f(x, t)](\alpha, \beta) - M_P(\beta) \Psi(f, n) - M_P(\alpha) \Phi(f, m).$$

In particular

$$S_x S_t \left[ \frac{\partial^m f}{\partial x^m} \right] = \frac{1}{\alpha^m} S_x S_t [f(x, t)](\alpha, \beta) - \left( \frac{1}{\alpha^m}, \frac{1}{\alpha^{m-1}}, \dots, \frac{1}{\alpha} \right) \begin{pmatrix} a_1 & a_2 & \cdot & \cdot & \cdot & a_n \\ a_2 & a_3 & \cdot & \cdot & a_n & 0 \\ a_3 & \cdot & \cdot & a_n & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} f_0(0, \beta) \\ \frac{\partial}{\partial x} f(0, \beta) \\ \cdot \\ \cdot \\ \frac{\partial^{m-1}}{\partial x^{m-1}} f(0, \beta) \end{pmatrix}$$

and similarly we have

$$S_x S_t \left[ \frac{\partial^n f}{\partial t^n} \right] = \frac{1}{\beta^n} S_x S_t [f(x, t)](\alpha, \beta) - \left( \frac{1}{\beta^n}, \frac{1}{\beta^{n-1}}, \dots, \frac{1}{\beta} \right) \begin{pmatrix} b_1 & b_2 & \cdot & \cdot & \cdot & b_n \\ b_2 & b_3 & \cdot & \cdot & b_n & 0 \\ b_3 & \cdot & \cdot & b_n & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_n & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} f_0(\alpha, 0) \\ \frac{\partial}{\partial y}(\alpha, 0) \\ \cdot \\ \cdot \\ \frac{\partial^{n-1}}{\partial y^{n-1}}(\alpha, 0) \end{pmatrix}$$

for  $n = 2$  and  $m = 2$  we have

$$S_x S_t \left[ \frac{\partial^2 f}{\partial x^2} \right](\alpha, \beta) = \frac{1}{\alpha^2} S_x S_t [f(x, t)](\alpha, \beta) - \begin{pmatrix} 1 & 1 \\ \alpha & \alpha^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_0(0, \beta) \\ \frac{\partial}{\partial x} f(0, \beta) \end{pmatrix}$$

and

$$S_x S_t \left[ \frac{\partial^2 f}{\partial t^2} \right](\alpha, \beta) = \frac{1}{\beta^2} S_x S_t [f(x, t)](\alpha, \beta) - \begin{pmatrix} 1 & 1 \\ \beta & \beta^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_0(\alpha, 0) \\ \frac{\partial}{\partial y}(\alpha, 0) \end{pmatrix}$$

for  $n = 1$  and  $m = 1$ , we have

$$(1.8) \quad S_x S_t \left[ \frac{\partial f}{\partial x} \right](\alpha, \beta) = \frac{1}{\alpha} S_x S_t [f(x, t)](\alpha, \beta) - \frac{1}{\alpha} f(0+, \beta)$$

and

$$(1.9) \quad S_x S_t \left[ \frac{\partial f}{\partial t} \right] (\alpha, \beta) = \frac{1}{\beta} S_x S_t [f(x, t)] (\alpha, \beta) - \frac{1}{\beta} f(\alpha, 0+)$$

*Proof.* We use induction on  $n, m$ . The result is trivially true if  $n = 0, m = 0$ , and the case  $n = 1, m = 1$  are equivalent to (1.8) and (1.9) respectively. Suppose now that the result is true for some  $n, m > 0$  and let

$$P(x, y) = \sum_{k=0}^{n+1} \frac{a_k}{x^k} + \sum_{l=0}^{m+1} \frac{b_l}{t^l}$$

having degree  $(n+1, m+1)$ . The first two statements follow by putting  $h = \frac{\partial f}{\partial x}$ ,  $z = \frac{\partial f}{\partial t}$  and using the induction hypothesis and (1.8) and (1.9). Now write  $P(x, t) = a_0 + \frac{1}{x}\zeta(x) + b_0 + \frac{1}{t}\eta(t)$ , where  $\zeta(x) = \sum_{k=0}^n \frac{a_{k+1}}{x^k}$  and  $\eta(t) = \sum_{l=0}^m \frac{b_{l+1}}{t^l}$ . Then

$$P(\dot{D})f(x, t) = a_0 f + \zeta(\dot{D})h + b_0 f + \eta(\dot{D})z$$

and therefore the double Laplace transform given by

$$\begin{aligned} S_x S_t [P(\dot{D})f] (p, s) &= a_0 S_x S_t [f] + S_x S_t [\zeta(\dot{D})h] + b_0 S_x S_t [f] + S_x S_t [\eta(\dot{D})z] \\ &= a_0 S_x S_t [f] + \zeta(p) S_x S_t [h] + b_0 S_x S_t [f] + \eta(s) S_x S_t [z] \\ &\quad - M_\zeta(p) \Psi(f, n) - M_\eta(s) \Phi(f, m) \\ &= a_0 S_x S_t [f] (\alpha, \beta) + b_0 S_x S_t [f] (\alpha, \beta) + \zeta(\alpha) \left( \frac{1}{\alpha} S_x S_t [f] (\alpha, \beta) - \frac{1}{\alpha} f(0+, \beta) \right) \\ &\quad + \eta(\beta) \left( \frac{1}{\beta} S_x S_t [f] (\alpha, \beta) - \frac{1}{\beta} f(\alpha, 0+) \right) \\ &\quad - \sum_{i=1}^n \frac{1}{\alpha^i} \sum_{k=0}^{n-i} a_{i+k+1} f^{(k+1)}(0+, \beta) - \sum_{j=1}^m \frac{1}{\beta^j} \sum_{l=0}^{m-j} b_{j+l+1} f^{(l+1)}(\alpha, 0+) \end{aligned}$$

on using (1.6) and  $h = f^{(k+1)}$  and  $z = f^{(l+1)}$ . The summation above can also be written

$$\begin{aligned}
& \sum_{i=1}^n \frac{1}{\alpha^i} \sum_{k=1}^{n-i+1} a_{i+k} f^{(k)}(0+, \beta) + \sum_{j=1}^m \frac{1}{\beta^j} \sum_{l=1}^{m-j+1} b_{j+l} f^{(l)}(\alpha, 0+) \\
= & \sum_{i=1}^n \frac{1}{\alpha^i} \sum_{k=0}^{n-i+1} a_{i+k} f^{(k)}(0+, \beta) + \sum_{j=1}^m \frac{1}{\beta^j} \sum_{l=0}^{m-j+1} b_{j+l} f^{(l)}(\alpha, 0+) \\
& - \sum_{i=1}^n \frac{1}{\alpha^i} a_i f(0+, \beta) - \sum_{i=1}^n \frac{1}{\beta^i} b_i f(\alpha, 0+) \\
= & \sum_{i=1}^{n+1} \frac{1}{\alpha^i} \sum_{k=0}^{n-i+1} a_{i+k} f^{(k)}(0+, \beta) - \frac{1}{\alpha^{n+1}} a_{n+1} f(0+, \beta) - \sum_{i=1}^n \frac{1}{\alpha^i} a_i f(0+, \beta) \\
& + \sum_{j=1}^{m+1} \frac{1}{\beta^j} \sum_{l=0}^{m-j+1} b_{j+l} f^{(l)}(\alpha, 0+) - \frac{1}{\beta^{m+1}} b_{m+1} f(\alpha, 0+) - \sum_{i=1}^n \frac{1}{\beta^i} b_i f(\alpha, 0+) \\
= & M_P(\alpha) \Psi(f, n+1) - \zeta(\alpha) f(0+, \beta) + M_P(\beta) \Phi(f, m+1) - \eta(\beta) f(\alpha, 0+).
\end{aligned}$$

Thus

$$\begin{aligned}
S_x S_t [P(D)f](\alpha, \beta) &= a_0 S_x S_t [f](\alpha, \beta) + b_0 S_x S_t [f](\alpha, \beta) + \zeta(\alpha) \left( \frac{1}{\alpha} S_x S_t [f](\alpha, \beta) - \frac{1}{\alpha} f(0+, \beta) \right) \\
&+ \eta(\beta) \left( \frac{1}{\beta} S_x S_t [f](\alpha, \beta) - \frac{1}{\beta} f(\alpha, 0+) \right) \\
&- M_P(\alpha) \Psi(f, n+1) - \zeta(\alpha) f(0+, \beta) + M_P(\beta) \Phi(f, m+1) - \eta(\beta) f(\alpha, 0+)
\end{aligned}$$

finally we have

$$S_x S_t [P(D)f(x, y)](\alpha, \beta) = P(\alpha, \beta) S_x S_t [f(x, t)](\alpha, \beta) - M_P(\beta) \Psi(f, n) - M_P(\alpha) \Phi(f, m).$$

■

In particular, if we consider the wave equation in the form of

$$\begin{aligned}
(1.10) \quad u_{tt} - u_{xx} &= -3e^{2x+t} & (x, t) \in \mathbb{R}_+^2 \\
u(x, 0) &= e^{2x} + e^x, & u_t(x, 0) = e^{2x} + e^x \\
u(0, t) &= 2e^t, & u_x(0, t) = 3e^t
\end{aligned}$$

then on using the above proposition we have



$$\begin{aligned}
A(\alpha, \beta) S_x S_y [u(x, y)](\alpha, \beta) &= -\frac{3}{(1-\beta)(1-2\alpha)} \\
&+ \left(\frac{1}{\alpha} \frac{1}{\alpha^2}\right) \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{(1-\beta)} \\ \frac{3}{(1-\beta)} \end{pmatrix} \\
&+ \left(\frac{1}{\beta} \frac{1}{\beta^2}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{2-3\alpha}{(1-2\alpha)(1-\alpha)} \\ \frac{2-3\alpha}{(1-2\alpha)(1-\alpha)} \end{pmatrix}
\end{aligned}$$

where  $A(\alpha, \beta) = \frac{1}{\beta^2} - \frac{1}{\alpha^2}$  by simplify and arrangement, we have

$$\begin{aligned}
S_x S_y [u(x, y)](\alpha, \beta) &= -\frac{3\alpha^2\beta^2}{(\alpha^2 - \beta^2)(1-2\alpha)(1-\beta)} - \frac{\beta^2(2+3\alpha)}{(\alpha^2 - \beta^2)(1-\beta)} \\
(1.11) \quad &+ \frac{\alpha^2(2-3\alpha-3\alpha\beta+2\beta)}{(\alpha^2 - \beta^2)(1-2\alpha)(1-\alpha)}
\end{aligned}$$

in the next step we replace the complex variables  $\alpha, \beta$  in eq(1.11) by  $\frac{1}{p}, \frac{1}{q}$  respectively and multiply each terms by  $\frac{1}{pq}$  then by applying the inverse transform and we compute the poles by using MAPLE software, then we obtain the solution of eq(1.10) as follows

$$u(x, t) = e^{2x+t} + e^{x+t}.$$

In the following example we apply the above proposition and double Sumudu transform to the Poisson's equation in the form of

$$\begin{aligned}
u_{xx} + u_{yy} &= -4e^{x+y} \sin(x+y) \\
u(x, 0) &= e^x \cos(x), \quad u(0, y) = e^y \cos(y) \\
u_x(0, y) &= e^y \cos(y) - e^y \sin(y) \\
(1.12) \quad u_y(x, y) &= e^x \cos(x) - e^x \sin(x),
\end{aligned}$$

then on using the above proposition and inverse double Sumudu transform we obtain the solution of eq(1.12) in the form of

$$u(x, y) = e^{x+y} \cos(x+y).$$

Thus, the same result can easily be obtained on using the double Laplace transform.

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#### REFERENCES

- [1] G.-Q. Chen and M. Feldman, Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type, *J. Amer. Math. Soc.*, 16 (2003), pp. 461–494.
- [2] H. Eltayeb and A. Kılıçman. A Note on Solutions of Wave, Laplace's and Heat Equations with Convolution Terms by Using Double Laplace Transform: *Appl. Math, Lett.* **21** (2008) 1324–1329.
- [3] H. Eltayeb and A. Kılıçman. A note on double Sumudu transform and double Laplace transform, *Int. Pure and Appl. Math.*, to appear.
- [4] A. Kılıçman and H. Eltayeb. A Note on Defining Singular Integral as Distribution And Partial Deferential Equation with Convolution term: *Mathematical and Computer Modelling* **49** (2009) 327–336.
- [5] A. Kılıçman and H. Eltayeb. A note on classification of hyperbolic and elliptic equations with polynomial coefficients, *Applied Mathematics Letters* 21(11) (2008), pp. 1124–1128.
- [6] A. Kılıçman and H. E. Gadain. An Application of Double Laplace Transform and Double Sumudu Transform, *Lobachevskii Journal of Mathematics*, 2009, Vol. 30, No. 3, pp. 214–223.
- [7] A. Kılıçman and H. Eltayeb. On finite product of convolutions and classifications of hyperbolic and elliptic equations, arXiv: 0901.2422v1 [math.AP], 16 jan 2009.
- [8] R. C. Merton. Theory of rational option pricing, *Bell Journal of Economics and Management Science* 4(Spring)(1973), pp. 141–183.

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