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A Result in the Ho and Lee's Model

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Abstract

A limit which arise in an empirical test of the Ho and Lee's model is calculated. This empirical test is that carry out on the French market by J.F.Boulier, J. Sikorav, *Yield Curve Fluctuations Does French Market Fit the Ho and Lee's Model*, 2nd AFIR Colloquium 1991, 1:225-236.

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1 Preliminary notes

In this paper I want to calculate the value of a limit that appears in [2], where the Ho and Lee's model is tested on the French market. For a presentation of this model, which is used to pricing interest rate contingent claims and which is a discrete time financial model, I remind to the original paper [1]. Nevertheless it needs to recall some definitions and some fundamental quantities that arise in this model. First of all we define a discount function $P : \mathrm{R}^+ \times \mathrm{R}^+ \to \mathrm{R}^+$ such that, if $(t, T) \in \mathrm{R}^+ \times \mathrm{R}^+$, where t is the first instant of the investment and T is the time to maturity,

$$P(t, 0) = 1 \ \forall \ t \in \mathrm{R}^+ \tag{1}$$

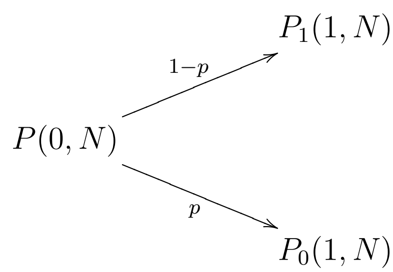
$$\lim_{T\to\infty} P(t, T) = 0 \ \forall \ t \in \mathrm{R}^+ \tag{2}$$

The Ho and Lee's model is a discrete time model, therefore the temporal variables t and T assume discrete values which have distance $\Delta \in \mathrm{R}$; hence $t = n\Delta$ and $T = N\Delta$, with $n, N \in \mathrm{N}$. We can assume $\Delta = 1$. The discount function can be redefined as $P : \mathrm{N} \times \mathrm{N} \to \mathrm{R}^+$ and the conditions (1) and (2) begin

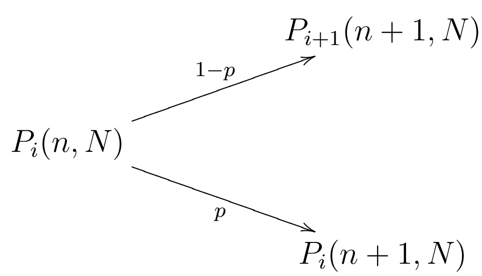
$$P(n,0) = 1 \ \forall \ n \in \mathrm{N} \tag{3}$$

$$\lim_{N\to\infty} P(n,N) = 0 \ \forall \ n \in \mathrm{N} \tag{4}$$

In this model the discount function evolves along a binomial lattice: at time $n = 1$ it is specified by two possible function, $P_0(1,N)$ e $P_1(1,N)$; $P(0,N)$ goes in this two states with probability respectively p and $1 - p$.



At a generic time n the diagram is



The assumption of the model is that two functions $h_+ : \mathrm{N} \to \mathrm{R}$ and $h_- : \mathrm{N} \to \mathrm{R}$ exist such that

$$P_{i+1}(n+1,N) = \frac{P_i(n,N+1)}{P_i(n,1)} h_+(N) \tag{5}$$

$$P_i(n+1,N) = \frac{P_i(n,N+1)}{P_i(n,1)} h_-(N) \tag{6}$$

From (3) it follows that

$$h_+(0) = h_-(0) = 1 \tag{7}$$

If the market is arbitrage-free and complete, an unique measure of probability p^* equivalent to p exists such that ([3], [4])

$$(1-p^*)h_+(N) + p^* h_-(N) = 1 \tag{8}$$

It is possible to find explicit expressions for $h_+(0)$ and $h_-(0)$; they are [1]

$$h_+(N) = \frac{1}{1+p^*(\sigma^N - 1)} \tag{9}$$

$$h_-(N) = \frac{\sigma^N}{p^*(\sigma^N - 1)+1} \tag{10}$$

where

$$\sigma \equiv \frac{1-(1-p^*)h(1)}{p^* h(1)} \tag{11}$$

As it is shown in [1], it is also possible to find

$$P_i(n,N) = \frac{P(0,n+N)\sigma^{N(n-i)} \prod_{s=1}^{n} h_+(N+n-s)}{P(0,n)\prod_{s=1}^{n-1} h_+(n-s)} \tag{12}$$

2 The limit $H(N)$

In [2] it is introduced the function $Y : (n,N) \subseteq \mathrm{N}\times\mathrm{N} \to \mathrm{R}$ such that

$$Y_i(n,N) = \ln\left[P_i(n,N)\frac{P(0,n)}{P(0,n+N)}\right] \tag{13}$$

Y is a measure of the distance between the forward rate and the future rate and, at first approximation, we can identify it with the risk premium ([2]). By equation (12) we have the following expression for $Y(n,N)$:

$$Y_i(n,N) = \ln \frac{\prod_{s=1}^{n} h_+(N+n-s)}{\prod_{s=1}^{n-1} h_+(n-s)} + (n-i)\ln\sigma = H(n,N) + (n-i)\ln\sigma^N \tag{14}$$

where it is defined

$$H(n,N) \equiv \ln \frac{\prod_{s=1}^{n} h_+(N+n-s)}{\prod_{s=1}^{n-1} h_+(n-s)} \tag{15}$$

Proposition 2.1. *In the limit for n to infinity it is*

$$H(N) \equiv \lim_{n\to\infty} H(n,N) = \sum_{s=0}^{N-1} \ln\left[1 + \frac{p^*}{1-p^*}\sigma^s\right] \tag{16}$$

Proof. Inserting the expression (9) of h_+ (9) in the definition of $H(n,N)$ and defining $n-s=j$, we find

$$H(n,N)=\ln\frac{\prod_{j=1}^{n-1}[1+p^*(\sigma^j-1)]}{\prod_{j=0}^{n-1}[1+p^*(\sigma^{N+j}-1)]}\Rightarrow$$

$$\Rightarrow H(n,N)=\sum_{j=1}^{n-1}\ln[1+p^*(\sigma^j-1)]-\sum_{j=0}^{n-1}\ln[1+p^*(\sigma^{N+j}-1)]$$

Then

$$H(n,N)=\sum_{j=1}^{n-1}\ln[1+p^*(\sigma^j-1)]-\sum_{j=0}^{n-1}\ln[1+p^*(\sigma^{N+j}-1)]\quad if\ \ N\geqslant n \tag{17}$$

$$H(n,N)=\sum_{j=1}^{N-1}\ln[1+p^*(\sigma^j-1)]-\sum_{j=n}^{N+n-1}\ln[1+p^*(\sigma^j-1)]\quad if\ \ N<n \tag{18}$$

Because $N<\infty$, from (18) it follows that

$$\lim_{n\to\infty}H(n,N)=\lim_{n\to\infty}\left\{\sum_{j=1}^{N-1}\ln[1+p^*(\sigma^j-1)]-\sum_{j=n}^{N+n-1}\ln[1+p^*(\sigma^j-1)]\right\} \tag{19}$$

If we redefine $j=s+n$ in the second sum of (19), we have

$$\lim_{n\to\infty}H(n,N)=\lim_{n\to\infty}\left\{\sum_{j=1}^{N-1}\ln[1+p^*(\sigma^j-1)]-\sum_{s=0}^{N-1}\ln[1+p^*(\sigma^{s+n}-1)]\right\} \tag{20}$$

Because $0\leqslant\sigma<1$, as shown in [1], (the case $\sigma=1$ is trivial) in the limit we have

$$\lim_{n\to\infty}\sigma^n=0$$

and then (20) begin

$$H(N)=\sum_{j=1}^{N-1}\ln[1+p^*(\sigma^j-1)]-N\ln[1-p^*]=\sum_{s=0}^{N-1}\ln\left[1+\frac{p^*}{1-p^*}\sigma^s\right] \tag{21}$$

□

Proposition 2.2. *The following inequalities are true:*

$$\int_0^N \ln\left[1+\frac{p^*}{1-p^*}\sigma^x\right]dx \leqslant H(N) \leqslant \int_0^N \ln\left[1+\frac{p^*}{1-p^*}\sigma^x\right]dx - \ln[1+p^*(\sigma^N-1)] \tag{22}$$

Proof. The function $\ln[1+p(\sigma^x-1)]$ for $p<1$ and $\sigma<1$ is negative and strictly decreasing. Hence for the summation in the definition of $H(N)$ it is true

$$\int_0^N \ln\left[1+\frac{p^*}{1-p^*}\sigma^x\right]dx \leqslant \sum_{s=0}^{N-1} \ln\left[1+\frac{p^*}{1-p^*}\sigma^s\right] \tag{23}$$

The (16) can rewritten in the following manner:

$$H(N) = \sum_{s=0}^{N} \ln\left[1+\frac{p^*}{1-p^*}\sigma^s\right] - \ln\left[1+\frac{p^*}{1-p^*}\sigma^N\right] \Rightarrow$$

$$\Rightarrow H(N) = \sum_{s=0}^{N} \ln\left[1+\frac{p^*}{1-p^*}\sigma^s\right] - \ln[1+p^*(\sigma^N-1)] + \ln(1-p^*) \Rightarrow$$

$$\Rightarrow H(N) = \sum_{s=1}^{N} \ln\left[1+\frac{p^*}{1-p^*}\sigma^s\right] - \ln[1+p^*(\sigma^N-1)]$$

For the summation in the previous equality the following inequality is true:

$$\sum_{s=0}^{N-1} \ln\left[1+\frac{p^*}{1-p^*}\sigma^s\right] \leqslant \int_0^N \ln\left[1+\frac{p^*}{1-p^*}\sigma^x\right]dx \tag{24}$$

and because the term $-\ln[1+p^*(\sigma^N-1)]$ is positive, we have

$$H(N) = \sum_{s=1}^{N} \ln\left[1+\frac{p^*}{1-p^*}\sigma^s\right] - \ln[1+p^*(\sigma^N-1)] \leqslant$$

$$\leqslant \int_0^N \ln\left[1+\frac{p^*}{1-p^*}\sigma^x\right]dx - \ln[1+p^*(\sigma^N-1)] \tag{25}$$

Therefore we obtain the statement of the proposition. □

References

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