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Optimization of a Problem of Optimal Control with Free Initial State

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Abstract

The theory of control analyzes the proprieties of commanded systems. Problems of optimal control (OC) have been intensively investigated in the world literature for over forty years. During this period, series of fundamental results have been obtained, among which should be noted the maximum principle [1] and dynamic programming [2]. For many of the problems of the optimal control theory (OCT) adequate solutions are found [4, 5, 7]. Results of the theory were taken up in various fields of science, engineering, and economics. The present paper aims at extending the constructive methods of [6] that were developed for the problems of optimal control with the bounded initial state is not fixed are considered.

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1 Introduction

Problems of optimal control (OC) have been intensively investigated in the world literature for over forty years. During this period, series of fundamental results have been obtained, among which should be noted maximum principle [1] and dynamic programming [2]. For many of the problems of the optimal control theory (OCT) adequate solutions are found [4, 5, 7, 8]. Results of the theory were taken up in various fields of science, engineering, and economics.

The aim of this paper is to solve a problem of optimal with free initial state. The problem has the following sense, the initial state of the optimized system is not known exactly, a priori information on the initial state is exhausted by inclusion $x_0 \in X_0$, by analogy with the theory of filtration, we say that the set X_0 is a priori distribution of the initial state of the control system.

The paper has the following structure: In section 1, the canonical OC problem is formulated. In section 2, we give some definitions whose we can need in our problem. In section 3, we defined support control of the problem and these accompanying elements. In section 4, Optimality and suboptimality criteria are formulated. In section 5, Optimality and ε-optimality criteria. In section 6, Numerical algorithm for solving the problem ;The iteration consists of three procedures: change of control, change of a support,at the end final procedure. In section 7, the results are illustrated by an example.

2 Statement of the problem

Let us consider the optimal control problem for a linear system at the time interval $T = [0, t^*]$:

$$c'x(t^*) \to max \tag{1}$$

$$\dot{x} = Ax + bu, x(0) = z \in X_0 = \{z \in R^n, \quad Gz = \gamma, \quad d_* \leq z \leq d^*\}, \tag{2}$$

$$Hx(t^*) = g, \tag{3}$$

$$f_* \leq u(t) \leq f^*, \quad t \in T = [0, t^*]. \tag{4}$$

Here $x \in R^n$ is a state of control system (2); $u(.) = (u(t), t \in t), T = [0, t^*]$, is a piecewise continuous function; $A \in R^{n \times n}$; $b, c \in R^n$; $g \in R^{m \times n}, rankH = m \leq n; f_*, f^*$ are scalars; $d_* = (d_{*j}, j \in J), d^* = d^*(J) = (d^*_j, j \in J)$ are $n-$vectors; $G \in R^{l \times n}, rankG = l \leq n, \gamma \in R^l, I = \{1, ..., m\}, J = \{1, ..., n\}, L = \{1, ..., l\}$ are sets of indices.

By using the Cauchy formula, we obtain the solution of the system (2) :

$$x(t) = F(t)(z + \int_0^t F^{-1}(\vartheta)bu(\vartheta)d\vartheta), t \in T, \tag{5}$$

where $F(t) = e^{At}, t \in T = [0, t^*]$ is defined by the relations:

$$\begin{cases} \dot{F}(t) = AF(t) & , \\ F(0) = I_n & . \end{cases}$$

Substituting (5) into $(1)-(4)$, we obtain the following equivalent formulation of the problem:

$$\tilde{c}'z + \int_0^{t^*} c(t)u(t)dt \longrightarrow max, \tag{6}$$

$$D(I,J)z + \int_0^{t^*} \varphi(t)u(t)dt = g, \tag{7}$$

$$G(L,J)z = \gamma, \ d_* \leq z \leq d^*, \tag{8}$$

$$f_* \leq u(t) \leq f^*, t \in T, \tag{9}$$

where $\tilde{c}' = c'F(t^*), \ c(t) = c'F(t^*)F^{-1}(t)b, \ D(I,J) = HF(t^*), \ \varphi(t) = HF(t^*)F^{-1}(t)b.$

3 Essentials Definitions

Definition 3.1 *A pair $v = (z, u(.))$ formed of an $n-$vector z and a piecewise continuous function $u(.)$ is called a generalized control.*

Definition 3.2 *A generalized control $v = (z, u(.))$ is said to be an admissible control if it satisfied the constraints (2)-(4).*

Definition 3.3 *An admissible control $v^0 = (z^0, u^0(.))$ is said to be an optimal open-loop control if a control criterion reaches its maximal value*

$$J(v^0) = \max_v J(v).$$

Definition 3.4 *For a given $\varepsilon \geq 0$, an $\varepsilon-$optimal control $v^\varepsilon = (z^\varepsilon, u^\varepsilon(.))$ is defined by the inequality*

$$J(v^0) - J(v^\varepsilon) \leq \varepsilon.$$

4 Support control and accompanying elements

Let us choose an arbitrary subset $T_B \subset T$ of $k \leq m$ elements and an arbitrary subset $J_B \subset J$ of $m + l - k$ elements. Form the matrix

$$P_B = \begin{pmatrix} D(I, J_B) & \varphi(t), t \in T_B \\ G(L, J_B) & 0 \end{pmatrix} \tag{10}$$

A set $S_B = \{T_B, J_B\}$ is said to be a support of problem $(1) - (4)$ if $detP_B \neq 0$.

A pair $\{v, S_B\}$ of an admissible control $v = (z, u(.))$ and a support S_B is said to be a support control. A support control $\{v, S_B\}$ is said to be primally not degenerate if $d_{*j} < z_j < d^*_j, j \in J_B, f_* < u(t) < f^*, t \in T_B$.
Let us consider another admissible control $\overline{v} = (\overline{z}, \overline{u}(.)) = v + \Delta v$, where $\overline{z} = z + \Delta z, \overline{u}(t) = u(t) + \Delta u(t), t \in T$, and let us calculate the increment of the cost functional

$$\Delta J(v) = J(\overline{v}) - J(v) = \tilde{c}'\Delta z + \int_{t \in T} c(t)\Delta u(t).$$

Since

$$D(I, J)\Delta z + \int_{t \in T} \varphi(t)\Delta u(t) = 0,$$

and

$$G(L, J)\Delta z = 0,$$

then the increment of the functional equals:

$$\Delta J(v) = (\tilde{c}' - \nu' \begin{pmatrix} D(I, J) \\ G(L, J) \end{pmatrix})\Delta z + \int_{t \in T} (\varphi(t) - \nu' c(t))\Delta u(t)$$

where $\nu = \begin{pmatrix} \nu_u \\ \nu_z \end{pmatrix} \in R^{m+l}, \nu_u \in R^m, \nu_z \in R^l$ is a function of the Lagrange multipliers called potentials, is calculated as a solution to the equation: $\nu' = q'_B Q$, where $Q = P_B^{-1}, q_B = (\tilde{c}_j, j \in J_B, c(t), t \in T_B)$. Introduce an n-vector of estimates $\Delta' = \nu' \begin{pmatrix} D(I, J) \\ G(L, J) \end{pmatrix} - \tilde{c}'$, and a function of cocontrol $\Delta(.) = (\Delta(t) = \nu'_u \varphi(t) - c(t), t \in T)$.
By using these notions, the value of the cost of functional increment takes the form:

$$\Delta J(v) = \Delta' \Delta z - \int_{t \in T} \Delta(t)\Delta u(t). \tag{11}$$

A support control $\{v, S_B\}$ is dually not degenerate if $\Delta(t) \neq 0, t \in T_H, \Delta_j \neq 0, j \in J_H$, where $T_H = T/T_B,\ J_H = J/J_B$.

5 Calculation of the value of suboptimality

The new control $\overline{v}(t)$ is admissible, if it satisfies the constraints:

$$d_* - z \leq \Delta z \leq d^* - z;\ f_* - u(t) \leq \Delta u(t) \leq f^* - u(t), t \in T. \tag{12}$$

The maximum of functional (11) under constraints (12) is reached for:

$$\begin{cases} \Delta z_j = d_{*j} - z_j & \text{if } \Delta_j > 0 \\ \Delta z_j = d^*_j - z_j & \text{if } \Delta_j < 0 \\ d_{*j} - z_j \leq \Delta z_j \leq d^*_j - z_j, & \text{if } \Delta_j = 0,\ j \in J. \end{cases}$$

$$\begin{cases} \Delta u(t) = f_* - u(t) & \text{if } \Delta(t) > 0 \\ \Delta u(t) = f^* - u(t) & \text{if } \Delta(t) < 0 \\ f_* \leq \Delta u(t) \leq f^*, & \text{if } \Delta(t) = 0, \ t \in T, \end{cases}$$

and is equal to:

$$\beta = \beta(v, S_B) = \sum_{j \in J_H^+} \Delta_j (z_j - d_{*j}) + \sum_{j \in J_H^-} \Delta_j (z_j - d_j^*) \tag{13}$$

$$+ \int_{t \in T^+} \Delta(t)(u(t) - f_*) + \int_{t \in T^-} \Delta(t)(u(t) - f^*)$$

where

$$T^+ = \{t \in T_H, \Delta(t) > 0\}, T^- = \{t \in T_H, \Delta(t) < 0\},$$
$$J_H^+ = \{j \in J_H, \Delta_j > 0\}, J_H^- = \{j \in J_H, \Delta_j < 0\}.$$

The number $\beta(v, S_B)$ is called a value of suboptimality of the support control $\{v, S_B\}$.
From there, $J(\overline{v}) - J(v) \leq \beta(v, S_B)$. Of this last inequality, the following result is deduced:

6 Optimality and ε-optimality criterion

Theorem 6.1 (7) *Following relations:*

$$\begin{cases} u(t) = f_*, & \text{if } \Delta(t) > 0 \\ u(t) = f^*, & \text{if } \Delta(t) < 0 \\ f_* \leq u(t) \leq f^*, & \text{if } \Delta(t) = 0, \ t \in T \\ z_j = d_{*j}, & \text{if } \Delta_j > 0 \\ z_j = d_j^*, & \text{if } \Delta_j < 0 \\ d_{*j} \leq z_j \leq d_j^*, & \text{if } \Delta_j = 0, \ j \in J. \end{cases} \tag{14}$$

are sufficient,and in the cases of non-degeneracy, they are necessary for the optimality of support control $\{v, S_B\}$.

proof 6.2 ***Sufficient condition.*** *If the relations* (14) *are satisfied, then from* (13)*, we obtain:* $\beta(v, S_B) = 0$,
Such as

$$\Delta J(v) \leq \beta(v, S_B) = 0,$$

then $\{v, S_B\}$ is an optimal support control.
Necessary condition. *Let us proceed by absurd:*
Let be $\{v, S_B\}$ a support control optimal non degeneracy and assume that the relations (14) *are not satisfied, i.e:*
$\exists t_0 \in T$ *such that:*

$$\Delta(t_0) > 0 \quad \textit{and} \quad u(t_0) > f_* \quad \textit{or} \quad u(t_0) < f^*.$$

Or $\exists j_0 \in J$ *such that:*

$$\Delta_{j0} > 0 \quad and \quad z_{j0} > d_* \quad or \quad z_{j0} < d^*.$$

It is easy to construct the admissible variations Δv*, which the value of the cost of functional increment is :*

$$\Delta J(v) = \Delta^{'} \Delta z - \int_{t \in T} \Delta(t) \Delta u(t) > 0,$$

then,

$$J(\overline{v}) - J(v) > 0.$$

And this contradict that $\{v, S_B\}$ *is optimal.*

Theorem 6.3 (7) *For any* $\varepsilon \geq 0$*, the admissible control* v *is* $\varepsilon-$*optimal if and only if there exists a support* S_B *such that* $\beta(v, S_B) \leq \varepsilon$*.*

7 Numerical algorithm for solving the problem

Suppose $\varepsilon > 0$ is a given number and $\{v, S_B\}$ is a known support control that does not satisfy optimality and $\varepsilon-$ optimality criterion. The method suggested is iterative, its aim is to construct an $\varepsilon-$ solution of problem $(1)-(4)$. As a support will be changing during the iterations together with an admissible control it is natural to consider them as a pair. The iteration of the method is to change initial support control $\{v, S_B\}$ for the "new" $\{\overline{v}, \overline{S}_B\}$ so that $\beta(v, S_B) \geq \beta(\overline{v}, \overline{S}_B)$. The iteration consists of three procedures:

1. Change of an admissible control $v \rightarrow \overline{v}$.

2. Change of support $S_B \rightarrow \overline{S}_B$.

3. Final procedure.

A construction of the initial support concerns with the first phase and can be solved using the algorithm described below.
At the beginning of each iteration the following information is stored:

1. An admissible control v.

2. A support $S_B = \{T_B, J_B\}$.

3. A value of suboptimality $\beta = \beta(v, S_B)$.

Before beginning iteration, we make sure that a support control $\{v, S_B\}$ does not satisfy criterion of $\varepsilon-$optimality.

7.1 Change of control.

Let us $\alpha_1 > 0, \alpha_2 > 0, h > 0, \mu > 0$ parameters of the method, and we construct the following sets:

$$J_0 = \{j \in J : |\Delta_j| \leq \alpha_2\}, J_* = \{j \in J : |\Delta_j| > \alpha_2\},$$

$$T_0 = \{t \in T : |\Delta(t)| \leq \alpha_1\}, T_* = \{t \in T : |\Delta(t)| > \alpha_1\}, where |J_0| = K,$$

and subdivide T_0 into subintervals $[\tau_i, \tau^i[, i = \overline{1, N}; \tau_i < \tau^i, T_0 = \bigcup_{i=1}^{N} [\tau_i, \tau^i], \tau^i - \tau_i \leq h, T_B \subset \{\tau_i, i = \overline{1, N}\}, u(t) = u_i = const, t \in [\tau_i, \tau^i[, i = \overline{1, N}$.
A new admissible control $\overline{v} = (\overline{z}, \overline{u}(t), t \in T)$ such that:

$$\begin{cases} \overline{z}_j = z_j + \kappa \Delta z_j, & j \in J \\ \overline{u}(t) = u(t) + \theta \Delta u(t), & t \in T, \end{cases} \tag{15}$$

Here

$$\Delta z_j = \begin{cases} d_j^* - z_j, & if \Delta_j < -\alpha_2 \\ d_{*j} - z_j, & if \Delta_j > \alpha_2, j \in J_* \\ 0, & if \Delta_j = 0, j \in J_0, \end{cases} \quad \Delta u(t) = \begin{cases} f^* - u(t), & if \Delta(t) < -\alpha_1 \\ f_* - u(t), & if \Delta(t) > \alpha_1, t \in T_* \\ u_i = const, & if t \in [\tau_i, \tau^i[, i = \overline{1, N}, t \in T_0. \end{cases}$$

We introduce the parameter vector:
$l_i = \theta u_i, i = \overline{1, N}, h_j = \kappa \Delta z_j, j \in J_0, h_{K+1} = \kappa$, and define these quantities:

$$g_i = -\int_{\tau_i}^{\tau^i} \Delta(t) dt, i = \overline{1, N}, g_{N+1} = -\int_{T_*} \Delta(t) \Delta u(t) dt,$$

$$\phi_i = -\int_{\tau_i}^{\tau^i} \varphi(t) dt, i = \overline{1, N}, \phi_{N+1} = -\int_{T_*} \varphi(t) \Delta u(t),$$

$$q_j = -\Delta_j, j \in J_0, q_{K+1} = \sum_{j \in J_*} -\Delta_j \Delta z_j, j \in J_*,$$

$$D_j = D(I, j), j \in J_0, D_{K+1} = \sum_{j \in J_*} D(I, j) \Delta z_j,$$

$$f_{*i} = f_* - u_i, f_i^* = f^* - u_i, i = \overline{1, N}, f_{*N+1} = 0, f_{N+1}^* = 1,$$

$$d_{*j} = d_* - z_j, d_j^* = d^* - z_j, j = \overline{1, K}, d_{*K+1} = 0, d_{K+1}^* = 1.$$

In order to find $(h_j, l_i), j = \overline{1, K+1}, i = \overline{1, N+1}$, we formulate the mathematical programming problem:

$$\begin{cases} \Delta J(v) = \sum_{j \in J_0 \cup \{K+1\}} q_j h_j + \sum i = 1^{N+1} g_i l_i \rightarrow max_{h_j, l_i}, \\ \sum_{J_0 \cup \{K+1\}} D(I, j) h_j + \sum_{i=1}^{N+1} \phi_i l_i = 0, \\ \sum_{j \in J_0 \cup \{K+1\}} G(l, j) h_j = 0, \\ f_{*i} \leq l_i \leq f_i^*, & i = \overline{1, N+1} \\ d_{*j} \geq h_j \geq d_j^*, & j = \overline{1, K+1}. \end{cases} \tag{16}$$

Problem (16) is solved by adaptive method. As a result, we obtain an $\varepsilon-$ optimal support plan $(h_j^\varepsilon, l_i^\varepsilon, \overline{J}_B, \overline{T}_B)$. The new control $(\overline{z}, \overline{u}(t), t \in T)$ are constructed according to the rules:

$$\overline{z}_j = \begin{cases} z_j + h_{K+1}\Delta z_j, & j \in J_* \\ z_j + h_j, & j \in J_0. \end{cases} \tag{17}$$

Here

$$\overline{u}(t) = \begin{cases} u(t) + l_{N+1}\Delta u(t), & t \in T_* \\ u(t) + l_i, & t \in [\tau_i, \tau^i[, i = \overline{1, N}. \end{cases} \tag{18}$$

It is clear that $J(\overline{v}) \geq J(v)$.

- If $K+1 \notin \overline{J}_B$ and $t_{N+1} \notin \overline{T}_B$, then we put:

$$\tilde{S}_B = \{\tilde{J}_B = \overline{J}_B, \tilde{T}_B = \overline{T}_B\}.$$

- If not, we would have the following cases:

1. If $K+1 \notin \overline{J}_B$ and $t_{N+1} \in \overline{T}_B$, we exclude index $N+1$ from the support in the following way: Let be:

$$\overline{\Delta}(t) = \Delta(t) + \sigma\delta(t),$$

where σ is the maximal dual step and $\delta(t)$ the direction . Let us determine i_* such that:

$$\sigma(t_{i*}) = min\sigma(t_i), t_i \in T_H,$$

with

$$\sigma(t_i) = \begin{cases} -\Delta(t_i)/\delta(t_i), & \text{if } \Delta(t_i) \times \delta(t_i) \leq 0, \delta(t_i) \neq 0 \\ +\infty, & \text{otherwise.} \end{cases}$$

$$\delta(t) = \begin{cases} 0, & \text{on } T_B/\{t_{N+1}\}; \\ 1, & \text{if } \overline{u}(t) = f_*; \\ -1, & \text{if } \overline{u}(t) = f^*. \end{cases}$$

$$\delta(t) = \delta_B' P_B^{-1}\phi(t), t \in T.$$

Then a new support is:

$$\tilde{J}_B = \overline{J}_B; \tilde{T}_B = (\overline{T}_B/\{t_{N+1}\}) \cup \{t_{i*}\}.$$

2. if $K+1 \in \overline{J}_B$ and $t_{N+1} \notin \overline{T}_B$, we exclude index $K+1$ from the support in the following way: Let be:

$$\overline{\Delta}_j = \Delta_j + \sigma_j\delta_j,$$

where σ_j is the maximal dual step and δ_j the direction
Let us determine j_* such that:

$$\sigma_{j*} = min \sigma_j, j \in J_H,$$

with

$$\sigma_j = \begin{cases} -\Delta_j/\delta_j, & \text{if } \Delta_j \times \delta_j leq 0, \delta_j \neq 0 \\ +\infty, & \text{otherwise.} \end{cases}$$

$$\delta_j = \begin{cases} 0, & \text{on } J_B/\{K+1\}; \\ 1, & \text{if } \overline{z}_j = d_*; \\ -1, & \text{if } \overline{z}_j = d^*. \end{cases}$$

$$\delta_j = \delta_B' P_B^{-1} \begin{pmatrix} D(I,J) \\ G(L,J) \end{pmatrix}, j \in J.$$

Then a new support is:

$$\tilde{J}_B = (\overline{J}_B/\{K+1\}) \cup \{j_*\}; \tilde{T}_B = \overline{T}_B.$$

3. The last case will be if $K+1 \in \overline{J}_B, t_{N+1} \in \overline{T}_B$, the new support will be:

$$\tilde{J}_B = (\overline{J}_B/\{K+1\}) \cup \{j_*\}; \tilde{T}_B = (\overline{T}_B/\{t_{N+1}\}) \cup \{t_{i*}\}.$$

At this stage, let us denote the new support $\tilde{S}_B$ and let us construct the support matrix $P(\tilde{S}_B)$ and let us check that it is not singular. Let us calculate the new suboptimality estimate $\beta(\tilde{v}, \tilde{S}_B)$.

- If $\beta(\tilde{v}, \tilde{S}_B) = 0$, then $\overline{v}$ is an optimal control.
- If $\beta(\tilde{v}, \tilde{S}_B) \leq \varepsilon$, then $\overline{v}$ is an $\varepsilon-$ optimal control.
- otherwise, we perform either a new iteration with $\{\overline{v}, \tilde{S}_B\}, \overline{\alpha}_1 < \alpha_1, \overline{\alpha}_2 < \alpha_2, \overline{h} < h$ or the procedure change of support.

7.2 Change of support.

let us assume that for the new control $\overline{v}$, we have $\beta(\overline{v}, \tilde{S}_B) > \varepsilon$, then we perform change of support. By using support $\tilde{S}_B$, let us construct the quasi-control $\tilde{v} = (\tilde{z}, \tilde{u}(t), t \in T)$:

$$\tilde{z}_j = \begin{cases} d_{j*} & \text{if } \tilde{\Delta}_j > 0 \\ d_j^* & \text{if } \tilde{\Delta}_j < 0 \\ \in [d_{j*}, d_j^*] & \text{if } \tilde{\Delta}_j = 0, \ j \in J \end{cases} \qquad \tilde{u}(t) = \begin{cases} f_*, & \text{if } \tilde{\Delta}(t) < 0 \\ f^*, & \text{if } \tilde{\Delta}(t) > 0, \\ \in [f_* f^*] & \text{if } \tilde{\Delta}(t) = 0, \ t \in T, \end{cases}$$

where: $\tilde{\Delta}(t) = -\tilde{\psi}'(t)b, t \in T, \tilde{\Delta}' = (\tilde{\Delta}_j, j \in J)' = \nu' \begin{pmatrix} D(I,J) \\ G(L,J) \end{pmatrix} - \tilde{c}'.$

Here, $\tilde{\psi}(t), t \in T$, the solution to the adjoint system corresponding to $\tilde{S}_B$.

Let us the quasi trajectory corresponding $\chi = (\chi(t), t \in T), \chi(0) = z \in X_0$ of the system $\dot{\chi} = A\chi + b\tilde{u}, \chi(0) = z \in X_0$.
If

$$D(I,J)\tilde{z} + \int_0^{t^*} \varphi(t)\tilde{u}(t)dt = g,$$

$$G(L,J)\tilde{z} = \gamma,$$

then $\overline{v}$ is optimal control, and if

$$D(I,J)\tilde{z} + \int_0^{t^*} \varphi(t)\tilde{u}(t)dt \neq g,$$

$$G(L,J)\tilde{z} \neq \gamma,$$

then construct a vector $\lambda(\tilde{J}_B, \tilde{T}_B)$ as follows:

$$P(\tilde{S}_B) \cdot \lambda(\tilde{J}_B, \tilde{T}_B) = \begin{pmatrix} D(I,J)\tilde{z} + \int_0^{t^*} \tilde{u}(t)dt - g \\ G(L,J)\tilde{z} - \gamma \end{pmatrix}$$

$$\lambda(\tilde{J}_B, \tilde{T}_B) = P_B^{-1}(\tilde{S}_B) \begin{pmatrix} D(I,J)\tilde{z} + \int_0^{t^*} \tilde{u}(t)dt - g \\ G(L,J)\tilde{z} - \gamma \end{pmatrix}.$$

Now, we studies the following cases:

- If $\| \lambda(\tilde{J}_B, \tilde{T}_B) \| = 0$, then the quasi-control $\tilde{v}$ is optimal for the problem $(1)-(4)$.
- If $\| \lambda(\tilde{J}_B, \tilde{T}_B) \| > \mu$, then let us change a support $\tilde{S}_B$ to $\overline{S}_B$ by dual method.
- if $\| \lambda(\tilde{J}_B, \tilde{T}_B) \| < \mu$, then we perform final procedure.

Dual method.
Let be:

$$\lambda_0 = max_{\{j \in J_B, t \in T_B\}} |\lambda(t), \lambda_j|.$$

Let us consider two cases:

1. If $\lambda_0 = max_{\{j \in J_B, t \in T_B\}} |\lambda(t), \lambda_j| = |\lambda_{j0}|, j_0 \in \tilde{J}_B$,
   The dual step is:

$$\sigma_j = \begin{cases} -\tilde{\Delta}_j/\delta_j, & \text{if } \tilde{\Delta}_j\delta_j < 0, \delta_j \neq 0 \\ 0, & \text{if} \tilde{\Delta}_j = 0, \delta_j > 0, \overline{z} \neq d_* or \tilde{\Delta}_j = 0, \delta_j < 0, \overline{z} \neq d^*, \\ +\infty, & \text{otherwise}, j \in J. \end{cases}$$

   Let us construct the following set:

$$J(\sigma) = \{j \in J : \sigma_j < \sigma\},$$

let us the decreasing rate of the dual functional:

$$\alpha(\sigma) = -|\lambda_{j0}| + 2\sum_{J(\sigma)} |\delta_j|.$$

By construction:

$$\alpha(0) = -|\lambda_{j0}| < 0 \; and \; \alpha(\sigma) < \alpha(\overline{\sigma}) \; if \; \sigma < \overline{\sigma}, \; If \; \alpha(\sigma) < 0 \; for \; \sigma > 0,$$

then a problem (1) – (4) do not possessed an admissible control. Otherwise, research $\sigma_0 \geq 0$ such that:

$$\alpha(\sigma_0 - y) < 0, \; \alpha(\sigma_0 + 0) \geq 0, \; \forall \; 0 \leq y \leq \sigma_0.$$

Let us $j_* \in J/\tilde{J}_B$ such instant verify that:

$$\tilde{\Delta}(j_*) + \sigma^0 \delta_{j*} = 0, \; \delta_{j*} \neq 0,$$

then a new support $\tilde{S}_B$ change into $\hat{S}_B$.

$$\hat{S}_B = \{\hat{J}_B = (\tilde{J}_B/\{j_0\}) \cup \{j_*\}, \hat{T}_B = \tilde{T}_B\}.$$

2. If $\lambda_0 = max_{\{j \in J_B, t \in T_B\}} |\lambda(t), \lambda_j| = |\lambda(t_0)|, t_0 \in \tilde{T}_B$, and let us calculate the following quantities:

$$\begin{cases} \Delta\nu(I) = -P_B^{-1}(t_0, I) sign\lambda(t_0) = \delta_{T_B} \cdot P^{-1}(T_B), & \\ \delta(t) = \Delta\psi' b, & t \in T \\ \Delta\dot{\psi} = -A'\Delta\psi, & \\ \Delta\psi(t_1) = -H'\Delta\nu, & \end{cases}$$

with $P_B^{-1}(t_0, I)$: the t_0^{th} row of matrix $[P(I, T_B)]^{-1}$.
The dual step is:

$$\sigma(t) = \begin{cases} -\tilde{\Delta}(t)/\delta(t), & \text{if } \tilde{\Delta}\delta(t) < 0, \quad \delta(t) \neq 0 \\ 0, & \text{if } \tilde{\Delta}(t) = 0, \delta(t) > 0, \overline{u}(t) \neq f_* or \tilde{\Delta}(t) = 0, \delta(t) < 0, \overline{u}(t) \neq f^*, \\ +\infty, & \text{otherwise}, \; t \in T. \end{cases}$$

Let us construct the following set:

$$T(\sigma) = \{t \in T : \sigma(t) < \sigma\},$$

let us the decreasing rate of the dual functional:

$$\alpha(\sigma) = -|\lambda(t_0)| + 2\int_{T(\sigma)} |\delta(t)| dt.$$

By construction:

$$\alpha(0) = -|\lambda(t_0)| < 0 \; and \; \alpha(\sigma) < \alpha(\overline{\sigma}) \; if \quad \sigma < \overline{\sigma}, \; If \; \alpha(\sigma) < 0 \; for \; \sigma > 0,$$

then a problem $(1)-(4)$ do not possessed an admissible control. Otherwise, research $\sigma_0 \geq 0$ such that:

$$\alpha(\sigma_0 - y) < 0, \alpha(\sigma_0 + 0) \geq 0, \forall \; 0 \leq y \leq \sigma_0.$$

Let us $t_* \in T/\tilde{T}_B$ such instant verify that:

$$\tilde{\Delta}(t_*) + \sigma^0 \delta(t_*) = 0, \delta(t_*) \neq 0,$$

then a new support $\tilde{S}_B$ change into $\hat{S}_B$.

$$\hat{S}_B = \{\hat{J}_B = \tilde{J}_B, \hat{T}_B = (\tilde{T}_B/\{t_0\}) \cup \{t_*\}\}.$$

Let us calculate the new suboptimality estimate $\beta(\overline{v}, \hat{S}_B)$:
1. If $\beta(\overline{v}, \hat{S}_B) = 0$, then the control $\overline{v}$ is optimal for problem (1)-(4) .
2. If $\beta(\overline{v}, \hat{S}_B) < \varepsilon$, then the control $\overline{v}$ is ε-optimal for problem (1)-(4) .
3. If $\beta(\overline{v}, \hat{S}_B) > \varepsilon$, then we perform the next iteration starting from the support control $\{\overline{v}, \hat{S}_B\}$.

7.3 final procedure.

Let us assume that for the new control $\overline{v}$, we have $\beta(\overline{v}, \hat{S}_B) > \varepsilon$. With the use of the support $\overline{S}_B$ we construct a quasicontrol $\widehat{v} = (\widehat{z}, \widehat{u}(t), t \in T)$:

$$\widehat{z}_j = \begin{cases} d_{j*} & \text{if } \Delta_j > 0 \\ d_j^* & \text{if } \Delta_j < 0 \\ \in [d_{j*}, d_j^*] & \text{if } \Delta_j = 0, \; j \in J \end{cases} \qquad \widehat{u}(t) = \begin{cases} f_*, & \text{if } \Delta(t) < 0 \\ f^*, & \text{if } \Delta(t) > 0, \quad t \in T. \end{cases}$$

If

$$D(I,J)\widehat{z} + \int_0^{t^*} \varphi(t)\widehat{u}(t)dt = g,$$
$$G(L,J)\widehat{z} = \gamma,$$

then $\widehat{v}$ is optimal, and if

$$D(I,J)\widehat{z} + \int_0^{t^*} \varphi(t)\widehat{u}(t)dt \neq g,$$
$$G(L,J)\widehat{z} \neq \gamma,$$

then denote $T^0 = \{t_i, \; i = \overline{1,s}\}, \; s = |T_B|$. Here, $t_i, \; i = \overline{1,s}$ are zeroes of the optimal cocontrol $\Delta(t) = 0, t \in T; \; t_0 = 0, \; t_{s+1} = t^*$. Suppose

$$\dot{\Delta}(t_i) \neq 0, \; i = \overline{1,s}.$$

Let us construct the following function:

$$f(\Theta) = \begin{pmatrix} D(I,J_B)z(J_B) + D(I,J_H)z(J_H) + \sum_{i=0}^{s}(\frac{f^*+f_*}{2} - \frac{f^*-f_*}{2} sign\dot{\Delta}(t_i)) \int_{t_i}^{t_{i+1}} \varphi(t)dt - g \\ G(L,J_B)z(J_B) + G(L,J_H)z(J_H) - \gamma \end{pmatrix}$$

where

$$z_j = \frac{d_j^* + d_j*}{2} - \frac{d_j^* - d_j*}{2} sign\Delta_j, \ j \in J_H.$$

$$\Theta = (t_i, \ i = \overline{1,s}; z_j, j \in J_B).$$

The final procedure consists in finding the solution

$$\Theta^0 = (t_i^0, i = \overline{1,s}; z_j^0, j \in J_B)$$

of the system of $m + l$ nonlinear equations

$$f(\Theta) = 0. \tag{19}$$

We solve this system by the Newton method using as an initial approximation the vector

$$\Theta^{(0)} = (\bar{t}_i, \ i = \overline{1,s}; \bar{z}_j, j \in J_B).$$

The $(k+1)^{th}$ approximation $\Theta^{(k+1)}$,equal:

$$\Theta^{(k+1)} = \Theta^{(k)} + \Delta\Theta^{(k)} \qquad \Delta\Theta^{(k)} = -\frac{\partial f^{-1}(\Theta^{(k)})}{\partial \Theta^{(k)}} \cdot f(\Theta^{(k)}).$$

Let us compute the Jacobi matrix for equation (19)

$$\frac{\partial f(\Theta^{(k)})}{\partial \Theta^{(k)}} = \begin{pmatrix} D(I,J_B) & (f_* - f^*) sign\dot{\Delta}(t_i^{(k)})\varphi(t_i^{(k)}), \ i = \overline{1,s} \\ & \\ G(L,J_B) & 0 \end{pmatrix}.$$

As $det P_B \neq 0$, we can easily show that

$$det \frac{\partial f(\Theta^{(0)})}{\partial \Theta^{(0)}} \neq 0. \tag{20}$$

For instants $t \in T_B$ there exists a small $\eta > 0$ that for any $\tilde{t}_i \in [t_i - \eta, t_i + \eta], i = \overline{1,s}$, the matrix $(\varphi(\tilde{t}_i), i = \overline{1,s})$ is not degenerate and the matrix $\frac{\partial f(\Theta^{(k)})}{\partial \Theta^{(k)}}$ is also not degenerate, if elements $t_i^{(k)}, i = \overline{1,s}, k = 1,2,...$ do not leave the η-vicinity of t_i, $i = \overline{1,s}$. Vector $\Theta^{(k^*)}$ is taken as solution of equation (19) if

$$\| f(\Theta^{(k^*)}) \| \leq \eta, \texttt{for a given } \eta > 0. \tag{21}$$

for a given $\eta > 0$.
So we put $\theta^0 = \theta^{(k^*)}$.
The suboptimal control for problem (1)-(4) is computed as

$$z_j^0 = \begin{cases} z_j^0, & j \in J_B \\ \widehat{z}_j, & j \in J_H; \end{cases}$$

$$u^0(t) = \frac{f^* + f_*}{2} - \frac{f^* - f_*}{2} sign\dot{\Delta}(t_i^0), \; t \in [t_i^0, t_{i+1}^0[, \; i = \overline{1, s}.$$

If the Newton method does not converge, we decrease the parameter $\alpha_1 > 0, \alpha_2 > 0, h > 0$ and perform the iterative process again.

8 Example.

We illustrate the results obtained in the paper using the following numerical example:

$$c'x(2) \rightarrow max \tag{22}$$

$$\dot{x}_1 = x_2, \dot{x}_2 = u, z \in X_0 = \{z \in \mathtt{R}^2 : Gz = \gamma, -2 \leq z_i \leq 2, i = 1, 2\} \tag{23}$$

$$Hx(2) = g, \tag{24}$$

$$|u(t)| \leq 1, t \in [0, 2]. \tag{25}$$

Where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, H = \begin{pmatrix} 1 & -2 \end{pmatrix}, c' = \begin{pmatrix} 0 & 1 \end{pmatrix}, g = 2, G = \begin{pmatrix} 1 & 2 \end{pmatrix}, \gamma = 3, f_* = -1, f^* = 1, d_* = (-2, -2), d^* = (2, 2), n = 2, m = 1, l = 1, t^* = 2.$
let the initial control and state be known, where:

$$u(t) = \begin{cases} 1/2, & t \in [0, 1[\\ -1/2, & t \in]1, 2]. \end{cases}, z = (z_1, z_2) = (3/2, 3/4) \in X_0.$$

$c(t) = 1, \varphi(t) = -t, \tilde{c} = \begin{pmatrix} 0 & 1 \end{pmatrix}, D(I, J) = \begin{pmatrix} 1 & 0 \end{pmatrix}.$
On the trajectory $x(t), t \in [0, 2]$ corresponding (23), we have :

$$Hx(t^*) = 2, J(v) = 3/4.$$

The first stage of the algorithm is run with $h = 0.25, \alpha_1 = 0.25, \alpha_2 = 0.25, \mu = 0.4, \varepsilon = 0.2$. To find the parameter vector (h_j, l_i), a series of linear programming

problems (16) are formed and solved by the adaptive method. Let the initial support S_B such that $T_B =]0.85, 1]$ correspond $\tau_B = 1, J = \{1, 2\}, J_B = \emptyset$.
The solution of the problem (16) is:$h_1 = 1, l_1 = 0.5, l_2 = 0.06, \overline{T}_B = [5/4, 2[$ correspond $\tau_B = 5/4, J_B = \emptyset$,
After three iteration, relations (21) are satisfied for the following set of parameters: $\lambda(\overline{T}_B, J_B) = (-0.35, 0)$, $\overline{T}_B = [5/4, 2[$ correspond $\tau_B = 5/4, J_B = \emptyset$,

$$\chi(t^*) = \begin{pmatrix} 7.4375 \\ 2.5 \end{pmatrix}.$$

Denote by $t_i, i = \overline{1, 2}$ the zeroes of the function $\overline{\Delta}(t), t \in [0, 2]$. Then, the algorithm passes to the finishing procedure. The data obtained at the previous stage are used to form the parameter vector θ and its initial approximation $\theta^{(0)} = (\overline{z}_j = 0, t_i = 1)$ are found with prescribed accuracy $\eta = 0.6$ by solving system (19) using Newton's method. Note that the form of system (19) is uniquely determined by the set of parameters $\overline{T}_B = [5/4, 2[$ correspond $\tau_B = 5/4, J_B = \emptyset, s = 2$. At the second iteration of Newton's method, the condition (21) is satisfied for the parameter vector $\theta^{(*)} = (\overline{z}_j = 0, t_i = 1)$ the condition We resolve the system (19) by Newton's method which taken as an approximation solution of system (19),

$$\theta^{(1)} = (\overline{z}_j = 0, t_i = 1.19),$$

Which is taken as an approximation solution of system(19). This vector is used to restore the control $\widehat{v}$, and trajectory $\chi(t^*)$ and to verify the satisfaction of the constraints and the condition of the ε-optimality. We obtain

$$\chi(t^*) = \begin{pmatrix} 7.3439 \\ 2.38 \end{pmatrix}, H\chi(t^*) = 2.5839.$$

consequently :

$$J(v^0) = 2.28.$$

The resulting control is admissible, and the ε-optimality is satisfied at $\varepsilon = 0.2$. Therefore, the original problem has been solved.

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