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Computation of Lyapunov Exponents for Dynamical System with Impact

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Abstract

Consider the dynamical system $\ddot{u} + 2\alpha\dot{u} + u = a\cos\omega t$ where the position u is constrained to remain above an obstacle of height $u_{\min}$; when u reaches the obstacle, its velocity is reversed and multiplied by a restitution coefficient $e \in [0, 1]$. For certain choices of parameters, the solutions are chaotic. We compute the Lyapunov exponents by three different methods, and we compare the results. The computation of these numbers is very sensitive to the method, and to the numerical parameters for a given method, even with a very accurate method.

Mathematics Subject Classification: Dynamical Systems

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1 Introduction

In this article, we consider a class of non smooth dynamical systems, which describe the motion of a mechanical system with one degree of freedom, subject to a unilateral constraint on the position. When this constraint is saturated, the velocity is reversed and multiplied by a restitution coefficient.

More precisely, let f be a continuous function from $\mathbb{R}^3$ to $\mathbb{R}$, which is Lipschitz continuous with respect to its last two arguments. Assume that a real number $u_{\min}$ and a restitution coefficient e are given. The initial data are $u_0 \geq u_{\min}$ and v_0, such that $v_0 \geq 0$ if $u_0 = u_{\min}$. The solutions of the Cauchy problem, are defined as follows: u is a continuous function from $[t_0, T]$ whose second derivative (in the sens of distributions) is a measure. This condition implies that the first derivative $\dot{u}$ of u is a function of bounded variation, and in particular, it is continuous almost everywhere, and it is continuous from the left and from the right at all points of $(t_0, T]$ and $[t_0, T)$ respectively. Thus, the difference $\ddot{u} - f(t, u, \dot{u})$ will be a measure μ; the function u and the measure μ have to satisfy the following relations:

$$\ddot{u} = f(t, u, \dot{u}) + \mu \tag{1}$$

$$u \geq u_{\min} \tag{2}$$

$$\text{supp}\mu \subset \{t : u(t) = u_{\min}\} \tag{3}$$

$$\text{if } u(t) = u_{\min}, \text{ then } \dot{u}(t+0) = -e\dot{u}(t-0) \tag{4}$$

$$u(t_0) = u_0, \dot{u}(t_0) = v_0 \tag{5}$$

A few words of comment on this definition are in order: the first derivative of u is expected to be discontinuous at impacts, for obvious geometric reasons; therefore, it makes sense to assume that the second derivative of u has Dirac masses, or more generally is a measure. The measure μ is the reaction of the obstacle at impact, as can be seen from (1), which is basically Newton's law. Condition (2) means that $u(t)$ remains inside the convex of constraints $[u_{\min}, \infty)$. The reaction can be different from 0 only when there is a contact: this is relation (3). Finally, condition (5) describes the constitutive law of the impact, with the help of the restitution coefficient e.

It has been proved in [5] that problem (1)-(5) possesses a solution. However, uniqueness is not always true and an example has been given in [12]. Nevertheless, generic uniqueness has been proved in a special case [2] for n degrees of freedom; uniqueness has also been proved for one degree of freedom when f is analytic with respect to all its arguments [13].

A simple case of $(1) - (4)$ is a forced vibrating system with one degree of freedom defined by

$$f(t, u, v) = a\cos(\omega t) - 2\alpha v - u. \tag{6}$$

In this case, the uniqueness theorem of [13] applies.

For particular values of the parameters α, $u_{\min}$ and e, the problem (1)–(5) with the choice (6) of function f shows typically non linear phenomena of sensitivity to initial data, and there are values of the parameter for which there exists a “strange” attractor.

In this article, we compute Lyapunov exponents for the system (1)–(6), using two different numerical methods.

The first numerical method is the impact detection method, and it is very straightforward: suppose we are given initial data $u(t_k) = u_k$ and $\dot{u}(t_k+0) = v_k$ at an impact time t_k. We use the elementary and formulæ for the free flight solution, and we seek the first time $t_{k+1} > t_k$ for which $u(t_{k+1}) - u_{\min}$ vanishes. We perform this search by sweeping forward in time so as to find a small interval where $u - u_{\min}$ changes sign; on this small interval, we find the impact time by Newton's method. The accuracy is limited only by the capability of the computer. At instant t_{k+1}, we reverse the velocity according to rule (4), and we start the process again using the new values of the initial data.

The second numerical method is the [4,6] numerical scheme defined in [4,6]. Let us describe this scheme: denote P_K the projection on the convex set $K = [(1+e)u_{\min}, +\infty)$; it is given by

$$P_K(x) = \max(x, (1+e)u_{\min}), \tag{7}$$

and a sequence F_n, which is defined by:

$$F^n = f(nh, U^n, (U^{n+1} - U^n)/h). \tag{8}$$

The numerical scheme is given by the following relation:

$$U^{n+1} + eU^{n-1} = P_K[2U^n + (e-1)U^{n-1} + h^2 F^n]. \tag{9}$$

It turns out that the computations using (7)–(9) are much faster than the computations by the impact detection method. However, the scheme is not very accurate; it is not better than first order with respect to the position.

It has been observed that in a chaotic case, the attractor of (1)–(6) calculated by the impact detection method is well approximated by the attractor calculated by the numerical scheme (7)–(9).

We would like to estimate how well the numerical scheme (7)–(9) approximates more refined information, namely the Lyapunov exponents of (1)–(6): we reduce our continuous time dynamical system in three dimensions to a discrete time dynamical system, by using a Poincaré map; this is easy because we cut the phase space by the planes $t = t_0 + kT$, where $T = 2\pi/\omega$ is the period of the forcing. In the two dimensional case, the largest Lyapunov exponent, if it exists, describes the average rate of divergence of two infinitesimally close trajectories; the sum of the two Lyapunov exponents, if it exists, describes the average rate of evolution of infinetesimal volume in phase space.

There are classical methods for calculating numerically Lyapunov exponents; they are described for instance in [7], and consist essentially in giving a reasonably stable numerical implementation of the definition of the Lyapunov exponents. The calculations of Lyapunov exponents are very close to the approximation of eigenvalues of a matrix by the power method, and it has been known for a long time that renormalizations are needed to make the computation possible.

The fact that our system is not smooth is not a serious problem, because the system can be linearized whenever the impact takes place with a strictly positive velocity, and we never observed tangential impacts for choices of the parameters leading to a chaotic behavior.

The article is organized as follows: in section 2, we recall the definition of the Lyapunov exponents, and we define a program which could lead to a theoretical treatment of the qualitative questions we consider, and we describe its state of advancement. In section 3, we differentiate the flow with respect to the initial data: we consider separately the flow in continuous time and the flow in discrete time, and we show that a crucial term disappears in the discrete time case. In section 4, we explain the three methods of computation of the Lyapunov exponents. In section 5, we describe and compare our numerical results. In section 6, we conclude.

2 Definition of Lyapunov exponents

Let us recall the definition of Lyapunov exponents in a smooth case: let $S^n x$ be a trajectory of a discrete dynamical system; if δ is very small, the largest Lyapunov exponent is the asymptotic rate of evolution of $|S^n(x+\delta) - S^n(x)|$ as time n tends to infinity, if it exists; similarly if δ_1 and δ_2 are independant, the sum of the first two Lyapunov exponents describes the asymptotic rate of evolution of the area of the paralellogram built on the vectors $S^n(x+\delta_1) - S^n(x)$ and $S^n(x+\delta_2) - S^n(x)$. That such objects can be defined and exist is a non trivial fact, which is proved only in rather particular cases. Let us state just a few of the results which would enable one to prove that the Lyapunov exponents exist. Our sources for the following description have been [9], [10], [11] and [14].

We give first some definitions which are classical in ergodic theory: consider a measurable space, i.e. a pair $(X, \mathfrak{X})$ where X is an abstract set, and $\mathfrak{X}$ is a σ-algebra of its subsets. A mapping S from X to itself is an endomorphism of $(X, \mathfrak{X})$ if for all C belonging to $\mathfrak{X}$, the set $T^{-1}C$ belongs also to $\mathfrak{X}$. A measure μ on $(X, \mathfrak{X})$ is invariant under the endomorphism S iff for all $C \in \mathfrak{X}$, $\mu(S^{-1}C) = \mu(C)$.

The celebrated Birkhoff-Khinchin ergodic theorem states that if μ is invariant under S, then for almost every x and for all $f \in L^1(X, \mathfrak{X}, \mu)$ the following limit

$$\bar{f}(x) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} f(S^j x)$$

exists. Moreover, the limiting function $\bar{f}$ is invariant, i.e;

$$\bar{f}(Sx) = \bar{f}(x),$$

for μ-almost all $x \in X$.

Let S be an endomorphism of $(X, \mathfrak{X})$ and let μ be a probability measure that is invariant under S. The symmetric difference of two sets A and B is denoted $A\Delta B$, and a set $A \in \mathfrak{X}$ is said to be invariant mod 0 iff $\mu(A\Delta(S^{-1}A)) = 0$.

An endomorphism S is called ergodic if any invariant modulo 0 set is of measure 0 or 1. An equivalent formulation of this definition is that in the Birkhoff-Khinchin theorem, $\bar{f}$ is a constant:

$$\bar{f}(x) = \int f(x)\, d\mu(x).$$

Alternatively, it is possible to say that μ is ergodic iff the time and the space averages coincide.

The existence of an invariant measure can be proved if X is a compact metric space, $\mathfrak{X}$ is the Borel σ-algebra and S is an homeomorphism of X. This result is due to Bogoliubov and Krilov and can be found for instance in [9], page 8, Lemma 1.2.

In order to prove that in our problem we have an invariant measure, we would have to follow the following program: first observe that with a periodic forcing, it makes sense to think of time as a periodic variable of period $2\pi/\omega = T$; the mapping S maps the position and velocity at time t_0 to their image by the flow at time $t_0 + T$. The second step is to show that the system defined by (1)–(6) has a bounded invariant set in $(\mathbb{R}/T) \times V_e$; consider the ω-limit set

$$A = \bigcap_{k \geq 0} \overline{\{\cup_{n \geq k} \{S^n x\}\}}$$

for some initial data $x = (u_0, v_0)^\bullet$ and show that A is compact.

This step has been performed in a slightly different case by Anglès, in his thesis [1], and we believe that the methods of Anglès apply with very little adjustement to the present case.

Then the really difficult task would be to prove that restricted to A, S is an homeomorphism; it is clear that it is continuous, thanks to the continuous dependence with respect to data; but its inverse could be very bad: S could even lack an inverse if $(0,0)^\bullet$ belonged to A. Proving or disproving that $(0,0)^\bullet$ belongs to A looks like a very hard problem.

Even if this result were proved, our invariant measure on A could be non unique and very badly behaved. However, the existence of such an invariant measure would be an important fact, because it would build the foundations for the following question: consider a function of x, such as the average kinetic energy: it is defined by

$$E(x) = \frac{1}{T}\int_0^T |\Phi_2(t_0+s, t_0, u_0, v_0)|^2\, ds$$

where (u_0, v_0) is any representative of the class x and Φ_2 is the second component of Φ. The existence of an average kinetic energy, i.e. of the limit

$$\lim_{n\to\infty} \frac{1}{(n+1)T}\int_0^{(n+1)T} |\Phi_2(t_0+s, t_0, u_0, v_0)|^2\, ds$$

is equivalent to the existence of the limit

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{j=0}^{n} E(S^n x).$$

Clearly, this is a question in ergodic theory.

Now, we state Oseledec's theorem concerning existence of Lyapunov exponents [9, theorem 2.1, page 23]:

Theorem 2.1 *(Oseledec). Let S be a C^1-diffeomorphism of a compact manifold X, and let μ be an ergodic measure. Then there are two possibilities:*

(i) there exists $\lambda \in \mathbb{R}$ such for all v in the tangent space T_xM to M at x,

$$\lim_{n\to\infty} \frac{1}{n} \ln \|D_x S^n v\| = \lambda \tag{10}$$

for almost all x in M, or

(ii) there exists $\lambda_1 > \lambda_2$ and a splitting $T_xM = E_x^1 \oplus E_x^2$ (with the maps $x \mapsto E_x^1, E_x^2$ being measurable) such that for all v_1 in E_x^1, and all v_2 in E_x^2:

$$\lim_{n\to\infty} \frac{1}{n} \ln \|D_x S^n v_1\| = \lambda_1 \tag{11}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \ln \|D_x S^n v_2\| = \lambda_2 \tag{12}$$

for almost all x in M

In our case, S is not a C^1-diffeomorphism, and we have not proved that there exists an ergodic measure μ. Therefore, Oseledec's theorem does not enables us to establish the existence of Lyapunov exponents.

However, our problem is not totally devoid of regularity: let us say that a strict impact time is an impact time for which the left limit of the velocity does not vanish. It is possible to calculate the differential of the mapping

$(u_0, u_1) \mapsto \Phi(t_1 - 0, t_0, u_0, u_1)$, provided that the interval (t_0, t_1) contains only a finite number of impact times, and that they are strict impact times.

When there is exactly one impact time, the differential is computed explicitly, and it is given by a 2×2 matrix; the most important term in this matrix comes from the differentiation of the impact time with respect to the initial data, which is possible only if the impact time is strict.

Thus, the Lyapunov exponents could be well defined though none of the conditions of Oseledec's theorem are satisfied; but their existence and their computation is a purely experimental matter. In what follows, it should be always understood that any object we consider should be complemented by the phrase "if it exists". But we shall not repeat it systematically.

In our case, the phase space is two dimensional, which leads to the computation of two Lyapunov exponents.

The larger one, λ_1, measures the rate of evolution of a one dimensional infinitesimal element of the phase space. It can be given by the evolution of the distance between the reference trajectory and a neighbouring one.

The sum of the smaller one, λ_2, and the larger one measures the rate of evolution of infinitesimal volume elements in the phase space. If $DS(x)$ had real distinct eigenvectors v_1 and v_2 at some point x, one could understand the definition of the second Lyapunov exponent in a more geometrical fashion: the parallelogram built on v_1 and v_2 is sent by the tangent mapping at x to S into another parallelogram built on $D_x S v_1 = \exp(\lambda_1 t)$ and $D_x S v_2 = \exp(\lambda_2 t) v_2$. Then the area of the parallelogram is multiplied by $\exp((\lambda_1 + \lambda_2)t)$.

In particular, if $\lambda_1 > 0$ and $\lambda_1 + \lambda_2 < 0$, the areas are contracted, while the distances can be expanded: this is precisely a situation which can lead to chaotic behavior.

For a dissipative dynamical systems the areas are contracted, as a rule.

3 Differentiation of the continuous and discrete flows

3.1 The continuous flow

In this section, we calculate the differential of the mapping $x_0 \mapsto \Phi(t_1, t_0, x_0)$ when there is exactly one impact time, in the interval (t_0, t_1), and the impact is strict, i.e. $\Phi_2(t_c - 0, t_0, x) < 0$. We denoted $x_0 = (u_0, v_0)$, the initial condition in the phase space.

To do this, we express the flow on such an interval as the composition of the flow from time t_0 to time t_c, the time of the impact, of a reflexion and of the flow from time t_c to time t_1. Since the reflexion law is given by

$$\dot{u}(t_c + 0) = -e\dot{u}(t_c - 0). \tag{13}$$

the phase after the impact is related to the phase before the impact by a linear transformation whose matrix is

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -e \end{pmatrix} \tag{14}$$

It is convenient to denote

$$\Phi(t_c - 0, t_0, x_0) = x_c. \tag{15}$$

Therefore, we may write now

$$\Phi(t_1, t_0, x_0) = \Phi(t_1, t_c + 0, Rx_c). \tag{16}$$

Theorem 3.1 *Assume that the impact at t_c is strict. Denote $\phi(t) = f(t, u(t), \dot{u}(t))$. Then the differential of $x \mapsto \Phi(t_1, t_0, x)$ is equal to*

$$D_1\Phi(t_1, t_c - 0, x_c) \circ \tilde{R} \circ D_1\Phi(t_c - 0, t_0, x_0), \tag{17}$$

where

$$\tilde{R} = \begin{pmatrix} -e & 0 \\ \dfrac{\phi(t_c + 0) + e\phi(t_c - 0)}{\dot{u}(t_c - 0)} & -e \end{pmatrix} \tag{18}$$

Proof. We differentiate the identity (16) with respect to x_0 and we obtain

$$D_3\Phi(t_1, t_0, x_0) = D_2\Phi(t_1, t_c + 0, Rx_c)Dt_c + D_3\Phi(t_1, t_c + 0, Rx_c)R.W, \tag{19}$$

with

$$W = \{D_1\Phi(t_c - 0, t_0, x_0)Dt_c + D_3\Phi(t_c - 0, t_0, x_0)\}$$

We calculate the different quantities which appear in (19): thanks to (15), t_c satisfies

$$\Phi_1(t_c - 0, t_0, x_0) = u_{\min},$$

which we differentiate with respect to x_0; as the impact is strict, we can see that

$$Dt_c(x_0) = -\frac{D_3\Phi_1(t_c - 0, t_0, x_0)}{\dot{u}(t_c - 0)}.$$

Let us calculate now $D_2\Phi$: we differentiate the differential equation and the initial conditions satisfied by $\Phi_1(t, s, x)$

$$(D_1^2 + 2\alpha D_1 + 1)\Phi_1(t, s, x) = f(t), \quad \Phi_1(s, s, x) = u, \quad D_1\Phi_1(s, s, x) = v,$$

with respect to the second argument, and we obtain

$$(D_1^2 + 2\alpha D_1 + 1)D_2\Phi_1(t, s, x) = 0, \quad D_2\Phi_1(s, s, x) = -v, \quad D_1D_2\Phi_1(s, s, x) = -f(s, u, v).$$

Therefore, if we observe that on intervals without impact the differential of Φ with respect to the third argument is simply the fundamental solution of the corresponding constant coefficient homogeneous linear system, we can see that

$$D_2\Phi(t_1, t_c+0, Rx_c) = D_3\Phi(t_1, t_c+0, Rx_c) \begin{pmatrix} -\dot{u}(t_c+0) \\ -\phi(t_c+0) \end{pmatrix}.$$

On the other hand

$$D_1\Phi(t_c-0, t_0, x_0) = \begin{pmatrix} \dot{u}(t_c-0) \\ \phi(t_c-0) \end{pmatrix}.$$

Moreover, we apply the matrix identity

$$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} c & d \\ \hat{c} & \hat{d} \end{pmatrix}$$

to transform the expressions containing $D_3\Phi_1$ into expressions containing $D_3\Phi$. Therefore we may write (6.7) as

$$D_3\Phi(t_1, t_0, x_0) = D_3\Phi(t_1, t_c+0, Rx_c) U D_3\Phi(t_c-0, t_0, x_0). \qquad (20)$$

with

$$U = R + R\begin{pmatrix} \dot{u}(t_c-0) & 0 \\ \phi(t_c-0) & 0 \end{pmatrix} \frac{1}{\dot{u}(t_c-0)} + R\begin{pmatrix} \dot{u}(t_c+0) & 0 \\ \phi(t_c+0) & 0 \end{pmatrix} \frac{1}{\dot{u}(t_c-0)} \qquad (21)$$

A direct computation shows that the quantity in brackets is equal to

$$\begin{pmatrix} -e & 0 \\ \dfrac{\phi(t_c+0) + e\phi(t_c-0)}{\dot{u}(t_c-0)} & -e \end{pmatrix}$$

which is precisely our claim.

3.2 The discrete flow

The discrete problem corresponding to (1) - (4) is defined by (7), (8) and (9). Let the discrete velocity be

$$V^n = \frac{U^{n+1} - U^n}{h}. \qquad (22)$$

The initial data for (6) and (7) are U^0 and

$$V^0 = \frac{U^1 - U^0}{h}.$$

The perturbed solution $\tilde{U}^n$ corresponds to perturbed initial conditions $\tilde{U}^0$ and $\tilde{V}^0$.

Let $\mathbb{N}$ (resp $\tilde{\mathbb{N}}$) be the set of integers n such that the constraint is active on U^n (respectively on $\tilde{U}^n$), i.e.

$$\mathbb{N} = \{n \geq 1 : 2U^n + (e-1)U^{n-1} + h^2 F^n \leq (1+e)u_{\min}\},$$

and

$$\tilde{\mathbb{N}} = \{n \geq 1 : 2\tilde{U}^n + (e-1)\tilde{U}^{n-1} + h^2 \tilde{F}^n \leq (1+e)u_{\min}\}.$$

Of course, for $n \in \mathbb{N}$, $U^{n+1} + eU^{n-1} = (1+e)u_{\min}$.

Lemma 3.2 *For all integer M such that for all $n \in \{1, \dots M\}$, $2U^n + (e-1)U^{n-1} + h^2 F^n$ is not equal to $(1+e)u_{\min}$, there exists a ρ such that for $|\tilde{U}^0 - U^0| + |\tilde{V}^0 - V^0| \leq \rho$, then $\tilde{\mathbb{N}} \cap \{0, \dots, M\}$ coincides with $\mathbb{N} \cap \{0, \dots, M\}$.*

Proof. The mapping which assigns to (u, v) the number w defined by

$$w + eu = P_{(1+e)K}(2v + (e-1)u + h^2 f(t, v, (w-u)/2h)) \tag{23}$$

is continuous; indeed, P_K is a contraction, and $(u, v) \mapsto h^2 f(t, v, (w-v)/h)$ is Lipschitz continuous with a Lipschitz ratio estimated by Lh, where L is the Lipschitz constant of f with respect to its last two arguments. Therefore, the principle of strict contractions implies that for $Lh < 1$, there exists a unique solution w of (23). If u' and v' are different data, we subtract (23) from

$$w' + eu' = P_{(1+e)K}(2v' + (e-1)u' + h^2 f(t, v', (w'-u')/2h)),$$

and we obtain

$$(1 - Lh)|w' - w| \leq (1 + hL)|u' - u| + (2 + h^2 L)|v - v'|,$$

which implies immediately our claim about continuity.

Therefore, for all $n \geq 1$, $\tilde{U}^n$ tends to U^n as $|\tilde{U}^0 - U^0| + |\tilde{V}^0 - V^0|$ tends to 0. If $2U^n + (e-1)U^{n-1} + h^2 F^n$ is not equal to $(1+e)u_{\min}$, then for ε small enough, $2\tilde{U}^n + (e-1)\tilde{U}^{n-1} + h^2 F^n$ is not equal to $(1+e)u_{\min}$, and therefore the constraints are saturated for U^n if they are saturated for $\tilde{U}^n$.

Let us define the discrete flow Φ^h as the mapping which assigns to the discrete times t^0 and $t^n = t(0 + nh)$ the solution $X^n = (U^n, V^n)^T$ of the numerical scheme at time t^n which satisfied the initial data $X^0 = (U^0, V^0)^T$ at the initial time t^0.

It should be remarked here that the phase space for the discrete flow has not been studied in the same detail as the phase space for the flow in continuous time. Our present understanding is that this phase space is $\mathbb{R}^2$.

Theorem 3.3 *Consider a discrete time interval $\{t^k, \ldots, t^l\}$ on which the solution U^n of the numerical scheme satisfies*

$$n \neq p, p+1 \quad \text{and} \quad t^k \leq t^n \leq t^l \Longrightarrow 2U^n + (e-1)U^{n-1} + h^2F^n > (1+e)u_{\min}, \tag{24}$$

and

$$n = p \text{ or } n = p+1 \Longrightarrow 2U^n + (e-1)U^{n-1} + h^2F^n < (1+e)u_{\min}. \tag{25}$$

The derivative of the flow with respect to its spatial argument is given by

$$D_3\Phi^h(t^l, t^k, X^k) = D_3\Phi^h(t^l, t^{p+1}, X^{p+1}) \circ R^h \circ D_3\Phi^h(t^{p-1}, t^k, X^k), \tag{26}$$

where

$$\hat{R} = \begin{pmatrix} -e & 0 \\ 0 & -e \end{pmatrix} \tag{27}$$

Proof. We can decompose the discrete flow on this interval as

$$X^l = \Phi^h(t^l, t^{p+1}, X^{p+1}),$$

and

$$X^{p-1} = \Phi^h(t^{p-1}, t^k, X^k).$$

Then, the differentiation of X^l with respect to X^k is performed by composition of differentiations. The assumption on the saturation of constraints at discrete times p and $p+1$ implies

$$U^{p+1} + eU^{p-1} = (1+e)u_{\min}, \quad V^{p+1} + eV^{p-1} = 0,$$

so that the differential of (U^{p+1}, V^{p+1}) with respect to (U^{p-1}, V^{p-1}) has matrix

$$\hat{R} = \begin{pmatrix} -e & 0 \\ 0 & -e \end{pmatrix}.$$

Then, by composition we obtain

$$D_3\Phi^h(t^l, t^k, X^k) = D_3\Phi^h(t^l, t^{p+1}, X^{p+1}) \circ \hat{R} \circ D_3\Phi^h(t^{p-1}, t^k, X^k).$$

4 The three computational methods

In this section, we compute numerically Lyapunov exponents for the problem (1)–(6) using three different methods. The first two methods are based on the impact detection scheme. The third method is based on the scheme (7)-(9).

We have to calculate $\ln(|D(S^k)(x)v_0|)/k$. In order to obtain a reliable computation, we have to normalize at each step the iterates; otherwise, the norm of the vectors $DS^k(x)v_0$ increases exponentially with k, and the computation leads to an overflow. This situation is reminiscent of the power method for the computation of eigenvalues of matrices ([8], [3]), and the method employed is essentially the same: let x_j be the sequence

$$x_0 = x, \quad x_j = S(x_{j-1}).$$

Then, by composition of differentials,

$$D(S^k)(x) = DS(x_{k-1}) \cdots DS(x_0).$$

Define a sequence of vectors

$$\hat{v}_j = DS(x_{j-1})v_{j-1}, \quad v_j = \frac{\hat{v}_j}{\|\hat{v}_j\|}.$$

Then we have the identitites

$$DS(x_0) = \hat{v}_1, \quad DS(x_1)DS(x_0)v_0 = \hat{v}_2\|\hat{v}_1\|,$$

and by an immediate induction

$$DS(x_{k-1}) \cdots DS(x_0) = \hat{v}_k \prod_{j=1}^{k-1} \|\hat{v}_j\|.$$

Thus, we will compute

$$\lambda_{1,k} = \frac{1}{k} \sum_{j=1}^{k} \ln \|\hat{v}_j\|, \tag{28}$$

as an approximation to the true Lyapunov exponent, if it exists.

Similarly, we have to calculate the second Lyapunov exponent: once again, the comparison with the methods used for the computation of matrix eigenvalues is illuminating: it is enough to compute $\ln|\det D(S^k)(x)|$, and to subtract from this number $\lambda_{1,k}$. Computationnally, this procedure would be highly unreliable, since the image of a basis by $D(S^k)(x)$ becomes extremely singular as k tends to infinity; thus, we modify the computation as follows:

$$\det D(S^k)(x) = \prod_{j=1}^{k} \det DS(x_{j-1}),$$

and we calculate the determinant of $DS(x_{j-1})$ with the help of a second sequence of vectors w_j defined by

$$\hat{w}_j = DS(x_{j-1})w_{j-1}, \quad w_j = \frac{\hat{w}_j - (\hat{w}_j \cdot v_j)v_j}{\|\hat{w}_j - (\hat{w}_j \cdot v_j)v_j\|}.$$

In other words, the basis $\{v_{j-1}, w_{j-1}\}$ is transformed into $\{\hat{v}_j, \hat{w}_j\}$ by the linearization of the Poincaré map S, and then submitted to Gram-Schmidt orthogonalization. Then

$$|\det DS(x_{j-1})| = \|\hat{v}_j\| \|\hat{w}_j - (\hat{w}_j \cdot v_j)v_j\|.$$

Therefore, we define

$$\lambda_{2,k} = \frac{1}{k} \sum_{j=1}^{k} \ln \|\hat{w}_j - (\hat{w}_j \cdot v_j)v_j\|, \tag{29}$$

to be the numerical approximation of the second Lyapunov exponent.

The numerical differentiation of the continuous time and the discrete time flows is implemented in a very simple fashion: let ε be a small positive number, and let

$$\delta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Given X_0 and t_0, we define the numerical differential $D_\varepsilon S(X_0)$ of the Poincaré map S as the matrix given by its column vectors

$$D_\varepsilon S(X_0) = \frac{1}{\varepsilon} \begin{pmatrix} \Phi(t_1, t_0, X_0 + \varepsilon\delta_1) - \Phi(t_1, t_0, X_0) & \Phi(t_1, t_0, X_0 + \varepsilon\delta_2) - \Phi(t_1, t_0, X_0) \end{pmatrix}.$$

The same definition applies for the discrete time flow:

$$D_\varepsilon S^h(X_0) = \frac{1}{\varepsilon} \begin{pmatrix} \Phi^h(t_1, t_0, X_0 + \varepsilon\delta_1) - \Phi^h(t_1, t_0, X_0) & \Phi^h(t_1, t_0, X_0) + \varepsilon\delta_2) - \Phi^h(t_1, t_0, X_0) \end{pmatrix}.$$

The three numerical schemes will be denoted

- IDED **I**mpact **D**etection, **E**xact **D**ifferentiation,
- IDND **I**mpact **D**etection, **N**umerical **D**ifferentiation,
- NSND **N**umerical **S**cheme, **N**umerical **D**ifferentiation.

In IDED, we use compute an approximation of DS to machine precision of DS, which is given by formula (6.6) when $t_1 = t_0 + T$, and there is only one impact in the time interval $(t_0, t_0 + T)$. In IDND, we replace DS by $D_\varepsilon S$, and in NSND, we use $D_\varepsilon S^h$. We did not try to compute exactly the derivative of S^h: it would have required the integration of a linear difference equation which admits an explicit solution because of its very simple nature. However, we claim that for ε small with respect to h, the results of our computation must be bad; therefore, it would be of little interest to let ε tend to 0: this is the reason why we did not perform the relevant calculation.

5 Numerical results

All our computations were performed in double precision.

All the results that are described in this section are obtained with $e = 0.5$ and the initial data $u_0 = 0$ and $v_0 = 0.1$. We start with a comparison of IDED and IDND: we performed the computation for 3000 periods, a time step h used as a sweeping parameter of $h = T/2513$, and $\varepsilon = 10^{-8}$ in IDND; of course T is the period $T = 2\pi/50$. By convention, $\lambda_{j,k}$ is the numerical approximation of λ_j obtained by a computation on k periods. The IDND give this results

$$\lambda_{1,3000} = 3.05694 \text{ and } \lambda_{2,3000} = -15.06065,$$

and the IDED give the results

$$\lambda_{1,3000} = 3.05695 \text{ and } \lambda_{2,3000} = -15.06066$$

The results obtained by respectively by IDED and IDND have 5 common significative digits at least.

We observe that the first Lyapunov exponent is positive, and the second is negative, and their sum is negative: this is in agreement with the chaotic character of the dissipative system we considered. Another computational observation is that IDED is much faster than IDND.

If we use values of ε which are very small with respect to h in the NSND scheme, the sum of the two Lyapunov exponents is positive; for instance, when $h = T/2513$ and $\varepsilon = 10^{-8}$, we obtained

$$\lambda_{1,3000} = 65.11, \quad \lambda_{2,3000} = -5.55$$

This fact led us to compute the Lyapunov exponents with ε somewhat larger than h: with the choice $\varepsilon = 3h$, $h = T/40000$, the NSND method gave reasonable results

6 Conclusion.

The comparison of the different methods used for the computation of the Lyapunov exponents has showed a number of discrepancies and difficulties of numerical origin. It could be argued that NSND gives bad results because it is of low order; however, IDED does not perform too well. Thus, we observed that strictly numerical and apparently harmless parameters can seriously alter the result of a computation.

Therefore, the conclusion is a caveat: any computations of a Lyapunov exponent should be subjected to a serious evaluation performed by applying several different numerical methods, and by appraising the effect of all the numerical parameters, including the ones which seem innocuous.

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