An EOQ Model for Deteriorating Items with Weibull Distribution Deterioration, Unit Production Cost with Quadratic Demand and Shortages

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Abstract

This article contains order level inventory models for deteriorating items with quadratic demand. The finite production rate is proportional to the demand rate and the deterioration is time proportional. The unit production cost is inversely proportional to the demand rate. We have investigated inventory-production system where the deteriorating items follow two parameters Weibull deterioration. The objective of the model is to develop an optimal policy that minimizes the total average cost. Numerical examples are used to illustrate the two developed models. Sensitivity analysis of the optimal solution with respect to major parameters is carried out.

1 Introduction

An important problem confronting a supply manager in any modern organization is the control and maintenance of inventories of deteriorating items. Deterioration is defined as change, damage, decay, spoilage, obsolescence, pilferage, loss of utility or loss of marginal value of goods that results in decrease of the usefulness of the original one. Maximum items deteriorate over time. The rate of deterioration is very small in some items like hardware, glassware, toys and steel that there is little need for considering deterioration in the determination of the economic lot size. Some items such as fish, medicine, vegetables, blood, gasoline, alcohol, radioactive chemicals and food grains like wheat, potato, onion etc. have finite shelf life and deteriorate rapidly over time. The effect of deterioration of physical goods can not be disregarded in many inventory systems.

Wagner and Whitin [8] considered an inventory model for fashionable products deteriorating at the end of a prescribed storage period. Ghare and Schrader [20] developed an economic order quantity (EOQ) model with an exponential decaying inventory in modified form. This model was extended by Covert and Philip [22] by considering Weibull distribution deterioration. A complete note on inventory literature for deteriorating inventory models was given by Goyal and Giri [25] and Raafat [5]. The constant rate of deterioration is assumed in most researches for deteriorating inventory. But, the Weibull deterioration is used to show the product in stock deteriorates with time. Wee [9] developed a deterministic inventory model with quantity discount, pricing and partial backordering when the product in stock deteriorates with times. Misra [21] adopted a two-parameter Weibull distribution deterioration to develop an inventory model with finite rate of replenishment. Fitting empirical data in mathematical distribution shows the way to many researchers to use the Weibull distribution to model the deterioration rate. The items in which the deterioration rate follows the Weibull distribution are roasted ground coffee, corn seed, frozen foods, pasteurized milk, refrigerated meats and ice creams. While discussing the fitting empirical data to mathematical distribution, Berrotoni [13] noticed that the leakage failure of dry batteries and life expectancy of ethical drugs could be expressed in terms of Weibull distribution. In both the cases, the deterioration rate increased with age or the longer the items remained unused and the failure rate was high. Beside these items, reservoir systems are subject to deteriorate in the form of evaporation. Papachristos and Skouri [28] reconsidered the work of Wee [9] and assumed a model where the demand rate is a convex decreasing function of the selling price and the backlogging rate is a time-dependent function. Philip[6] developed a generalized EOQ model with a three-parameter Weibull distribution to represent the time of deterioration. Some researchers (Wu and Lee[14]; Mondal

et al. [2]; Chen and Lin [12]; Ghosh and Chaudhuri[23]; Mahapatra and Maiti[19]) extended the models for deterioration which follows Weibull distribution.

In recent years, many researchers have given considerable attention towards the situation where the demand rate is dependent on the level of the on-hand inventory. The assumption of constant demand rate is not always applicable to many inventory items such as fashionable clothes, electronic equipments, tasty foods etc. as they are fluctuated in the demand rate. Donaldson [32] derived an analytical solution to problems for obtaining the optimal number of replenishments and the optimal replenishment times of an EOQ model with a linearly time-dependent demand pattern over a finite time horizon. Demand of a product may vary with time or price or ever with the instantaneous level of inventory displayed in a retail shop. With the progress of time, researchers developed inventory models with deteriorating items and time-dependent demand rates. In this area, the work done by various authors (Silver[3],Ritchie[4],Deb and Chaudhuri[17], Goel and Aggarwal[31], Hargia and Benkherouf[16] and Jalan et al.[1]). Goyal et al.[26] suggested a new replenishment policy in which shortages are permitted in every cycle. Inventory models for deteriorating items with linearly trended demand and no-shortage were considered by Dave and Patel [30], Bahari-Kashani [7], Chung and Ting [15] etc. Demands for spare parts of new aeroplanes, computer chips of advanced computer machines etc. increased widely while the demands for spare parts of the obsolete aircrafts, computers chips etc. decreased very rapidly with time. Some researchers represented this type of demand as an exponentially increasing/decreasing function of time. An exponential rate of change is very high and indeed in doubt that whether the real market demand of any product, it undergoes a rate of change, which is very high as an exponential rate. So the case of quadratic demand is considered. The quadratic demand technique is applied to control the problem in order to determine the optimal production policy. Deb and Chaudhuri [18] introduced the concept of inventory shortage to the model of Donaldson [32]. They developed a heuristic along the lines of Silver[3] and found the conclusion that the calculation of the shortage cost was erroneous. This error was corrected by Goyal[24], Murdeshwar[29] and they tried to modify the model of Donaldson[32] by considering shortages. Goyal [24], Murdeshwar [29] also used the incorrect cost-expression, which was derived by Deb and Chaudhuri [18]. An order level inventory model for deteriorating items with time-dependent deterioration rate, unit production cost and shortages developed by Manna and Chaudhuri[27]. They considered a linear trend in demand and assumed that the finite production rate is proportional to the time-dependent demand rate and the deterioration is time proportional. Wee and Law[10,11] developed a deterministic inventory model for deteriorating items with price-dependent demand rate, finite production rate and time varying deterioration rate taking into account the time value of money over a fixed time horizon

In this paper, we have discussed an economic order quantity model with the following considerations:
a) The deterministic demand rate is time dependent with quadratic demand.
b) The unit production cost is inversely proportional to the time dependent demand rate.

c) The production rate is finite.
d) Deterioration rate is time proportional which is two parameters Weibull distribution deterioration.

The paper constitutes of two models. The model allowing no shortage is solved in the first part of this paper. Further, the case of inventory shortage is discussed. Numerical examples are used to illustrate the solution procedure and the developed models. Sensitivity analysis is carried out to identify the most sensitive parameters in the system.

2 Model-I: deterministic model without shortage

A deterministic order-level inventory model with a finite rate of replenishment is developed with the following assumptions and notations:

a) The demand rate is assumed to be $R = f(t) = a + bt + ct^2$ at any time $t \geq 0, a > 0, b > 0, c > 0$. Here a is the initial rate of demand, b is the rate with which the demand rate increases and c is the rate of change at which the demand rate itself increases.

b) The production rate, say $K = rf(t)$, where $r > 1$.

c) A variable fraction $\theta(t)$ of the on-hand inventory deteriorates per unit of time where $\theta(t) = \alpha\beta t^{\beta-1}, 0 < \alpha << 1$ and $\beta \geq 1, t \geq 0$.

d) The lead-time is zero.

e) c_1 is the constant holding cost per unit item per unit of time.

f) c_2 is the shortage cost which is infinite, i.e. shortages in inventory are not allowed.

g) c_3 is the constant deterioration cost per unit per unit of time.

h) C is the total average cost for the production cycle and S is the stock level reached in the cycle.

The unit production cost v is inversely related to the demand rate as $v = \alpha_1 R^{-\gamma}$, where $\alpha_1 > 0, \gamma > 0$ and $\gamma = 1, \gamma \neq 2$. α_1 is positive as v and R are both non-negative; also higher demands result in lower unit costs of production. Therefore, v and R are inversely related and γ must be positive. So $\frac{dv}{dR} = -\alpha_1\gamma R^{-(\gamma+1)} < 0$ and $\frac{d^2v}{dR^2} = \alpha_1\gamma(\gamma+1)R^{-(\gamma+2)} > 0$. Therefore marginal unit cost of production is an increasing function of R. Thus these results imply that, as the demand rate increases at an increasing rate, the unit cost of production decreases. For this reason, the manufacturer is encouraged to produce more as the demand for the item increases. The necessity of the restriction $\gamma = 1$, $\gamma \neq 2$ arises from the nature of solution of problem.

The stock level is initially zero. Production begins just after $t = 0$, continues up to $t = t_1$ and stops as soon as the stock level becomes S. Then the inventory level decreases both due to demand and deterioration, till it becomes zero at $t = t_2$. Then the cycle repeats itself. The intensity of deterioration is very low

initially but it increases with time. However, it remains bounded for $t \gg 1$ since $0 < \alpha \ll 1$. Let $Q(t)$ be the inventory level of the system at any time $t(0 \le t \le t_2)$. The instantaneous state of the inventory level $Q(t)$ in the interval $[0, t_2]$ is governed by the differential equations,

$$\frac{dQ(t)}{dt} + \theta(t)Q(t) = K - f(t), \quad 0 \le t \le t_1 \tag{1}$$

$$\frac{dQ(t)}{dt} + \theta(t)Q(t) = -f(t), \quad t_1 \le t \le t_2 \tag{2}$$

where $\theta(t) = \alpha\beta t^{\beta-1}$ and $f(t) = a + bt + ct^2$. Using the values of $\theta(t)$ and $f(t)$, equation (1) and (2) become respectively

$$\frac{dQ(t)}{dt} + \alpha\beta t^{\beta-1}Q(t) = (r-1)(a + bt + ct^2), \quad 0 \le t \le t_1 \tag{3}$$

with the conditions $Q(0) = 0$ and $Q(t_1) = S$ and

$$\frac{dQ(t)}{dt} + \alpha\beta t^{\beta-1}Q(t) = -(a + bt + ct^2), \quad t_1 \le t \le t_2 \tag{4}$$

with the conditions $Q(t_1) = S$ and $Q(t_2) = 0$. The solution of (3) using the condition $Q(0) = 0$ is

$$Q(t) = (r-1)\left[\begin{array}{l} a\left(t - \alpha t^{1+\beta} + \dfrac{\alpha t^{1+\beta}}{1+\beta}\right) + b\left(\dfrac{t^2}{2} - \dfrac{\alpha t^{2+\beta}}{2} + \dfrac{\alpha t^{2+\beta}}{2+\beta}\right) \\[3mm] + c\left(\dfrac{t^3}{3} - \dfrac{\alpha t^{3+\beta}}{3} + \dfrac{\alpha t^{3+\beta}}{3+\beta}\right) \end{array}\right], \quad 0 \le t \le t_1 \tag{5}$$

neglecting powers of α higher than 1. This approximation is followed throughout the subsequent calculations. The solution of (4) using the condition $Q(t_1) = S$ is

$$Q(t) = S\left[1 + \alpha\left(t_1^{\beta} - t^{\beta}\right)\right] + a\left[(t_1 - t) + \frac{\alpha}{1+\beta}\left(t_1^{1+\beta} - t^{1+\beta}\right) + \alpha\left(t^{1+\beta} - t_1 t^{\beta}\right)\right]$$

$$+ b\left[\left(\frac{t_1^2}{2} - \frac{t^2}{2}\right) + \frac{\alpha}{2+\beta}\left(t_1^{2+\beta} - t^{2+\beta}\right) + \frac{\alpha}{2}\left(t^{2+\beta} - t_1^2 t^{\beta}\right)\right]$$

$$+ c\left[\left(\frac{t_1^3}{3} - \frac{t^3}{3}\right) + \frac{\alpha}{3+\beta}\left(t_1^{3+\beta} - t^{3+\beta}\right) + \frac{\alpha}{3}\left(t^{3+\beta} - t_1^3 t^{\beta}\right)\right] \quad t_1 \le t \le t_2 \quad . \tag{6}$$

As $Q(t_2) = 0$, equation (6), yields

$$S\left[1 + \alpha\left(t_1^{\beta} - t_2^{\beta}\right)\right] = a\left[(t_2 - t_1) + \frac{\alpha}{1+\beta}\left(t_2^{1+\beta} - t_1^{1+\beta}\right) + \alpha\left(t_1 t_2^{\beta} - t_2^{1+\beta}\right)\right]$$

$$+ b\left[\left(\frac{t_2^2}{2} - \frac{t_1^2}{2}\right) + \frac{\alpha}{2+\beta}\left(t_1^{2+\beta} - t_1^{2+\beta}\right) + \frac{\alpha}{2}\left(t_1^2 t_2^{\beta} - t_2^{2+\beta}\right)\right]$$

$$+ c\left[\left(\frac{t_2^3}{3} - \frac{t_1^3}{3}\right) + \frac{\alpha}{3+\beta}\left(t_2^{3+\beta} - t_1^{3+\beta}\right) + \frac{\alpha}{3}\left(t_1^3 t_2^{\beta} - t_2^{3+\beta}\right)\right]$$

For a first-order approximation over α, this relation gives

$$S = a\left[(t_2 - t_1) + \frac{\alpha}{1+\beta}\left(t_2^{1+\beta} - t_1^{1+\beta}\right) - \alpha\left(t_2 t_1^{\beta} - t_1^{1+\beta}\right)\right]$$

$$+ b\left[\left(\frac{t_2^{2}}{2} - \frac{t_1^{2}}{2}\right) + \frac{\alpha}{2+\beta}\left(t_2^{2+\beta} - t_1^{2+\beta}\right) - \frac{\alpha}{2}\left(t_2^{2} t_1^{\beta} - t_1^{2+\beta}\right)\right]$$

$$. \qquad + c\left[\left(\frac{t_2^{3}}{3} - \frac{t_1^{3}}{3}\right) + \frac{\alpha}{3+\beta}\left(t_2^{3+\beta} - t_1^{3+\beta}\right) - \frac{\alpha}{3}\left(t_2^{3} t_1^{\beta} - t_1^{3+\beta}\right)\right].$$

Therefore,

$$Q(t) = \begin{cases} (r-1)\left[\begin{array}{l} a\left(t - \alpha t^{1+\beta} + \dfrac{\alpha t^{1+\beta}}{1+\beta}\right) + b\left(\dfrac{t^2}{2} - \dfrac{\alpha t^{2+\beta}}{2} + \dfrac{\alpha t^{2+\beta}}{2+\beta}\right) \\ + c\left(\dfrac{t^3}{3} - \dfrac{\alpha t^{3+\beta}}{3} + \dfrac{\alpha t^{3+\beta}}{3+\beta}\right) \end{array}\right] & \text{if } 0 \le t \le t_1 \\[3em] \begin{array}{l} S\left[1 + \alpha\left(t_1^{\beta} - t^{\beta}\right)\right] + a\left[(t_1 - t) + \dfrac{\alpha}{1+\beta}\left(t_1^{1+\beta} - t^{1+\beta}\right) + \alpha\left(t^{1+\beta} - t_1 t^{\beta}\right)\right] \\[1.5em] b\left[\left(\dfrac{t_1^{2}}{2} - \dfrac{t^2}{2}\right) + \dfrac{\alpha}{2+\beta}\left(t_1^{2+\beta} - t^{2+\beta}\right) + \dfrac{\alpha}{2}\left(t^{2+\beta} - t_1^{2} t^{\beta}\right)\right] \\[1.5em] c\left[\left(\dfrac{t_1^{3}}{3} - \dfrac{t^3}{3}\right) + \dfrac{\alpha}{3+\beta}\left(t_1^{3+\beta} - t^{3+\beta}\right) + \dfrac{\alpha}{3}\left(t^{3+\beta} - t_1^{3} t^{\beta}\right)\right] \end{array} & \text{if } t_1 \le t \le t_2 \end{cases}$$

$$\text{(7)}$$

The total inventory in the cycle is

$$\int_0^{t_1} Q(t)dt + \int_{t_1}^{t_2} Q(t)dt = a\left[\frac{rt_1^{2}}{2} + \frac{t_2^{2}}{2} - t_1 t_2 + \frac{\alpha}{1(1+\beta)}\left(t_2 t_1^{1+\beta} - t_1 t_2^{1+\beta}\right)\right.$$

$$\left. + \frac{\alpha t_1^{2+\beta}}{(2+\beta)} - \frac{\alpha t_1^{2+\beta}}{(1+\beta)(2+\beta)} - \frac{\alpha t_1^{2+\beta} r}{(2+\beta)} + \frac{\alpha t_1^{2+\beta} r}{(1+\beta)(2+\beta)}\right]$$

$$+ b\left[\frac{rt_1^{3}}{6} + \frac{t_2^{3}}{3} - t_1\frac{t_2^{2}}{2} + \frac{\alpha}{2(1+\beta)}\left(t_2^{2} t_1^{1+\beta} - t_2^{3+\beta}\right) + \frac{\alpha\left(t_2^{3+\beta} - t_1 t_2^{2+\beta}\right)}{(2+\beta)} - \frac{\alpha t_2^{3+\beta}}{(2+\beta)(3+\beta)}\right.$$

$$\left. + \frac{\alpha t_2^{3+\beta}}{2(3+\beta)} - \frac{\alpha t_2^{3+\beta} r}{2(3+\beta)} + \frac{\alpha t_1^{3+\beta} r}{(2+\beta)(3+\beta)}\right]$$

$$+ c\left[\frac{rt_1^{4}}{12} + \frac{t_2^{4}}{4} - t_1\frac{t_2^{3}}{3} + \frac{\alpha}{3(1+\beta)}\left(t_2^{3} t_1^{1+\beta} - t_2^{4+\beta}\right) + \frac{\alpha\left(t_2^{4+\beta} - t_1 t_2^{3+\beta}\right)}{(3+\beta)}\right.$$

$$\left. - \frac{\alpha t_2^{4+\beta}}{(3+\beta)(4+\beta)} + \frac{\alpha t_2^{4+\beta}}{3(4+\beta)} - \frac{\alpha t_1^{4+\beta} r}{3(4+\beta)} + \frac{\alpha t_1^{4+\beta}}{(3+\beta)(4+\beta)}\right]$$

for a first-order approximation over α. The total number of deteriorated items $[0, t_2]$ is given by = production in $[0, t_1]$ – demand in $[0, t_2]$

$$= r\int_0^{t_1}\left(a + bt + ct^2\right)dt - \int_0^{t_2}\left(a + bt + ct^2\right)dt = a(rt_1 - t_2) + \frac{b}{2}(rt_1^2 - t_2^2) + \frac{c}{6}(rt_1^3 - t_2^3). \quad (8)$$

Since the production in $[u, u + du]$ is Kdu, the cost of production in $[u, u + du]$ is

$$Kvdu = \frac{\alpha_1 rR}{(a + bu + cu^2)^\gamma}du = \frac{\alpha_1 r}{(a + bu + cu^2)^{\gamma - 1}}du.$$

Hence the production cost in $[0, t_1]$ is

$$\int_0^{t_1}\frac{\alpha_1 r}{(a + bu + cu^2)^{\gamma - 1}}du = \alpha_1 rt_1, \quad \gamma = 1, \gamma \neq 2. \quad (9)$$

Therefore, the total average cost of the system is

$$C = \frac{1}{t_2}\left[ac_1\left(\frac{rt_1^2}{2} + \frac{t_2^2}{2} - t_1 t_2 + \frac{\alpha}{1(1+\beta)}\left(t_2 t_1^{1+\beta} - t_1 t_2^{1+\beta}\right)\right)\right.$$

$$+ \frac{\alpha t_1^{2+\beta}}{1(2+\beta)} - \frac{\alpha t_1^{2+\beta}}{(1+\beta)(2+\beta)} - \frac{\alpha t_1^{2+\beta}r}{(2+\beta)} + \frac{\alpha t_1^{2+\beta}r}{(1+\beta)(2+\beta)}\right)$$

$$+ bc_1\left(\frac{rt_1^3}{6} + \frac{t_2^3}{3} - t_1\frac{t_2^2}{2} + \frac{\alpha}{2(1+\beta)}\left(t_2^2 t_1^{1+\beta} - t_2^{3+\beta}\right) + \frac{\alpha\left(t_2^{3+\beta} - t_1 t_2^{2+\beta}\right)}{(2+\beta)} + \frac{\alpha t_2^{3+\beta}}{2(3+\beta)}\right.$$

$$\left. - \frac{\alpha t_2^{3+\beta}}{(2+\beta)(3+\beta)} - \frac{\alpha t_2^{3+\beta}r}{2(3+\beta)} + \frac{\alpha t_1^{3+\beta}r}{(2+\beta)(3+\beta)}\right)$$

$$+ cc_1\left(\frac{rt_1^4}{12} + \frac{t_2^4}{4} - t_1\frac{t_2^3}{3} + \frac{\alpha}{3(1+\beta)}\left(t_2^3 t_1^{1+\beta} - t_2^{4+\beta}\right) + \frac{\alpha\left(t_2^{4+\beta} - t_1 t_2^{3+\beta}\right)}{(3+\beta)}\right.$$

$$\left. + \frac{\alpha t_2^{4+\beta}}{3(4+\beta)} - \frac{\alpha t_2^{4+\beta}}{(3+\beta)(4+\beta)} - \frac{\alpha t_1^{4+\beta}r}{3(4+\beta)} + \frac{\alpha t_1^{4+\beta}}{(3+\beta)(4+\beta)}\right)$$

$$+ ac_3(rt_1 - t_2) + \frac{b}{2}c_3(rt_1^2 - t_2^2) + \frac{c}{6}c_3(rt_1^3 - t_2^3) + \alpha_1 rt_1 \right] \quad (10)$$

Using calculus, we now minimize C. The optimum values of t_1 and t_2 for the minimum average cost C are the solutions of the equation

$$\frac{\partial C}{\partial t_1} = 0 \quad \text{and} \quad \frac{\partial C}{\partial t_2} = 0 \quad (11)$$

provided that they satisfy the sufficient conditions

$$\frac{\partial^2 C}{\partial t_1^2} > 0, \quad \frac{\partial^2 C}{\partial t_2^2} > 0 \quad \text{and} \quad \frac{\partial^2 C}{\partial t_1^2}\frac{\partial^2 C}{\partial t_2^2} - \left(\frac{\partial^2 C}{\partial t_1 \partial t_2}\right)^2 > 0.$$

Equation (11) can be written as

$$ac_1\left(rt_1 - t_2 + \frac{\alpha}{(1+\beta)}\left((1+\beta)t_1^\beta t_2 - t_2^{1+\beta}\right) + \alpha t_1^{1+\beta} - \alpha t_1^{1+\beta}r + \frac{\alpha t_1^{1+\beta}r}{(1+\beta)}\right)$$

$$+ bc_1\left(\frac{rt_1^2}{2} - \frac{t_2^2}{2} - \frac{\alpha t_2^{2+\beta}}{(2+\beta)} + \frac{\alpha}{2}t_1^\beta t_2^2 - \frac{\alpha t_1^{2+\beta}r}{2} + \frac{\alpha t_1^{2+\beta}r}{(2+\beta)}\right)$$

$$+ cc_1 \left(\frac{rt_1^3}{3} - \frac{t_2^3}{3} - \frac{\alpha t_2^{3+\beta}}{(3+\beta)} + \frac{\alpha}{3} t^{\beta} t_2^3 - \frac{\alpha t_1^{3+\beta} r}{3} + \frac{\alpha t_1^{3+\beta} r}{(3+\beta)} \right)$$

$$+ c_3 \left(ar + brt_1 + \frac{c}{2} rt_1^2 \right) + \alpha_1 r = 0 \tag{12}$$

$$ac_1 \left(t_2 - t_1 + \frac{\alpha}{(1+\beta)} \left(t_1^{1+\beta} - (1+\beta) t_1 t_2^{\beta} \right) - \frac{\alpha}{(1+\beta)} t_2^{1+\beta} \right)$$

$$+ bc_1 \left(t_2^2 - t_1 t_2 + \frac{\alpha}{(2+\beta)} \left((3+\beta) t_2^{2+\beta} - (2+\beta) t_1 t_2^{1+\beta} \right) \right.$$

$$+ \frac{\alpha}{2(1+\beta)} \left(2t_1^{1+\beta} t_2 - (3+\beta) t_2^{2+\beta} \right) - \frac{\alpha t_2^{2+\beta}}{(2+\beta)} + \frac{\alpha t_2^{2+\beta}}{2} \right)$$

$$+ cc_1 \left(t_2^3 - t_1 t_2^2 + \frac{\alpha}{(3+\beta)} \left((4+\beta) t_2^{3+\beta} - (3+\beta) t_1 t_2^{2+\beta} \right) \right.$$

$$+ \frac{\alpha}{3(1+\beta)} \left(3t_1^{1+\beta} t_2 - (4+\beta) t_2^{2+\beta} \right) - \frac{\alpha t_2^{2+\beta}}{(3+\beta)} + \frac{\alpha t_2^{2+\beta}}{3} \right)$$

$$- c_3 \left(a + bt_2 + \frac{c}{2} t_2^2 \right) - C = 0. \tag{13}$$

3 Model-II: Deterministic model with shortage

In this section, we have developed an order-level model for deteriorating items with finite rate of replenishment with the assumptions described in the previous model. Another addition assumption is that shortages in inventory are allowed and backlogged completely. c_2 is the constant shortage cost per unit per unit of time.

At $t = 0$, the stock is zero initially. Production begins at time $t = 0$ and continues up to $t = t_1$ when the stock attains a level S. The production is then stopped at $t = t_1$. Inventory gathered in $[0, t_1]$ after meeting the demands is used in $[t_1, t_2]$. The stock level attains a level zero at time t_2. Again at $t = 0$, the shortage starts and accumulate to the level P at $t = t_3$. The production inventory level starts at $t = t_3$. The running demands as well as the backlog for $[t_2, t_3]$ are satisfied in $[t_3, t_4]$. The inventory level becomes zero at the time $t = t_4$. This decrease in level occurs due to the demand. After time t_4, the repetition of the inventory cycle occur. Our aim is to determine the optimum values of t_1, t_2, t_3, t_4 and C with the assumptions given above. Let $Q(t)$ represent the instantaneous inventory level at any time $t (0 \le t \le t_4)$. The differential equations governing the instantaneous states of $Q(t)$ in the interval $[0, t_4]$ is as follows:

$$\frac{dQ(t)}{dt} + \alpha \beta t^{\beta-1} Q(t) = (r-1)(a + bt + ct^2), \quad 0 \le t \le t_1, \tag{14}$$

with the conditions $Q(0) = 0$ and $Q(t_1) = S$;

$$\frac{dQ(t)}{dt} + \alpha\beta t^{\beta-1}Q(t) = -(a + bt + ct^2), \quad t_1 \leq t \leq t_2, \tag{15}$$

with the conditions $Q(t_1) = S$ and $Q(t_2) = 0$;

$$\frac{dQ(t)}{dt} = -(a + bt + ct^2), \quad t_2 \leq t \leq t_3, \tag{16}$$

with the conditions $Q(t_2) = S$ and $Q(t_3) = -S$;

$$\frac{dQ(t)}{dt} = (r-1)(a + bt + ct^2), \quad t_3 \leq t \leq t_4, \tag{17}$$

with the conditions $Q(t_3) = -P$ and $Q(t_4) = 0$. The solution of (14) and (15) can be found from equation (7) as described in model-I. The solution of (16) and (17) using the conditions $Q(t_2) = 0$ and $Q(t_4) = 0$ respectively are

$$Q(t) = a(t_2 - t) + \frac{1}{2}b(t_2^2 - t^2) + \frac{1}{3}c(t_2^3 - t^3), \quad t_2 \leq t \leq t_3, \tag{18}$$

$$Q(t) = a(r-1)(t - t_4) + \frac{1}{2}b(r-1)(t^2 - t_4^2) + \frac{1}{3}c(r-1)(t^3 - t_4^3), \quad t_3 \leq t \leq t_4, \tag{19}$$

As there is no deteriorated items during the period $[t_2, t_4]$, therefore the total number of deteriorated items in $[0, t_4]$ is the same as given in (8). The total shortage in $[t_2, t_4]$ is given by

$$\int_{t_2}^{t_4}[-Q(t)]dt = \int_{t_2}^{t_3}[-Q(t)]dt + \int_{t_3}^{t_4}[-Q(t)]dt = \frac{1}{2}a[t_2^2 + rt_3^2 + (r-1)t_4^2 - 2t_2t_3 - 2(r-1)t_3t_4]$$

$$+ \frac{1}{6}b[2t_2^3 + rt_3^3 + 2(r-1)t_4^3 - 3t_2^2t_3 - 3(r-1)t_3t_4^2]$$

$$+ \frac{1}{12c}b[3t_2^4 + rt_3^4 + 2(r-1)t_4^4 - 3t_2^3t_3 - 3(r-1)t_3t_4^3].$$

The production cost in $[t_3, t_4]$ is

$$\int_{t_3}^{t_4} Kvdu = \alpha_1 r \int_{t_3}^{t_4} (a + bu + cu^2)^{1-\gamma} du = \alpha_1 r(t_4 - t_3), \quad \gamma = 1, \gamma \neq 2$$

Hence the production cost in $[0, t_4]$ is

$$\alpha_1 r[t_1 + t_4 - t_3], \quad \gamma = 1, \gamma \neq 2$$

The total average cost of the system in $[0, t_4]$ is

$$C = \frac{1}{t_2}\left[ac_1\left(\frac{rt_1^2}{2} + \frac{t_2^2}{2} - t_1t_2 + \frac{\alpha}{1(1+\beta)}(t_2t_1^{1+\beta} - t_1t_2^{1+\beta}) \right.\right.$$

$$+ \frac{\alpha t_1^{2+\beta}}{1(2+\beta)} - \frac{\alpha t_1^{2+\beta}}{(1+\beta)(2+\beta)} - \frac{\alpha t_1^{2+\beta}r}{(2+\beta)} + \frac{\alpha t_1^{2+\beta}r}{(1+\beta)(2+\beta)} \right)$$

$$+ bc_1\left(\frac{rt_1^3}{6} + \frac{t_2^3}{3} - t_1\frac{t_2^2}{2} + \frac{\alpha}{2(1+\beta)}(t_2^2t_1^{1+\beta} - t_2^{3+\beta}) + \frac{\alpha(t_2^{3+\beta} - t_1t_2^{2+\beta})}{(2+\beta)} + \frac{\alpha t_2^{3+\beta}}{2(3+\beta)} \right.$$

$$- \frac{\alpha t_2^{3+\beta}}{(2+\beta)(3+\beta)} - \frac{\alpha t_1^{3+\beta} r}{2(3+\beta)} + \frac{\alpha t_1^{3+\beta} r}{(2+\beta)(3+\beta)} \bigg)$$

$$+ cc_1 \bigg(\frac{rt_1^4}{12} + \frac{t_2^4}{4} - t_1 \frac{t_2^3}{3} + \frac{\alpha}{3(1+\beta)} \big(t_2^3 t_1^{1+\beta} - t_2^{4+\beta} \big) + \frac{\alpha \big(t_2^{4+\beta} - t_1 t_2^{3+\beta} \big)}{(3+\beta)}$$

$$+ \frac{\alpha t_2^{4+\beta}}{3(4+\beta)} - \frac{\alpha t_2^{4+\beta}}{(3+\beta)(4+\beta)} - \frac{\alpha t_1^{4+\beta} r}{3(4+\beta)} + \frac{\alpha t_1^{4+\beta} r}{(3+\beta)(4+\beta)} \bigg)$$

$$+ \frac{a}{2} c_2 \Big[t_2^2 + rt_3^2 + (r-1)t_4^2 - 2t_2 t_3 - 2(r-1)t_3 t_4 \Big]$$

$$+ \frac{b}{6} c_2 \Big[2t_2^3 + rt_3^3 + 2(r-1)t_4^3 - 3t_2^2 t_3 - 3(r-1)t_3 t_4^2 \Big]$$

$$+ \frac{c}{12} c_2 \Big[3t_2^4 + rt_3^4 + 2(r-1)t_4^4 - 3t_2^3 t_3 - 3(r-1)t_3 t_4^3 \Big]$$

$$+ ac_3 (rt_1 - t_2) + \frac{b}{2} c_3 (rt_1^2 - t_2^2) + \frac{c}{6} c_3 (rt_1^3 - t_2^3) + \alpha_1 r \big(t_1 + t_4 - t_3 \big) \bigg], \gamma = 1, \gamma \neq 2 \ (20)$$

The optimum values of t_1, t_2, t_3 and t_4 which minimize the cost function C are the solutions of the equations

$$\frac{\partial C}{\partial t_1} = 0, \frac{\partial C}{\partial t_2} = 0, \frac{\partial C}{\partial t_3} = 0, \frac{\partial C}{\partial t_4} = 0 \tag{21}$$

Provided that these values of $t_i (i = 1,2,3,4)$ satisfy the conditions $D_i > 0$ $(i = 1,2,3,4)$, where D_i is the Hessian determinant of order i is given by

$$D_i = \begin{vmatrix} C_{11} & C_{21} & \ldots\ldots & C_{1i} \\ C_{21} & C_{22} & \ldots\ldots & C_{2i} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ C_{i1} & C_{i2} & \ldots\ldots & C_{ii} \end{vmatrix}, \quad c_{ij} = \frac{\partial^2 C}{\partial t_i \partial t_j} \quad (i, j = 1,2,3,4).$$

The expanded forms of the (21) are

$$ac_1 \bigg(rt_1 - t_2 + \frac{\alpha}{(1+\beta)} \big((1+\beta) t_1^\beta t_2 - t_2^{1+\beta} \big) + \alpha t_1^{1+\beta} - \alpha t_1^{1+\beta} r + \frac{\alpha t_1^{1+\beta} r}{(1+\beta)} \bigg)$$

$$+ bc_1 \bigg(\frac{rt_1^2}{2} - \frac{t_2^2}{2} + \frac{\alpha}{2} t_1^\beta t_2^2 - \frac{\alpha}{(2+\beta)} t_2^{2+\beta} - \frac{\alpha t_2^{2+\beta} r}{2} + \frac{\alpha t_1^{2+\beta} r}{(2+\beta)} \bigg)$$

$$+ cc_1 \bigg(\frac{rt_1^3}{3} - \frac{t_2^3}{3} + \frac{\alpha}{3} t_1^\beta t_2^3 - \frac{\alpha}{(3+\beta)} t_2^{3+\beta} - \frac{\alpha t_2^{23\beta} r}{3} + \frac{\alpha t_1^{2+\beta} r}{(3+\beta)} \bigg)$$

$$+ ac_3 r + b c_3 t_1 r + \frac{cc_3}{2} rt_1^2 + \alpha_1 r = 0, \tag{22}$$

$$ac_1 \bigg(t_2 - t_1 + \frac{\alpha}{(1+\beta)} \big(t_1^{1+\beta} - (1+\beta) t_1 t_2^{1+\beta} \big) + \frac{\alpha t_2^{1+\beta}}{(1+\beta)} \bigg)$$

$$+ bc_1 \bigg(t_2^2 - t_1 t_2 + \frac{\alpha}{2(1+\beta)} \big(2t_1^{1+\beta} t_2 - (3+\beta) t_2^{2+\beta} \big) + \frac{\alpha}{(2+\beta)} \big((3+\beta) t_2^{2+\beta} - (2+\beta) t_1 t_2^{2+\beta} \big) - \frac{\alpha}{(2+\beta)} t_2^{2+\beta}$$

$$+\frac{\alpha}{2}t_2^{2+\beta}\bigg)+cc_1\bigg(t_2^3-t_1t_2+\frac{\alpha}{3(1+\beta)}\big(3t_1^{1+\beta}t_2^2-(4+\beta)t_2^{3+\beta}\big)+\frac{\alpha}{(3+\beta)}\big((4+\beta)t_2^{3+\beta}-(3+\beta)t_1t_2^{2+\beta}\big)$$

$$-\frac{\alpha}{(2+\beta)}t_2^{3+\beta}+\frac{\alpha}{3}t_2^{3+\beta}\bigg)+ac_2(t_2-t_3)+bc_2\big(t_2^2-t_2t_3\big)+cc_2\big(t_2^3-t_2^2t_3\big)$$

$$-ac_3-bc_3t_2-\frac{cc_3}{2}t_2^2=0, \tag{23}$$

$$ac_2\big(-t_2+rt_3-(r-1)t_4\big)+\frac{b}{2}c_2\big(-t_2^2+rt_3^2-(r-1)t_4^2\big)$$

$$+\frac{c}{3}c_2\big(-t_2^3+rt_3^3-(r-1)t_4^3\big)-\alpha_1r=0, \tag{24}$$

$$(r-1)\big[ac_2(t_4-t_3)+bc_2(t_4^2-t_3t_4)+cc_2(t_4^3-t_3t_4^2)\big]+\alpha_1r-C=0 \tag{25}$$

4 Numerical Examples

Example 1: Let $a=25$, $b=20$, $c=15$, $c_1=8$, $c_3=10$, $\alpha=0.01$, $\beta=5.5$, $\alpha_1=35$ and $r=4$, in appropriate units. Solving the non-linear equations (12) and (13) of model-I; we obtain the optimum values of t_1 and t_2 are $t_1^*=2.45584$, $t_2^*=3.34311$. Substituting the values of t_1^* and t_2^* in (10), the optimum average cost is $C^*=1989.87$.

Example 2: Let $a=25, b=20, c=15, c_1=8, c_3=10, c_2=5, \alpha=0.01, \beta=5.5$, $\alpha_1=35$ and $r=4$, in appropriate units. Solving the non-linear equations (22), (23), (24) and (25) of model-II; we obtain the optimum values of t_1, t_2, t_3 and t_4 are $t_1^*=2.49908$, $t_2^*=3.33924$, $t_3^*=4.32765$ and $t_4^*=4.56441$. Substituting the values of t_1^*, t_2^*, t_3^* and t_4^* in (20), the optimum average cost is $C^*=1662.83$.

5 Sensitivity Analysis

We now study the effects of changes in the system parameters a,b,c $\alpha,\beta,r,c_1,c_2,c_3$ and α_1 on the optimal values of t_1,t_2,t_3,t_4 and the minimal optimum cost C^* by the two models. The sensitivity analysis is performed by changing each of the parameter by $+50\%,+25\%,+10\%,-10\%,-25\%$ and -50%, taking one parameter at a time and keeping the remaining parameters unchanged. The results are shown in Table-1 and Table-2. On the basis of the results in Table-1, the following observations are taken into account:
1) With increase in the value of a, b; t_1^*, t_2^* and minimal optimal cost C^* increase, but in the value of c, t_1^* and optimal cost C^* increases with a

decrease in the value of t_2^*.

2) With increase in value of c_1 and c_3; t_1^*, t_2^* and the optimal cost C^* decreases.

3) The changes in the value of t_1^*, t_2^* and C^* more seen when we increase the percentage of α and β. A decrease in optimal cost C^* occurs. So α and β are more sensitive.

4) A slight changes seen in the value of t_1^*, t_2^* and C^* with the parameter α_1. i.e. α_1 is less sensitive.

5) With increase in the value of r, the optimal values of t_1^*, t_2^* and C^* are moderately sensitive.

6) On the basis of the results in Table-2, the following observations are taken into account

7) Increase in the value of a,b; the value of t_1^*, t_2^*, t_3^*, t_4^* and C^* increases. While increase the value of c; t_1^* and optimal cost C^* increase with a decrease in the value of t_2^*, t_3^*, t_4^*.

8) Increase in value of the holding cost c_1 with this Model-I increase the value of t_3^*, t_4^* and C^* and decreases the values of t_1^*, t_2^*. While increase of the shortage cost c_2; the value of t_2^* and the optimal cost C^* are increased and decreases the values of t_1^*, t_3^*, t_4^*. c_2 is more sensitive to this model. The rise of the deterioration cost c_3 increases the values of t_1^*, t_2^*, t_3^*, t_4^* and C^*.

9) With increase in the value of the parameters α and β, decrease in the value of t_1^*, t_2^*, t_3^*, t_4^* and C^*. When β value decreases to 50%, it results in complex roots, which is also seen in Table-1 also. This is a peculiar case arise in these model.

10) A slight increase seen in the value of t_1^*, t_2^*, t_3^*, t_4^* and C^* while change in the parameter α_1 and r i.e. α_1 and r are less sensitive.

Table-1. Without Shortage

Parameter Changing	Change (%)	t_1^*	t_2^*	C^*	Change in C^* (%)
a	+50	2. 33537	3. 36567	2129. 55	+7. 019
	+25	2. 39561	3. 35665	2063. 64	+ 3. 707
	+10	2. 4318	3. 34915	2020. 42	+1. 535
	-10	2. 47976	3. 33615	1957. 8	-1. 611
	-25	2. 51527	3. 32375	1906. 6	-4. 184
	-50	2. 57296	3. 29692	1811. 82	-8. 947
b	+50	2. 4922	3. 35271	2387. 34	+19. 974
	+25	2. 47565	3. 34866	2188. 19	+9. 966
	+10	2. 46421	3. 34555	2069. 08	+3. 9806
	-10	2. 4468	3. 34034	1910. 84	-3. 9716
	-25	2. 43178	3. 33543	1792. 69	-9. 909
	-50	2. 402	3. 32469	1597. 11	-19. 737
c	+50	2. 51159	3. 30276	2363. 7	+18. 786
	+25	2. 48767	3. 3208	2180. 6	+9. 585
	+10	2. 46976	3. 33358	2067. 29	+3. 890
	-10	2. 43993	3. 35361	1910. 59	-3. 984
	-25	2. 41125	3. 37148	1787. 31	-10. 179
	-50	2. 34377	3. 40874	1564. 91	-21. 356
c_1	+50	2. 38364	3. 22591	2417. 02	+21. 466
	+25	2. 41369	3. 27574	2203. 23	+10. 722
	+10	2. 43709	3. 31347	2075. 11	+4. 283
	-10	2. 47789	3. 37738	1904. 85	-4. 272
	-25	2. 51939	3. 44038	1778. 1	-10. 642
	-50	2. 62721	3. 59702	1571. 78	-21. 0109
c_3	+50	2. 54664	3. 4801	2511. 31	+26. 204
	+25	2. 50319	3. 41558	2249. 61	+13. 053
	+10	2. 47529	3. 37319	2093. 55	+5. 2103
	-10	2. 43565	3. 31139	1886. 43	-5. 198
	-25	2. 40386	3. 26027	1731. 66	-12. 976
	-50	2. 3464	3. 16326	1474. 23	-25. 913
α	+50	2. 26766	3. 13947	1768. 56	-11. 121
	+25	2. 35076	3. 22944	1864. 42	-6. 304
	+10	2. 41049	3. 29407	1935. 14	-2. 750
	-10	2. 50679	3. 3982	2052. 43	+3. 143
	-25	2. 59704	3. 49572	2166. 11	+8. 856
	-50	2. 80761	3. 72307	2445. 97	+22. 921
β	+50	1. 71225	2. 42308	1360. 87	-31. 610
	+25	1. 98464	2. 76617	1571. 7	-21. 014
	+10	2. 23098	3. 07044	1781. 38	-10. 477
	-10	2. 75877	3. 70357	2297. 83	+15. 476
	-25	3. 47602	4. 53241	3159. 05	+58. 756
	-50	---	---	---	---
α_1	+50	2. 45808	3. 34765	2041. 28	+25. 835
	+25	2. 45696	3. 34539	2015. 57	+12. 915
	+10	2. 45629	3. 34402	2000. 15	+0. 516
	-10	2. 45539	3. 34220	1979. 58	-0. 517
	-25	2. 45472	3. 34083	1964. 15	-12. 925
	-50	2. 45359	3. 33854	1938. 43	-25. 850
r	+50	2. 5121	3. 45701	3516. 26	+76. 708
	+25	2. 49014	3. 40784	2755. 01	+38. 451
	+10	2. 47168	3. 37157	2296. 31	+15. 400
	-10	2. 43578	3. 30977	1683. 12	-15. 415
	-25	2. 39281	3. 24548	1223. 31	-38. 523
	-50	2. 23101	3. 04377	467. 722	-76. 494

Table-2. With Shortage

Parameter Changing	Change (%)	t_1^*	t_2^*	t_3^*	t_4^*	C^*	Change in C^* (%)
a	+50	2.38294	3.36906	4.37127	4.61425	1777.0	+6.866
	+25	2.44122	3.35675	4.35288	4.5930	1723.16	+3.628
	+10	2.47604	3.34695	4.33867	4.57684	1687.83	+1.503
	-10	2.52194	3.33048	4.31525	4.5505	1636.59	-1.578
	-25	2.55573	3.31511	4.29376	4.52653	1594.65	-4.100
	-50	2.61017	3.2857	4.2489	4.47686	1516.92	-8.774
b	+50	2.5344	3.3448	4.38194	4.63399	1977.97	+18.952
	+25	2.51832	3.34268	4.35664	4.60144	1820.37	+9.474
	+10	2.50721	3.34081	4.33974	4.57981	1725.84	+3.789
	-10	2.49031	3.33738	4.31482	4.54813	1599.86	-3.786
	-25	2.47574	3.3339	4.29405	4.5219	1505.48	-9.462
	-50	2.44688	3.32575	4.25466	4.47255	1348.55	18.900
c	+50	2.5508	3.29568	4.43271	4.49127	1994.18	+19.926
	+25	2.52874	3.31513	4.26251	4.49127	1831.81	+10.162
	+10	2.5121	3.32893	4.29974	4.53306	1731.39	+4.123
	-10	2.48413	3.3506	4.35849	4.59909	1592.7	-4.217
	-25	2.45697	3.36997	4.41141	4.65866	1483.82	+10.765
	-50	2.39214	3.41039	4.52346	4.78524	1288.33	-22.521
c_1	+50	2.43501	3.22168	4.38438	4.65589	1945.34	+16.989
	+25	2.46128	3.27171	4.35068	4.60572	1805.27	+8.566
	+10	2.48215	3.30954	4.3351	4.57942	1720.07	+3.442
	-10	2.5192	3.37356	4.32341	4.55225	1605.34	-3.457
	-25	2.55755	3.43661	4.32561	4.54178	1518.88	-8.656
	-50	2.659234	3.59329	4.37057	4.56298	1376.96	-17.191
c_2	+50	2.4905	3.34033	4.07319	4.259	1738.09	+4.256
	+25	2.49428	3.33987	4.18036	4.38828	1705.68	+2.576
	+10	2.49701	3.33952	4.2625	4.48671	1681.55	+1.125
	-10	2.5014	3.33892	4.40365	4.65467	1641.46	-1.285
	-25	2.50545	3.33833	4.54538	4.82193	1602.89	-3.604
	-50	2.51448	3.33687	4.90924	5.24604	1510.95	-9.133
c_3	+50	2.59075	3.47524	4.59842	4.86693	2063.73	+24.109
	+25	2.54685	3.4112	4.46859	4.72178	1863.69	+12.079
	+10	2.5187	3.36911	4.38553	4.62901	1743.3	+4.839
	-10	2.47874	3.30774	4.26751	4.49733	1582.16	-4.851
	-25	2.44674	3.25693	4.17249	4.39145	1460.65	-12.158
	-50	2.38897	3.16046	3.99830	4.19768	1256.29	-24.448
α	+50	2.3095	3.13746	4.09696	4.32338	1470.79	-11.548
	+25	2.39323	3.22662	4.19898	4.43002	1553.88	-6.552
	+10	2.45341	3.29066	4.27216	4.50647	1615.28	-2.859
	-10	2.5504	3.39381	4.38991	4.6294	1717.25	+3.272
	-25	2.64127	3.4904	4.50004	4.7443	1816.23	+9.225
	-50	2.85323	3.71553	4.7563	5.01138	2060.47	+23.913
β	+50	1.74149	2.42271	3.39935	3.6123	1075.85	-35.300
	+25	2.01936	2.76431	3.74275	3.96502	1273.96	-23.386
	+10	2.27029	3.06743	4.04992	4.27992	1469.51	-11.625
	-10	2.80695	3.69873	4.69818	4.9438	1947.12	+17.096
	-25	3.53457	4.52577	5.5621	5.82819	2738.67	+64.699
	-50	---	---	---	---	---	---
α_1	+50	2.50211	3.3436	4.35326	4.58399	1704.68	+2.516
	+25	2.5006	3.34143	4.34051	4.57424	1683.78	+1.259
	+10	2.49969	3.34012	4.3328	4.56835	1671.22	+0.504
	-10	2.49848	3.33837	4.32247	4.56045	1654.44	-0.504
	-25	2.49757	3.33705	4.31468	4.55448	1641.83	-1.262
	-50	2.49606	3.33484	4.30159	4.54446	1620.76	-2.530
r	+50	2.56113	3.42795	4.88585	5.08576	2782.28	+67.321
	+25	2.53689	3.39149	4.63028	4.84742	2235.22	+34.422
	+10	2.51653	3.36278	4.45574	4.68444	1895.12	+13.969
	-10	2.47705	3.31061	4.18757	4.43241	1425.73	-14.258
	-25	2.43008	3.25263	3.94697	4.20247	1060.49	-36.223
	-50	2.25476	3.05556	3.38924	3.63152	427.899	-74.266

--- indicates the solution are infeasible.

5 Conclusions

In this paper, two deterministic order-level inventory models have been considered. Here the demand rate is of quadratic type. The model consists of time-proportional deterioration rate, which is a two-parameter Weibull deterioration. The production rate is assumed to be finite and proportional to the quadratic demand rate. The unit production cost is inversely related to the demand rate by a factor γ. The paper is accompanied by Model-I and Model-II. Model-I is being solved by allowing no shortages in inventory. Model-II is developed by allowing shortages and backlogged. In both of the models some non-linear algebraic equations are raised and are also solved to minimize the total average cost. Sensitivity analysis is carried out for the two models with numerical examples.

From the numerical results of Model-I and Model-II, we concluded that the optimum average cost in Model-II is 16.435% less than that of Model-I. The quadratic demand technique is applied to control the problem in order to determine the optimal production policy. Quadratic demand seems to be a better representation of time-varying market demands. Some researchers suggest that rapidly increasing demand can be represented by an exponential function of time. The assumption of an exponential rate of change in demand is high and the fluctuation or variation of any commodity in the real market cannot be so high. Thus this accelerated growth in demand rate in the situations like demands of computer chips of computerized machines, spare parts of new aeroplanes etc. is changing the demand more rapidly. Therefore this situation can be best represented by a quadratic function of time. Some researchers consider the demand rate functions in the form of linear demand as $R(t) = a + bt$, $a \geq 0, b \neq 0$ or exponential type demand rate like $R(t) = \alpha e^{\beta t}$, $\alpha > 0, \beta \neq 0$. The linear type demand show steady increase $(b > 0)$ or decrease $(b < 0)$ in the demand rate, which is rarely seen in real market. Also the exponential rate is being very high. i.e. it increases $(\beta > 0)$ or decreases $(\beta < 0)$ exponentially with the demand rate. Therefore the real market demand of any product may rise or fall exponentially. The demand rate function of the form $R(t) = a + bt + ct^2$, $a \geq 0, b \neq 0, c \neq 0$.

$$\frac{dR(t)}{dt} = b + 2ct , \quad \frac{d^2 R(t)}{dt^2} = 2c$$

Then we have the following cases depending on b and c:
a) For $b > 0$ and $c > 0$, the rate of increase of demand rate $R(t)$ is itself an increasing function of time which is termed as accelerated growth in demand.
b) For $b > 0$, $c < 0$, there is retarded growth in demand for all time.
c) For $b < 0$ and $c < 0$, the demand rate $R(t)$ decreases at a decreasing rate which we may call it as accelerated decline demand. This case usually happens to the spare parts of an obsolete aircraft model or microcomputer chip of high technology products substituted by another.
d) For $b < 0$ and $c > 0$, the demand rate falls at an increasing rate for

$$t > \left(-\frac{b}{2c} \right).$$

Thus we may have different types of realistic demand patterns from the functional form $R(t) = a + bt + ct^2$ depending on the signs of b and c. Therefore the quadratic time-dependence of demand is more realistic than its linear or exponential time-dependent demand.

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