Applied Mathematical Sciences, Vol. 4, 2010, no. 1, 13 - 20

Modules of which All Proper Factor Modules are Finitely Cogenerated

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Abstract

A module is almost finitely cogenerated, or 1-critical, if it is not finitely cogenerated but all its proper factors are finitely cogenerated. In this paper, we study almost finitely cogenerated modules over a commutative rings.

Mathematics Subject Classification: 13C05, 16D70

Keywords: Artinian modules, Fractional ideals, Critical modules, Finitely cogenerated modules

1 Introduction

The rings considered are associative with unit, and the modules considered are left unitary modules.

Modules that are not finitely generated but whose proper submodules are finitely generated are called "almost finitely generated" by W. D. Weakly, and were studied in [3] and [9]. This can be dualized as follows. A module M is almost finitely cogenerated (a.f.cog.) if M is not finitely cogenerated, but for any nonzero submodule N of M, M/N is finitely cogenerated. Recall that a module M is finitely cogenerated if for any family $(M_i)_{i\in I}$ of submodules of M with $\bigcap_{i\in I} M_i = 0$, there is a finite subset J of I such that $\bigcap_{j\in J} M_j = 0$.

In fact, it turns out that the a.f.cog. modules are the 1-critical modules. Recall that for an ordinal α, a module M is called α-critical, if $\alpha = K\text{-dim}(M/N)$ (the Krull dimension of M), and $\alpha > K\text{-dim}(M/N)$, for each nonzero submodule N.This paper explores the structure and properties of a.f.cog. modules. In section 2, we establish some properties of a.f.cog. modules and we give some examples in the noncommutative case. In section 3, we study a.f.cog. modules over commutative rings. The main result states that a commutative ring R has a faithful a.f.cog. module iff R is a noetherian integral domain of dimension 1 and when R is as above, an R-module M is a.f.cog. iff M is isomorphic to a nonzero R-submodule of a fractional ideal of R of the form $\bigcup_{k\geqslant 1}(I^k)^{-1}$ where I is an intersection of a finite number of maximal ideals of R. In particular if R is semilocal noetherian integral domain of dimension 1, then an R-module M is a.f.cog. iff M is isomorphic to a nonzero fractional ideal of R.

2 Properties of a.f.cog. modules

Proposition 2.1 *For any R-module M, the following are equivalent:*
(1) M is a.f.cog.
(2) M is not artinian, but M/N is artinian for any nonzero submodule N of M.
(3) M is 1-critical.

Proof. (1)$\Longleftrightarrow$(2). From ([1], Proposition 10.10).
(2)$\Longleftrightarrow$(3). See for instance [7].

Proposition 2.2 *Let M be an a.f.cog. R-module, then:*
(1) M is uniform
(2) Any nonzero submodule of M is an a.f.cog. module.
(3) For any $f \in End_R(M)$, $f = 0$ or $Ker f$=0 and if $f(M) \neq M$ then $\bigcap_{n\geqslant 1} f^n(M) = 0$.

Proof. The assertions (1) and (2) follow from ([7], Lemma 6.2.11). For (3), let f be an R-endomorphism of M. Since we have, $f(M) \simeq M/Ker f$, $Ker f \neq 0$ implies $f(M) = 0$. On the other hand, if $N = \bigcap_{n\geqslant 1} f^n(M)$ and $N \neq 0$, then $\ker f = 0$ and M/N is artinian and so there exists a nonzero integer n

such that $f^{n+1}(M) = f^n(M)$, it follows That $f(M) = M$.

Proposition 2.3 *If a ring R is a.f.cog. (as a left R-module) then,*
(1) R is left Ore domain, and
(2) Every nonzero prime left ideal of R is maximal.

Proof. (1). Proposition 2.2 implies that R is a domain by taking the multiplications from right by the elements of R . Let $(x, y) \in R^2$ with $x \neq 0$ and $y \neq 0$ and let $I = Rx + Ry$. Since I is uniform, then $Rx \cap Ry \neq 0$.
(2). See ([8], Proposition 3.5.45).

Proposition 2.4 *For any ring R, the following are equivalent:*
(1) R is a.f.cog. (as a left R-module).
(2) R is an integral domain of Krull dimension 1.

Proof. (1) $\Longrightarrow$ (2) is trivial.
(2) $\Longrightarrow$ (1). R is not artinian and by ([8], Lemma 3.5.43) for any nonzero left ideal I of R, R/I is an artinian R-module.

Proposition 2.5 *If R is a commutative ring, then the following are equivalent :*
(1) R is an a.f.cog. R-module.
(2) R is a noetherian integral domain of dimension 1.

Proof. (1) $\Longrightarrow$ (2) by Proposition 2.3 and Hopkins-Levitzki theorem.
(2) $\Longrightarrow$ (1). R is not artinian and for any nonzero element x of R, R/Rx is artinian ([5], Lemma 8.4).

Examples 2.6
(1) $\mathbb{Z}$ is a. f. cog.

(2) If R is a left discrete valuation ring ([4]*, Exercise 19.7), then R is a.f.cog. left R-module. As an example of noncommutative left discrete valuation ring, we take a field K with an endomorphism σ, $\sigma \neq id$, and let $R = K[[X, \sigma]]$ be the ring of formal power series of the form* $\sum_{i \geqslant 0} \alpha_i X^i$ $(\alpha_i \in K)$, with multiplication induced by the twist $X\alpha = \sigma(\alpha)X$ for *all $\alpha \in K$. Then R is a nocommutative left discrete valuation ring.*

(3) The Weyl algebra $A_1(K)$ is an a.f.cog. $A_1(K)$-module for any field K of characteristic 0 *(*[8]*; p. 462)*

(4) If R is left hereditary, noetherian integral domain which is not a

division ring, then R is an a.f.cog. R-module ([7]; p. 197).

(5) Let K be a field with an automorphism σ and δ a σ-derivation, if $R = K[X, \sigma, \delta]$ or $R = K[X, X^{-1}, \sigma]$, then R is an a.f.cog. R-Module *([7]; p. 197).*

(6) If M is a noetherian R-module with K- $\dim(M) \geqslant 1$, then there exists a submodule N of M such that M/N is an a.f.cog. R-module ([7]; p. 185).

3 A.f.cog. modules over a commutative rings

Throughout section 3, R will be a commutative ring with unit, $Q(R)$ will denote the total ring of quotients of R and M an unitary R-module.

Some notations and definitions.

(1) For any subset X of M, the annihilator in R of X (denoted $ann_R(X)$) is the set $\{r \in R : for\ all\ x \in X, rx = 0\}$.

(2) A prime ideal P of R is said to be associated to M if $P = ann_R(x)$ for some $x \in M$. The set of prime ideals of R associated to M is denoted $Ass_R(M)$.

(3) Let I be an ideal of R, the I-torsion submodule of M is the set $T_I(M) = \big\{x \in M : I^k x = 0\ for\ some\ \text{integer}\ k \geqslant 1\big\}$.

(4) The R-submodules of $Q(R)$ are called fractional ideals of R. For a fractional ideal F, the set $F^{-1} = \{x \in Q(R) : Fx \subseteq R\}$ is also a fractional ideal , and if for two fractional ideals $F_1$and F_2 ; $F_1 \subseteq F_2$ then $F_2^{-1} \subseteq F_1^{-1}$.

(5) Let I be an ideal of R. We will write $F(I)$ for the fractional ideal $\bigcup_{k \geqslant 1} (I^k)^{-1}$. Then $F(I) = \big\{x \in Q(R) : I^k x \subseteq R\ for\ some\ \text{integer}\ k \geqslant 1\big\}$.

The next proposition enables us to focus our attention on the a.f.cog. faithful R-modules.

Proposition 3.1 *Let M be an a.f.cog. R-module, and let P the annihilator in R of M. Then:*

(1) P is prime ideal of R which is not a maximal ideal. In particular, if R is of dimension 1, then $P = 0$.

(2) M is a.f.cog., faithful and torsionfree R/P-module.

Proof. The multiplications by the elements of R are an R-endomorphisms of M. By Proposition 2.2 (3), if $rx = 0$ $(r \in R, x \in M)$ then $x = 0$ or $rM = 0$, which implies that P is a prime ideal of R and M, as R/P-module, is a.f.cog., faithful and torsionfree. Now because of Propositions 2.2 and 2.5, a vector space canot be a.f.cog. Therefore P is not a maximal ideal.

Lemma 3.2 *(*[9]*, Lemma 1.7) If M is a nonzero artinian R-module. Then:*

(1) $Ass_R(M) = \{P_1, ..., P_n\}$ is a finite set of maximal ideals.

(2) $M = \underset{1\leqslant i\leqslant n}{\oplus} T_{P_i}(M)$.

Lemma 3.3 *Let M be a fractional ideal of R. If $R \subseteq M$ and M/R is a nonzero artinian R-module, then there is a finite number $P_1, ..., P_n$ of maximal ideals of R such that: $M \subseteq F(\underset{1\leqslant i\leqslant n}{\cap} P_i)$.*

Proof. By Lemma 3.2, $M/R = \underset{1\leqslant i\leqslant n}{\oplus} T_{P_i}(M/R)$ for some maximal ideals $P_1, ..., P_n$ of R. Let $x \in M$, there exist $x_1, ..., x_n \in M$ and $k \geqslant 1$ such that $P_i^k x_i \subseteq R$ $(1 \leqslant i \leqslant n)$ and $x - \underset{1\leqslant i\leqslant n}{\Sigma} x_i \in R$. If $I = \underset{1\leqslant i\leqslant n}{\cap} P_i$, then $I^k x = P_1^k ... P_n^k x \subseteq R$.

Remark 3.4 *In Lemma 3.3, the inclusion can be strict, for example if $R = \mathbb{Z}$ and $M = \frac{1}{2}\mathbb{Z}$ then $M/\mathbb{Z} = \left\{\mathbb{Z}, \frac{1}{2}\mathbb{Z}\right\}$, $Ass_{\mathbb{Z}}(M/\mathbb{Z}) = \{2\mathbb{Z}\}$ and $F(2\mathbb{Z}) = \mathbb{Z}\left[\frac{1}{2}\right]$.*

Lemma 3.5 *Suppose that R is a noetherian integral domain of dimension 1 and P a maximal ideal of R. Then $F(P)/R$ is an artinian R-module.*

Proof. Let A be the localization of R at P. We have $R \subseteq A$ and $Q(A) = Q(R)$. Let $N_1 \supseteq N_2 \supseteq ... \supseteq N_j \supseteq ...$ be a descending chain of submodules of $F(P)$ such that $N_j \supseteq R$ for all $j \geqslant 1$. If $S = R \setminus P$, $S^{-1}N_j$ is A-submodule of $Q(R)$ for all $j \geqslant 1$. Since $Q(R)/A$ is an artinian A-module ([6], Theorem 5.5), then there is an integer m such that $S^{-1}N_j = S^{-1}N_m$ for all $j \geqslant m$. If $x \in N_m$ and $j \geqslant m$, then $sx \in N_j$ and $P^k x \subseteq R$ for some $s \in S$ and $k \geqslant 1$. Since P is maximal, there is $t \in P^k$ and $a \in R$ such that $1 = t + as$ hence $x = tx + asx \in R + N_j \subseteq N_j$.

Corollary 3.6 *If R is a noetherian integral domain of dimension 1 and $P_1, ..., P_n$ are maximal ideals of R, then $F(\underset{1\leqslant i\leqslant n}{\cap} P_i)/R$ is an artinian R-module.*

Proof. It will suffice to see that $F(\underset{1\leqslant i\leqslant n}{\cap} P_i)/R = \underset{1\leqslant i\leqslant n}{\oplus} F(P_i)/R$.

We are now ready to prove the main theorem.

Theorem 3.7 *For any R-module M, the following are equivalent:*

(1) M is a.f.cog. and faithful.

(2) R is a noetherian integral domain of dimension 1, and there exist maximal ideals $P_1, ..., P_n$ of R such that M is isomorphic to a nonzero R-submodule of the fractional ideal $F(\underset{1\leqslant i\leqslant n}{\cap} P_i)$.

Proof. (1) $\Longrightarrow$ (2). By Proposition 3.1, M is torsionfree. Then it follows from Proposition 2.5 that R is a noetherian integral domain of dimension 1. Fix a nonzero element x_0 of M. By Proposition 2.2 (1), for any $x \in M$ there exist a, $b \in R$ with $b \neq 0$, such that $bx = ax_0$. If we put $\sigma(x) = \frac{a}{b} \in Q(R)$, then $\sigma : M \to Q(R)$ is an injective R-homomorphism and $\sigma(M) \supseteq R$. Now by Proposition 3.3, there exist maximal ideals $P_1, ..., P_n$ of R such that $\sigma(M) \subseteq F(\underset{1\leqslant i\leqslant n}{\cap} P_i)$.
(2) $\Longrightarrow$ (1). By Proposition 2.2, it suffices to show that if $P_1, ..., P_n$ are maximal ideals of R, then $F = F(\underset{1\leqslant i\leqslant n}{\cap} P_i)$ is an a.f.cog. R-module. Since R is not artinian, F is also not artinian. Let N be a nonzero submodule of F. From Corollary 3.6 , F/R is artinian. Since we have $N \cap R \neq 0$, $R/(N \cap R)$ is artinian. Therefore $F/(N \cap R)$ is artinian and so F/N is artinian.

Corollary 3.8 *An abelian group A is a.f.cog. $\mathbb{Z}$-module iff A is isomorphic to a nonzero subgroup of $\mathbb{Z}\left[\frac{1}{n}\right]$ for some nonzero integer n.*

Proof. It suffices to observe that if $n = \underset{1\leqslant i\leqslant r}{\Pi} p_i^{e_i}$ is the decomposition of n into prime numbers, then $F(\underset{1\leqslant i\leqslant r}{\cap} \mathbb{Z}p_i) = \mathbb{Z}\left[\frac{1}{\Pi p_i}\right] = \mathbb{Z}\left[\frac{1}{n}\right]$.

Corollary 3.9 *For any ring R, the following are equivalent:*
(1) Any nonzero fractional ideal of R is an a.f.cog. R-module.
(2) $Q(R)$ is an a.f.cog. R-module.
(3) R is semilocal noetherian integral domain of dimension 1*.*

Proof. (1) $\Longrightarrow$ (2) is trivial, and (2) $\Longrightarrow$ (1) follows from Proposition 2.2 (2).
(2) $\Longrightarrow$ (3). Since $Q(R)$ is a faithful R-module, then from Theorem 3.7, R is noetherian integral domain of dimension 1 and there exists an injective R-homomorphism $\sigma : Q(R) \to F(\underset{1\leqslant i\leqslant n}{\cap} P_i)$, where $P_1, ..., P_n$ are maximal ideals of R. We can suppose that $R \subseteq \sigma(Q(R))$ (see the proof of Theorem 3.7). Since we have $1 \in \sigma(Q(R))$, it is easy to see that $Q(R) \subseteq \sigma(Q(R))$, and so $Q(R) = F(\underset{1\leqslant i\leqslant n}{\cap} P_i)$. Let I be a maximal ideal of R, we will show that $I = P_i$ for some $i\,(1 \leqslant i \leqslant n)$. According to ([2], Proposition 1.11), it will suffice to prove that $I \subseteq \underset{1\leqslant i\leqslant n}{\cup} P_i$. Assume for the moment that there exists $a \in I$ with $a \notin \underset{1\leqslant i\leqslant n}{\cup} P_i$. For any $i(1 \leqslant i \leqslant n)$, we can find $x_i \in P_i$ and $r_i \in R$ such that $1 = x_i + r_i a$. If k is a nonzero integer such that $(\underset{1\leqslant i\leqslant n}{\cap} P_i)^k \frac{1}{a} \subseteq R$, then $1 = (\underset{1\leqslant i\leqslant n}{\Pi} (x_i + r_i a))^k = x + ra$ where $x \in (\underset{1\leqslant i\leqslant n}{\cap} P_i)^k$ and $r \in R$. Now $\frac{1}{a} = \frac{x}{a} + r \in R$, a contradiction.

(3) $\Longrightarrow$ (2). Let $P_1, ..., P_n$ be the maximal ideals of R. Fix an element $x \in Q(R)$ and let A_i be the localization of R at P_i $(1 \leqslant i \leqslant n)$. Since $Q(R)/A_i$ is an artinian A_i-module ([6] ,Theorem 5. 5), it follows from Lemma 3.3, that there exists $k_i \geqslant 1$ such that $P_i^{k_i} x \subseteq A_i$. If $k \geqslant k_i$, for any i, then $(\bigcap_{1\leqslant i\leqslant n} P_i)^k x \subseteq \bigcap_{1\leqslant i\leqslant n} A_i = R$. Hence $Q(R) = F(\bigcap_{1\leqslant i\leqslant n} P_i)$ and so, from Theorem 3.7, $Q(R)$ is an a.f.cog. R-module.

From Proposition 3.1, Theorem 3.7 and Corollary 3.9, we can deduce the two following results:

Corollary 3.10 *Let R be semilocal noetherian integral domain of dimension* 1*. Then an R-module M is a.f.cog. iff M is isomorphic to a nonzero fractional ideal of R.*

Corollary 3.11 *For any R-module M, the following are equivalent:*

(1) M is a.f.cog. and injective.

(2) $R/ann_R(M)$ is a semilocal noetherian integral domain of dimension 1 *and M is isomorphic to $Q(R/ann_R(M))$.*

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Received: April, 2009