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On a Conjecture for a Sixth Order Overdetermined Elliptic Boundary Value Problem

Sulbha Goyal and Vinod Goyal

Department of Mathematics, Tuskegee University
Tuskegee, AL 36088, USA
su36832@yahoo.com

Abstract

In this note, we prove the conjecture of Payne and Schaefer [2], regarding an overdetermined boundary value problem for the triharmonic operator, $\Delta^3 = \Delta\Delta\Delta$. It is deduced that if a solution of the problem exists, then the domain must be a ball in $\mathbb{R}^N$, $N \geq 2$.

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1 Introduction

In [2], the authors considered a number of overdetermined elliptic boundary value problems of second, fourth, and higher orders. First, an integral identity equivalent to the problem was obtained, and then this integral dual was used to conclude that the domain in the problem must be a ball. The authors also conjectured that if the problem

$$\Delta^3 u = -1 \text{ in } D \tag{1}$$

$$u = \frac{\partial u}{\partial n} = \Delta u = 0 \text{ on } \partial D \tag{2}$$

$$\frac{\partial \Delta u}{\partial n} = -C \text{ on } \partial D \tag{3}$$

where D is a bounded domain in $\mathbb{R}^N$, $(N > 2)$ with $C^{6+\epsilon}$ boundary ∂D, has a sufficiently smooth solution u in $C^6(\bar{D})$, then D is an N-ball.

Here, we prove the validity of this conjecture by the method of Weinberger [3] and Bennett [1]. We shall use the comma notation for partial differentiation and the summation convention, i.e., a repeated index in a term signifies summation over the index from 1 to N.

2 Proof of the Conjecture

First we prove the following Lemmas:

Lemma 1 *If u is a solution of the problem (1), (2), (3), then*

$$\int_D u dx = \frac{NVC^2}{N+6} \tag{4}$$

where V is the volume of D.

Proof. We note that if u satisfies (1) and r denotes the distance from x to the fixed origin of D, then

$$\Delta^3 \left(r\frac{\partial u}{\partial r} \right) = r\frac{\partial}{\partial r}(\Delta^3 u) + 6\Delta^3 u = -6 \tag{5}$$

From (5), we obtain

$$
\begin{aligned}
\int_D \left(6u - r\frac{\partial u}{\partial r}\right) dx &= \int_D \left(-u\Delta^3(r\frac{\partial u}{\partial r}) + r\frac{\partial u}{\partial r}\Delta^3 u\right) dx \\
&= \int_{\partial D} \left(r\frac{\partial u}{\partial r}\frac{\partial \Delta^2 u}{\partial n} - \Delta^2 u\frac{\partial}{\partial n}(r\frac{\partial u}{\partial r})\right. \\
&\quad \left. +\Delta(r\frac{\partial u}{\partial r})\frac{\partial \Delta u}{\partial n}\right) ds \\
&= \int_{\partial D} \left(r\frac{\partial r}{\partial n}\frac{\partial u}{\partial n}\frac{\partial \Delta^2 u}{\partial n} - \Delta^2 u\frac{\partial}{\partial n}(r\frac{\partial r}{\partial n}\frac{\partial u}{\partial n})\right. \\
&\quad \left. +(r\frac{\partial \Delta u}{\partial r} + 2\Delta u)\frac{\partial \Delta u}{\partial n}\right) ds \\
&= \int_{\partial D} \left[-\Delta^2 u\left(\frac{\partial}{\partial n}(r\frac{\partial r}{\partial n})\frac{\partial u}{\partial n} + r\frac{\partial r}{\partial n}\frac{\partial^2 u}{\partial n^2}\right)\right. \\
&\quad \left. +r\frac{\partial r}{\partial n}\frac{\partial \Delta u}{\partial n}\frac{\partial \Delta u}{\partial n}\right] ds \qquad (6)
\end{aligned}
$$

where in the second equality, we used the Green Identity for the tri-Laplacian, and in the last equality we used the fact that $\frac{\partial u}{\partial n} = \Delta u = 0$ on the boundary. Now in view of $u = \frac{\partial u}{\partial n} = 0$ on the boundary, we observe that $\Delta u = \frac{\partial^2 u}{\partial n^2}$ on ∂D. Consequently, (6) reduces to

$$
\int_D \left(6u - r\frac{\partial u}{\partial r}\right) dx = C^2 \int_{\partial D} r\frac{\partial r}{\partial n} ds = C^2 NV \qquad (7)
$$

by the second Green Identity, where V is the volume of D and N is the number of dimensions. Furthermore,

$$
\int_D r\frac{\partial u}{\partial r} dx = \int_D \text{ grad } (\frac{r^2}{2}) \text{ grad } u \, dx = -\int_D \Delta(\frac{r^2}{2}) u dx = -N\int_D u dx, \qquad (8)
$$

where in the equality before last, we used Green's first identity and the fact that $u = 0$ on ∂D. Consequently, by (7), and (8), we get

$$
\int_D u dx = \frac{NVC^2}{N+6}
$$

and this completes the proof of lemma 1.

Lemma 2 *The function* Φ *defined by*

$$
\Phi = (\Delta u)_{,i} (\Delta u)_{,i} - \Delta u \Delta^2 u - u + \Delta\alpha + \Delta\psi \qquad (9)
$$

attains its maximum value on ∂D provided the functions α and ψ are such that

$$\Delta^2\alpha = \frac{2N-3}{2N}(\Delta^2 u)^2 \quad in \quad D \tag{10}$$

$$\Delta\alpha = -\frac{N}{N+6}C^2 \quad on \quad \partial D \tag{11}$$

$$\frac{\partial\alpha}{\partial n} = -\frac{VC^2}{S} \quad on \quad \partial D \tag{12}$$

and

$$\Delta^2\psi = -\frac{1}{2}(\Delta u)_{,ij}\,(\Delta u)_{,ij} \quad in \quad D \tag{13}$$

$$\Delta\psi = \frac{6C^2}{N+6} \quad on \quad \partial D \tag{14}$$

$$\frac{\partial\psi}{\partial n} = \frac{18VC^2}{S(N+6)} \quad on \quad \partial D \tag{15}$$

where S denotes the surface area of D.

Proof.

First, we show that the problem (10), (11) and (12) has a solution. Evidently, if α is a solution, $\alpha+$ constant is also a solution. We assert that for fixed $\Delta^2\alpha$ and $\Delta\alpha$ there is a unique $\frac{\partial\alpha}{\partial n}$ to ensure the existence of α. To prove this, we let

$$\beta(x) = \Delta\alpha + \frac{N}{N+6}C^2.$$

Then β satisfies the Dirichlet problem

$$\begin{aligned} \Delta\beta &= \tfrac{2N-3}{2N}(\Delta^2 u)^2 \quad \text{in} \quad D \\ \beta &= 0 \quad \text{on} \quad \partial D. \end{aligned}$$

Thus, a unique β is guaranteed and by the maximum principle $\beta < 0$ in D. To determine α, we have

$$\begin{aligned} \Delta\alpha &= \beta - \tfrac{NC^2}{N+6} \quad \text{in} \quad D \\ \tfrac{\partial\alpha}{\partial n} &= -\tfrac{VC^2}{S} \quad \text{on} \quad \partial D. \end{aligned}$$

Integrating the equation over D and using the Second Green Identity, we have

$$\begin{array}{rcl} \int_{\partial D}\frac{\partial\alpha}{\partial n}dS & = & \int_D \beta dx - \frac{NC^2}{N+6}V \\ or, \left(-\frac{VC^2}{S}\right)S & = & \int_D \beta dx - \frac{NC^2}{N+6}V. \end{array}$$

Remeber that $\int_D \beta dx$ is uniquely determined by $\frac{2N-3}{2N}(\Delta^2 u)^2$, so for fixed $\frac{NC^2}{N+6}$ and $\frac{2N-3}{2N}(\Delta^2 u)^2$ there is only one $\frac{VC^2}{S}$, given by the relation above, to ensure the existence of α.

Likewise, in the case of (13), (14) and (15), we define

$$w(x) = \Delta\psi - \frac{6C^2}{N+6}$$

and, as above, for any fixed $\Delta^2\psi$ and $\Delta\psi$, there is a unique $\frac{\partial\psi}{\partial n}$ to ensure the existence of ψ.

Now we compute

$$\begin{array}{rcl} \Delta\Phi & = & 2(\Delta u)_{,ij}(\Delta u)_{,ij} + 2(\Delta u)_{,i}(\Delta^2 u)_{,i} - (\Delta^2 u)^2 - 2(\Delta u)_{,i}(\Delta^2 u)_{,i} \\ & & -\Delta u \Delta^3 u - \Delta u + \Delta^2\alpha + \Delta^2\psi \\ & = & \frac{3}{2}\left((\Delta u)_{,ij}(\Delta u)_{,ij} - \frac{1}{N}(\Delta^2 u)^2\right) + \left(\Delta^2\alpha - \frac{2N-3}{2N}(\Delta^2 u)^2\right) \\ & = & \frac{3}{2}\left((\Delta u)_{,ij}(\Delta u)_{,ij} - \frac{1}{N}(\Delta^2 u)^2\right) \\ & \geq & 0. \end{array}$$

Hence, Φ is subharmonic in D and therefore attains its maximum value on ∂D. This proves lemma 2.

Our next step is to show that Φ is constant in D. We note that by the boundary conditions (2), (3), (11) and (14)

$$\Phi = \frac{12C^2}{N+6} \quad \text{on} \quad \partial D$$

and, hence, by lemma 2

$$\Phi \leq \frac{12}{N+6}C^2 \quad \text{in} \quad D. \tag{16}$$

Now integrating ϕ on D

$$\int_D \Phi dx = \int_D u dx + \int_{\partial D}\frac{\partial\alpha}{\partial n}ds + \int_{\partial D}\frac{\partial\psi}{\partial n}ds = \frac{12C^2}{N+6}V \tag{17}$$

where we have used lemma 1, Green Identities and the boundary conditions (12) and (15). Hence with the help of (16) and (17)

$$\Phi \equiv \frac{12}{N+6} C^2 \quad \text{in} \quad \bar{D}.$$

This implies that $\Delta\Phi$ vanishes identically in $\bar{D}$ and therefore

$$(\Delta u)_{,ij} \, (\Delta u)_{,ij} - \frac{(\Delta^2 u)^2}{N} \equiv 0 \quad \text{in} \quad \bar{D}. \tag{18}$$

To prove that D is an N-ball, we first demonstrate the proof in 3 dimensions. Henceforth we shall use the notation $(\Delta u)_{,ij} = \Delta u_{ij}$. If $N = 3$, then (18) can be written as

$$\begin{gathered}(\Delta u_{11} - \Delta u_{22})^2 + (\Delta u_{11} - \Delta u_{33})^2 + (\Delta u_{22} - \Delta u_{33})^2 \\ +6(\Delta u_{12})^2 + 6(\Delta u_{13})^2 + 6(\Delta u_{23})^2 = 0\end{gathered}$$

which implies that

$$\Delta u_{11} = \Delta u_{22} = \Delta u_{33} \tag{19}$$

and

$$\Delta u_{12} = 0, \Delta u_{13} = 0, \Delta u_{23} = 0 \tag{20}$$

Now integrating $\Delta u_{12} = 0$, we get, $\Delta u = f(x_1) + h(x_2)$ where f and h are arbitrary functions. Using $\Delta u_{11} = \Delta u_{22}$, we obtain $f''(x_1) = h''(x_2)$ which is not possible unless $f''(x_1) = h''(x_2) = k$ for some constant k. Thus,

$$\Delta u = \frac{k}{2}[(x_1 - a_1)^2 - d_1 + (x_2 - a_2)^2 - d_2] \tag{21}$$

for suitable choices of a_1, a_2, d_1, and d_2.

In the same way, we integrate $\Delta u_{13} = 0$, and then with the help of $\Delta u_{11} = \Delta u_{33}$, we get

$$\Delta u = \frac{k}{2}[(x_1 - a_1)^2 - d_1 + (x_3 - a_3)^2 - d_3]. \tag{22}$$

Lastly, integrating $\Delta u_{23} = 0$, and using $\Delta u_{22} = \Delta u_{33}$, we obtain

$$\Delta u = \frac{k}{2}[(x_2 - a_2)^2 - d_2 + (x_3 - a_3)^2 - d_3]. \tag{23}$$

Now adding (21), (22), and (23), we have

$$\Delta u = \frac{k}{3}[(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 - d],$$

where $d = d_1 + d_2 + d_3$. Since $\Delta u = 0$ on ∂D, we finally get

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 = d \tag{24}$$

which shows that D is a 3-sphere. In N dimensions, we get

$$\Delta u_{11} = \Delta u_{22} = \Delta u_{33} = \cdots = \Delta u_{NN} \tag{25}$$

and $\frac{N(N-1)}{2}$ equations

$$\begin{gathered} \Delta u_{12} = 0, \Delta u_{13} = 0, \Delta u_{14} = 0, \ldots, \Delta u_{1N} = 0 \\ \Delta u_{23} = 0, \Delta u_{24} = 0, \ldots, \Delta u_{2N} = 0 \\ \Delta u_{34} = 0, \Delta u_{35} = 0, \ldots, \Delta u_{3N} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \Delta u_{N-2N-1} = 0, \Delta u_{N-2N=0} \\ \Delta u_{N-1N} = 0. \end{gathered} \tag{26}$$

In this case also, as before,

$$\Delta u = \frac{N-1}{2N} k[(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_N - a_N)^2 - d]$$

where $a_1, a_2, \ldots, a_N$, and d are suitably chosen constants. Again since $\Delta u = 0$ on ∂D, we conclude that D is an N-ball

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_N - a_N)^2 = d. \tag{27}$$

It is easily checked that when D is an N-Ball its radius R and the solution of the problem (1), (2), (3) are given respectively by

$$R = \{CN(N+2)(N+4)\}^{1/3} \tag{28}$$

and

$$\begin{aligned} u = &-\frac{r^6}{48N(N+2)(N+4)} + \left(\frac{C^2}{N(N+2)(N+4)}\right)^{1/3}\frac{r^4}{16} \\ &- \left(C^4N(N+2)(N+4)\right)^{1/3}\frac{r^2}{16} + \frac{C^2N(N+2)(N+4)}{48}. \end{aligned} \tag{29}$$

We summarize the foregoing in the following theorem

Theorem 1 *Let D be a bounded domain in $\mathbb{R}^N (N \geq 2)$ with $C^{6+\epsilon}$ boundary ∂D and suppose that the overdetermined problem (1) (2), (3) has a solution in $C^6(\bar{D})$. Then D is an open N-ball of radius R given by (28) and the solution by (29).*

As a consequence of theorem 1, we derive the following corollary:

Corollary 1 *Let D be a bounded domain in $\mathbb{R}^N$ with $C^{6+\epsilon}$ boundary ∂D of positive Gaussian curvature and suppose there is a real constant M such that*

$$\int_D B(1+uQ)dx = M\int_{\partial D} \Delta B ds \tag{30}$$

for every function B satisfying

$$\begin{aligned} \Delta^3 B - QB = 0 \quad &in \quad D \\ B = \frac{\partial B}{\partial n} = 0 \quad &on \quad \partial D \end{aligned} \tag{31}$$

where the function $Q \geq 0$ and $u \in C^6(\bar{D})$ is the solution of the boundary value problem

$$\begin{aligned} \Delta^3 u = -1 \quad &in \quad D \\ u = \frac{\partial u}{\partial n} = \Delta u = 0 \quad &on \quad \partial D \end{aligned} \tag{32}$$

then D is an N-ball.

Proof.

From the Green Identity for the triharmonic operator, (31) and (32), it follows that
$$\int_D B(1+uQ)dx = -\int_{\partial D} \Delta B \frac{\partial \Delta u}{\partial n} ds.$$
We see from (30) that

$$\int_{\partial D} \Delta B \left(\frac{\partial \Delta u}{\partial n} + M \right) ds = 0. \tag{33}$$

Now we choose $B \in C^6(\bar{D})$ to be the solution of

$$\Delta^3 B = QB \quad \text{in} \quad D$$
$$B = \tfrac{\partial B}{\partial n} = 0 \text{ and } \Delta B = \tfrac{\partial \Delta u}{\partial n} + M \quad \text{on} \quad \partial D.$$
It is immediate from (33) that
$$\frac{\partial \Delta u}{\partial n} = -M \quad \text{on} \quad \partial D.$$
Hence the theorem 1 implies that D is an open N-ball. This proves the Corollary 1.

3 Concluding Remark

An alternative proof of theorem 1 can be given by reformulating the problem in an equivalent integral form. As in [2], the integral dual of the problem (1), (2) and (3) is

$$\int_D t dx = C \int_{\partial D} \Delta t ds \tag{34}$$

for any triharmonic function t such that

$$\begin{aligned} \Delta^3 t = 0 \qquad & \text{in } D \\ t = \frac{\partial t}{\partial n} = 0 \quad & \text{on } \partial D. \end{aligned} \tag{35}$$

Now let $t = x_i u,_i - 6u$ in (34) where u solves (1), (2) and (3). Since $\Delta t = x_i(\Delta u),_i - 4\Delta u$, it is easily deduced that

$$\int_D u dx = \frac{C^2 NV}{N+6}. \tag{4}$$

Define the function λ such that

$$\begin{aligned} \Delta\lambda &= -1 \quad \text{in } D \\ \lambda &= 0 \quad \text{on } \partial D. \end{aligned} \tag{36}$$

With Φ as in Lemma 2 and λ in (36), we get, by second Green Identity

$$\int_D \lambda \Delta \phi dx = 0 \tag{37}$$

where we have used

$$\int_D \phi dx = \frac{12C^2 V}{N+6} \text{ and } \phi|_{\partial D} = \frac{12C^2}{N+6}.$$

Since $\lambda > 0$ and $\Delta\phi \geq 0$ in D, (37) yields

$$\Delta\Phi = (\Delta u)_{,ij} (\Delta u)_{,ij} - \frac{(\Delta u)^2}{N} = 0$$

in D. Hence, as in theorem 1, D is open N-ball ($N \geq 2$).

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