

On Estimation of the Exponentiated Pareto Distribution Under Different Sample Schemes

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Abstract

Bayes and classical estimators have been obtained for two parameters exponentiated Pareto distribution when sample is available from complete, type I and type II censoring scheme. Bayes estimators have been developed under squared error loss function as well as under LINEX loss function using non-informative type of priors for the parameters. It has been seen that the estimators obtained are not available in nice closed forms, although they can be easily evaluated for the given sample by using suitable numerical methods. The performance of the proposed estimators have been compared on the basis of their simulated risks obtained under squared error as well as under LINEX loss functions.

Keywords: Bayes estimators; Maximum likelihood estimator; Non-informative type priors; Type I censoring; Type II censoring ; Squared error loss function; LINEX loss function.

1. INTRODUCTION

The exponentiated Pareto distribution with cumulative distribution function is expressed as

$$F(x; \alpha, \theta) = [1 - (1 + x)^{-\alpha}]^{\theta} \quad x > 0, \alpha > 0, \theta > 0. \quad (1.1)$$

was introduced by Gupta et al. (1998) as a lifetime model. The probability density function with two shape parameters α and θ is given by

$$f(x; \alpha, \theta) = \alpha \theta \left[1 - (1+x)^{-\alpha} \right]^{\theta-1} (1+x)^{-(\alpha+1)} \quad x > 0, \alpha > 0, \theta > 0. \quad (1.2)$$

When $\theta = 1$, the above distribution corresponds to standard Pareto distribution of the second kind [see Johnson et al. (1994)].

The estimation procedure for exponentiated Pareto distribution under censoring case seems to be untouched and, therefore, we are interested to develop the estimation procedure for exponentiated Pareto distribution for censored sample case [see Lawless (1982)].

On another important issue, it is to be noted that the inferential procedures for lifetime models are often developed using squared error loss function. No doubt, the use of squared error loss function is well justified when the loss is symmetric in nature. Its use is also very popular, perhaps, because of its mathematical simplicity. But in life testing and reliability problems, the nature of losses are not always symmetric and hence the use of squared error loss function is forbidden and unacceptable in many situations. Inappropriateness of squared error loss function has also been pointed out by different authors. Ferguson (1967), Zellner and Geisel (1968) Aitchison and Dunsmore (1975), Varian (1975) and introduced LINEX loss function which is the simple generalization of squared error loss function and can be used in almost every situation. squared error loss function can also be considered as particular case of LINEX loss function [see Zellner (1986); Parsian (1990); Khatree (1992), etc.]. Gupta et al. (1998) showed that the exponentiated Pareto distribution can be used quite effectively in analyzing many lifetime data. The exponentiated Pareto distribution can have decreasing and upside-down bathtub shaped failure rates depending on the shape parameter θ . Shawky and Abu-Zinadah (2009) studied how the different estimators of the unknown parameters of an exponentiated Pareto distribution can behave for different sample sizes and for different parameter values. Finally; Singh et al. (2005) studied exponentiated Weibull family based on type II Censored Scheme.. Here, we mainly compare the maximum likelihood estimators with the other estimators such as the method of moment estimators, estimators based on percentiles, least squares estimators, weighted least squares estimators and the estimators based on the linear combinations of order statistics, mainly with respect to their biases and root mean squared errors using extensive simulation techniques.

Recently, it is very important to introduce and study point estimators for both the shape parameters of exponentiated Pareto distribution under complete, censored type I and censored type II samples respectively. The organization of the paper is as follows: Section 2 maximum likelihood estimators are discussed with regardless types of sample and Fisher information matrix will be obtained. Bayes estimators have been developed under squared error loss function as well as under LINEX loss function using non-informative type of priors for the parameters in section 3. Finally; in section 4 an example will be discussed to illustrate the application of results.

2. MAXIMUM LIKELIHOOD ESTIMATORS

In a typical life test, n specimens are placed under observation and as each failure occurs the time is noted. Finally at some pre-determined fixed time T or after pre-determined fixed number of sample specimens fail r , the test is terminated. In both of these cases the data collected consist of observations $x_{(1)}, x_{(2)}, \dots, x_{(r)}$ plus the information that $(n - r)$ specimens survived beyond the time of termination, T in the former case and $x_{(r)}$ in the latter. When T is fixed and r is thus a random variable, censoring is said to be of single censored type I, when r is fixed and the time of termination T is a random variable, censoring is said to be of single censored type II. In both type I and type II censoring, Cohen (1965) gave the likelihood function as

$$L(X; \alpha, \theta) = \frac{n!}{(n - r)!} \prod_{i=1}^r f(x_{(i)}; \alpha, \theta) [1 - F(x_0; \alpha, \theta)]^{n-r} \tag{2.1}$$

where $f(x; \alpha, \theta)$ and $F(x; \alpha, \theta)$ are the density and distribution functions respectively, and in the type I the time of termination at $x_0 = T$ and in the type II at $x_0 = x_{(r)}$. If $r = n$, then, equation (2.1) reduces to complete samples. By taking logarithm likelihood function with cumulative function (1.1) and probability density function (1.2) based on equation (2.1) is given by

$$\begin{aligned} \log L(X; \alpha, \theta) = & \log C + r \log \alpha + r \log \theta + (\theta - 1) \sum_{i=1}^r \log [1 - (1 + x_{(i)})^{-\alpha}] - (\alpha + 1) \sum_{i=1}^r \log(1 + x_{(i)}) \\ & + (n - r) \log [1 - F(x_0; \alpha, \theta)] \end{aligned} \tag{2.2}$$

where $C = n! / (n - r)!$.

Thus, the maximum likelihood estimates $\hat{\alpha}$ and $\hat{\theta}$ can be obtained by differentiating (2.2) with respect to α and θ and equating to zero; that is, by simultaneously solving the estimating equations,

$$\frac{r}{\alpha} + (\theta - 1) \sum_{i=1}^r \frac{(1 + x_{(i)})^{-\alpha} \log(1 + x_{(i)})}{1 - (1 + x_{(i)})^{-\alpha}} - \sum_{i=1}^r \log(1 + x_{(i)}) - (n - r) \frac{\partial / \partial \alpha . F(x_0; \alpha, \theta)}{1 - F(x_0; \alpha, \theta)} = 0 \tag{2.3}$$

$$\frac{r}{\theta} + \sum_{i=1}^r \log [1 - (1 + x_{(i)})^{-\alpha}] - (n - r) \frac{\partial / \partial \theta . F(x_0; \alpha, \theta)}{1 - F(x_0; \alpha, \theta)} = 0 \tag{2.4}$$

therefore

$$\hat{\alpha} = \frac{r.w}{w \sum_{i=1}^r \log(1+x_{(i)}) + (n-r) \partial/\partial \alpha F(x_0; \alpha, \theta) - w.(\theta-1) \sum_{i=1}^r \left(\frac{(1+x_{(i)})^{-\alpha} \log(1+x_{(i)})}{1-(1+x_{(i)})^{-\alpha}} \right)} \quad (2.5)$$

and

$$\hat{\theta} = \frac{-r.w}{w \sum_{i=1}^r \log[1-(1+x_{(i)})^{-\hat{\alpha}}] - (n-r) \partial/\partial \theta F(x_0; \hat{\alpha}, \hat{\theta})} \quad (2.6)$$

where

$$w = [1 - F(x_0; \hat{\alpha}, \hat{\theta})],$$

$$\frac{\partial}{\partial \alpha} F(x_0; \alpha, \theta) = \theta [1 - (1+x_0)^{-\alpha}]^{\theta-1} (1+x_0)^{-\alpha} \log(1+x_0),$$

and

$$\frac{\partial}{\partial \theta} F(x_0; \alpha, \theta) = [1 - (1+x_0)^{-\alpha}]^{\theta} \log[1 - (1+x_0)^{-\alpha}]$$

Note that if $r = n$ the normal equations in (2.3) and (2.4) will reduce to the normal equations from complete sample in Shawky and Abu-Zinadah (2009).

Again, to solve the system of the non linear equations (2.5) and (2.6), restoring to numerical techniques and mathematical packages.

The asymptotic variance covariance matrix of the estimators of the parameters is obtained by inverting the Fisher information matrix in which elements are negatives of expected values of the second partial derivatives of the logarithm of the likelihood function. The elements of the sample information matrix, for censored schemes sample will be

$$\begin{aligned} \frac{\partial^2}{\partial \alpha^2} \log L(X; \alpha, \theta) &= \frac{-r}{\alpha^2} - (\theta-1) \sum_{i=1}^r \frac{(1+x_{(i)})^{-\alpha} [\log(1+x_{(i)})]^2}{[1-(1+x_{(i)})^{-\alpha}]^2} \\ &\quad - (n-r) \frac{\partial^2/\partial \alpha^2 F(x_0; \alpha, \theta) [1 - F(x_0; \alpha, \theta)] + [\partial/\partial \alpha F(x_0; \alpha, \theta)]^2}{[1 - F(x_0; \alpha, \theta)]^2} \end{aligned}$$

$$\frac{\partial^2}{\partial \theta^2} \log L(X; \alpha, \theta) = \frac{-r}{\theta^2} - (n-r) \frac{\partial^2/\partial \theta^2 F(x_0; \alpha, \theta) [1 - F(x_0; \alpha, \theta)] + [\partial/\partial \theta F(x_0; \alpha, \theta)]^2}{[1 - F(x_0; \alpha, \theta)]^2}$$

and

$$\frac{\partial^2}{\partial \alpha \partial \theta} \log L(X; \alpha, \theta) = \sum_{i=1}^r \frac{(1+x_{(i)})^{-\alpha} \log(1+x_{(i)})}{1-(1+x_{(i)})^{-\alpha}} - (n-r) \frac{\partial^2 / \partial \alpha \partial \theta F(x_0; \alpha, \theta) [1-F(x_0; \alpha, \theta)] + \partial / \partial \alpha F(x_0; \alpha, \theta) \partial / \partial \theta F(x_0; \alpha, \theta)}{[1-F(x_0; \alpha, \theta)]^2}$$

where

$$\frac{\partial^2}{\partial \alpha^2} F(x_0; \alpha, \theta) = \theta(\theta-1) [1-(1+x_0)^{-\alpha}]^{\theta-1} [(1+x_0)^{-\alpha} \log(1+x_0)]^2, \\ -\theta [1-(1+x_0)^{-\alpha}]^{\theta-1} (1+x_0)^{-\alpha} [\log(1+x_0)]^2$$

$$\frac{\partial^2}{\partial \theta^2} F(x_0; \alpha, \theta) = [1-(1+x_0)^{-\alpha}]^\theta \log[1-(1+x_0)^{-\alpha}],$$

and

$$\frac{\partial^2}{\partial \alpha \partial \theta} F(x_0; \alpha, \theta) = [1-(1+x_0)^{-\alpha}]^{\theta-1} (1+x_0)^{-\alpha} \log(1+x_0) \\ + \theta [1-(1+x_0)^{-\alpha}]^{\theta-1} (1+x_0)^{-\alpha} \log(1+x_0) \log[1-(1+x_0)^{-\alpha}]$$

therefore the approximate sample information matrix will be

$$I(\hat{\alpha}, \hat{\theta}) = - \begin{bmatrix} \frac{\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha \partial \theta} \\ \frac{\partial^2 \ln L}{\partial \alpha \partial \theta} & \frac{\partial^2 \ln L}{\partial \theta^2} \end{bmatrix}_{\substack{\alpha=\hat{\alpha} \\ \theta=\hat{\theta}}} \tag{2.7}$$

[see Cohen (1963)]. For large n , ($n \geq 50$), matrix (2.7) is a reasonable approximation to the inverse of the Fisher information matrix. Note that closed form expressions of the expected values of these second order partial derivatives are not readily available. These terms can be evaluated by using numerical methods. Furthermore, define $V = \lim_{n \rightarrow \infty} n I_1^{-1}(\hat{\alpha}, \hat{\beta}, \hat{\theta})$. The joint asymptotic distribution of the maximum likelihood estimators of α and θ is multivariate normal [see Lawless(1982)].

3. Bayes Estimators

Consider independent non-informative type of priors for parameters α and θ as

$$g_1(\alpha) = \frac{1}{c}; \quad 0 < \alpha < c \tag{3.1}$$

and

$$g_2(\theta) = \frac{1}{\theta}; \quad \theta > 0. \quad (3.2)$$

respectively.

Combining (3.1) and (3.2) with likelihood function (2.1) with cumulative function (1.1) and probability density function (1.2) and using Bayes theorem, the joint posterior distribution is derived as follows

$$\pi(\alpha, \theta) = \frac{\theta^{r-1} \alpha^r \prod_{i=1}^r [1 - (1 + x_{(i)})^{-\alpha}]^{\theta-1} (1 + x_{(i)})^{-(\alpha+1)} \left\{ 1 - [1 - (1 + x_0)^{-\alpha}]^\theta \right\}^{n-r}}{j_1} \quad (3.3)$$

where

$$j_1 = \int_0^c \int_0^\infty \theta^{r-1} \alpha^r \prod_{i=1}^r [1 - (1 + x_{(i)})^{-\alpha}]^{\theta-1} (1 + x_{(i)})^{-(\alpha+1)} \left\{ 1 - [1 - (1 + x_0)^{-\alpha}]^\theta \right\}^{n-r} d\theta d\alpha \quad (3.4)$$

Marginal posterior of a parameter is obtained by integrating the joint posterior distribution with respect to the other parameter and hence the marginal posterior of α can be written, after simplification, as

$$\pi(\alpha / x) = \frac{\alpha^r \prod_{i=1}^r (1 + x_{(i)})^{-(\alpha+1)} j_2}{j_1} \quad (3.5)$$

where

$$j_2 = \int_0^\infty \theta^{r-1} \prod_{i=1}^r [1 - (1 + x_{(i)})^{-\alpha}]^{\theta-1} \left\{ 1 - [1 - (1 + x_0)^{-\alpha}]^\theta \right\}^{n-r} d\theta$$

Similarly integrating the joint posterior with respect to α , the marginal posterior θ can be obtained as

$$\pi(\theta / x) = \frac{\theta^{r-1} j_3}{j_1} \quad (3.6)$$

where

$$j_3 = \int_0^c \alpha^r \prod_{i=1}^r (1 + x_{(i)})^{-(\alpha+1)} \prod_{i=1}^r [1 - (1 + x_{(i)})^{-\alpha}]^{\theta-1} \left\{ 1 - [1 - (1 + x_0)^{-\alpha}]^\theta \right\}^{n-r} d\alpha$$

The Bayes estimators for parameters α and θ of exponentiated Pareto under squared error loss function may be defined as

$$\hat{\alpha}_{bs} = E(\alpha / x) = \int_0^c \alpha \pi(\alpha / x) d\alpha$$

$$\hat{\theta}_{bs} = E(\theta / x) = \int_0^\infty \theta \pi(\theta / x) d\theta$$

respectively. These estimation can be expressed as

$$\hat{\alpha}_{bs} = \frac{j_4}{j_1}$$

and

$$\hat{\theta}_{bs} = \frac{j_5}{j_1}$$

where

$$j_4 = \int_0^c \int_0^\infty \theta^{r-1} \alpha^{r+1} \prod_{i=1}^r [1 - (1 + x_{(i)})^{-\alpha}]^{\theta-1} (1 + x_{(i)})^{-(\alpha+1)} \left\{ 1 - [1 - (1 + x_0)^{-\alpha}]^\theta \right\}^{n-r} d\theta d\alpha \tag{3.7}$$

and

$$j_5 = \int_0^c \int_0^\infty (\theta\alpha)^r \prod_{i=1}^r [1 - (1 + x_{(i)})^{-\alpha}]^{\theta-1} (1 + x_{(i)})^{-(\alpha+1)} \left\{ 1 - [1 - (1 + x_0)^{-\alpha}]^\theta \right\}^{n-r} d\theta d\alpha \tag{3.8}$$

Following Zellner (1986), the Bayes estimators for the shape parameters α and θ of exponentiated Pareto under LINEX loss function are

$$\hat{\alpha}_{bl} = \frac{-1}{a} \log(E(e^{-a\alpha}))$$

and

$$\hat{\theta}_{bl} = \frac{-1}{a} \log(E(e^{-a\theta}))$$

respectively, where $E(.)$ denotes the posterior expectation. After simplification, we have

$$\hat{\alpha}_{bl} = \frac{-1}{a} \log\left(\frac{j_6}{j_1}\right) \tag{3.9}$$

and

$$\hat{\theta}_{bl} = \frac{-1}{a} \log\left(\frac{j_7}{j_1}\right) \tag{3.10}$$

where j_1 is given in (3.4),

$$j_6 = \int_0^c \int_0^\infty e^{-a\alpha} \theta^{r-1} \alpha^r \prod_{i=1}^r [1 - (1 + x_{(i)})^{-\alpha}]^{\theta-1} (1 + x_{(i)})^{-(\alpha+1)} \left\{ 1 - [1 - (1 + x_0)^{-\alpha}]^\theta \right\}^{n-r} d\theta d\alpha \tag{3.11}$$

and

$$j_7 = \int_0^c \int_0^\infty e^{-a\theta} \theta^{r-1} \alpha^r \prod_{i=1}^r [1 - (1 + x_{(i)})^{-\alpha}]^{\theta-1} (1 + x_{(i)})^{-(\alpha+1)} \left\{ 1 - [1 - (1 + x_0)^{-\alpha}]^\theta \right\}^{n-r} d\theta d\alpha \tag{3.12}$$

There are no explicit forms for obtaining estimators for the exponentiated Pareto distribution under censored schemes samples. Therefore, numerical solution and computer facilities are needed.

4. A Numerical Illustration

To illustrate the usefulness of the propose estimators obtained in section 2 and section 3. Using "MATHCAD" (2001), a sample of size 50 was generated from the exponentiated Pareto distribution, with parameters $\alpha = 2$, and $\theta = 0.5$.

Table 1 shows the different estimators for different sample schemes. It may be seen from table 1 that maximum likelihood estimators is closed to the true values of α and θ as compared to sample schemes. The change in the types of schemes does not effects the maximum likelihood estimates only, but on mean square error of the types of schemes also. It will be illogical and inappropriate to suppose that type I censored and type II censored perform better than complete sample.

Table 1
Maximum Likelihood Estimators

Estimators Samples	$\hat{\alpha}$	$\hat{\theta}$	$MSE(\hat{\alpha})$	$MSE(\hat{\theta})$
Complete	1.824	0.447	0.031	0.00279
Type I censored	1.767	0.445	0.054	0.00306
Type II censored	1.580	0.417	0.176	0.00682

Bayes estimators under different schemes have evaluated for the prior hyper parameter $c = 4, 10$ and 12 and their corresponding values have shown in table 2,3 and 4. these tables revealed that the Bayes estimators are not seems very sensitive with variation of "c". It also worth mentioned that thought the Bayes estimators developed with non informative prior yet the estimated values of Bayes estimators are not very far from the estimated values of maximum likelihood estimators. Also, these tables shows the different estimators under different sample schemes for $a = 1, 0.01$ and -1 . It may be seen from these tables that Bayes estimators under squared error loss function and LINEX loss function are close to the true values of α and θ as compared to maximum likelihood estimators in table 1. the change in the values of "a" does effect the Bayes estimators under LINEX loss function estimates only, but on the basis of single sample estimate, it will be illogical and inappropriate to infer that Bayes estimators under squared error loss function and LINEX loss function perform better than maximum likelihood estimators.

Table 2
Bayes Estimator Under Complete Sample

Hyper Parameter		c = 4		c = 10		c = 12	
Estimators		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$
Squared Error Loss		2.273	0.53	2.273	0.53	2.273	0.53
LINEX Loss Function	a = 1	0.937	0.229	0.937	0.228	0.937	0.228
	a = .01	0.831	0.21	0.831	0.21	0.831	0.21
	a = -1	1.02	0.238	1.02	0.238	1.02	0.238

Table 3
Bayes Estimator Under Type I Sample

Hyper Parameter		c = 4		c = 10		c = 12	
Estimators		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$
Squared Error Loss		2.221	0.431	2.272	0.434	2.272	0.434
LINEX Loss Function	a = 1	1.163	0.221	1.241	0.322	1.24	0.322
	a = .01	0.866	0.186	0.865	0.177	0.865	0.177
	a = -1	1.259	0.321	1.479	0.372	1.479	0.372

Table 4
Bayes Estimator Under Type II Sample

Hyper Parameter		c = 4		c = 10		c = 12	
Estimators		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$
Squared Error Loss		2.163	0.324	2.64	0.56	2.904	0.568
LINEX Loss Function	a = 1	0.551	0.356	0.549	0.252	0.509	0.241
	a = .01	0.431	0.24	0.41	0.21	0.41	0.21
	a = -1	0.772	0.472	0.707	0.332	0.7	0.332

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