

Ordering Subsets of (Partially) Ordered Sets: Representation Theorems

Karen Villaverde

Department of Computer Science
New Mexico State University
New Mexico State University
Las Cruces, NM 88003, USA
kvillave@cs.nmsu.edu

Olga Kosheleva

Department of Teacher Education
University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
olgak@utep.edu

Abstract

In many practical situations, we have a (partially) ordered set V of different values. For example, we may have the set of all possible values of temperature, or the set of all possible degrees of confidence in a statement. In practice, we are often uncertain about the exact value of the quantity. Due to this uncertainty, at best, we know a set $S \subseteq V$ of possible values of the quantity: e.g., an interval of possible values. For such sets, it is natural to define a relation “possibly larger” $S_1 \diamond \leq S_2$ meaning that $v_1 \leq v_2$ for some $v_1 \in S_1$ and $v_2 \in S_2$. In this paper, we prove that an arbitrary reflexive relation can be thus represented.

Similar representation theorems are proven for different versions of this relation.

Mathematics Subject Classification: 06A06

Keywords: partial orders, ordering subsets of a poset

1 Formulation of the Problem

In many practical situations, we have a (partially) ordered set V of different values. For example, we may have the set of all possible values of temperature,

or the set of all possible degrees of confidence in a statement (see, e.g., [1, 3]).

In practice, we are often uncertain about the exact value of the quantity. Due to this uncertainty, at best, we know a set $S \subseteq V$ of possible values of the quantity: e.g., an interval of possible values (see, e.g., [2, 4]).

It is necessary to define ordering between such sets.

2 Need for an Ordering Between Sets

In the original set V , we had a (partial) ordering $v_1 \leq v_2$ meaning that the value v_2 is larger than (or equal to) the value v_1 .

In practice, as we have mentioned, we do not know the actual value $v \in V$. Instead, for each statement, we only know the set $S \subseteq V$ of possible values. Let us assume that for two quantities, we know the corresponding sets of values S_1 and S_2 .

A natural question is: which set corresponds to the larger quantity? In this form, this question may not have a definite answer. For example,

- it may be that $v_1 < v_2$ for some $v_1 \in S_1$ and $v_2 \in S_2$, and
- it may also be that for some other values $v'_1 \in S_1$ and $v'_2 \in S_2$, we have $v_2 < v_1$.

For example, when $S_1 = [0, 1]$ and $S_2 = [0.5, 0.5]$:

- on one hand, for $v_1 = 0 \in [0, 1]$ and $v_2 = 0.5 \in [0.5, 0.5]$, we have $v_1 < v_2$;
- on the other hand, for $v_1 = 1 \in [0, 1]$ and $v_2 = 0.5 \in [0.5, 0.5]$, we have $v_2 < v_1$.

This example shows that in case of uncertainty, we are not always sure whether the value described by the set S_2 is larger than (or equal to) the value described by the set S_1 .

3 “Possibly Larger” Relation

Definition. What we *can* always check is whether it is *possible* that the value described by the set S_2 is larger than (or equal to) the value described by the set S_1 . The corresponding “possibly larger” relation means that there exist values $v_1 \in S_1$ and $v_2 \in S_2$ for which $v_1 \leq v_2$:

$$S_1 \diamond \leq S_2 \Leftrightarrow \exists v_1 \in S_1 \exists v_2 \in S_2 (v_1 \leq v_2). \quad (1)$$

Main problem. A natural question is: what are the properties of the “possibly more confident” relation (1)?

Example: it is not always an order. The “possibly larger” relation is not always an order. Indeed, for an order relation, $a \leq b$ and $b \leq a$ imply $a = b$.

However, here, we have

- $[0, 1] \diamond \leq [0.5, 0.5]$,
- $[0.5, 0.5] \diamond \leq [0, 1]$, but
- $[0, 1] \neq [0.5, 0.5]$.

Example: it is not always transitive. The “possibly larger” relation is not even always transitive. For example:

- on one hand, $[0.8, 1] \diamond \leq [0, 1]$: indeed,

$$0.8 \in [0.8, 1], \quad 1.0 \in [0, 1], \quad 0.8 < 1.0;$$

- on the other hand, $[0, 1] \diamond \leq [0, 0.2]$; indeed,

$$0 \in [0, 1], \quad 0.2 \in [0, 0.2], \quad 0 < 0.2;$$

- however, it is not true that $[0.8, 1] \diamond \leq [0, 0.2]$: indeed, every value $v_1 \in [0.8, 1]$ is larger than every value $v_2 \in [0, 0.2]$.

Analysis: it is always reflexive. The only thing we can conclude about this relation is that it is reflexive: $S \diamond \leq S$ for every set S , since $v \leq v$ for every $v \in S$.

What we do in this paper. In this paper, we show that reflexivity is all we can conclude about the “possibly larger” relation. Namely, we will prove that every reflexive relation can be represented as a “possibly larger” relation for some class of subsets of an appropriate fuzzy set V .

4 “Possibly Larger” Relation: Formulation and Proof

Theorem 4.1 *For every reflexive relation uRu' on a set U , there exists an ordered set V and a mapping $f : U \rightarrow 2^V - \{\emptyset\}$ that maps each element of U into a non-empty set $f(u) \subseteq V$ in such a way that*

$$uRu' \Leftrightarrow \exists v \in f(u) \exists v' \in f(u') (v \leq v'). \quad (2)$$

Comment. In other words, uRu' if and only if $f(u) \diamond \leq f(u')$.

Proof. As the desired set V , let us take

$$V \stackrel{\text{def}}{=} \{a_{uu'}^+ : uRu'\} \cup \{a_{uu'}^- : uRu'\}, \quad (3)$$

where $a_{uu'}^\pm$ are different elements. For example, we can consider them as triples $\langle \pm, u, u' \rangle$, then $V \subseteq \{+, -\} \times U^2$.

The ordering relation \leq on this set V is defined as follows: $v \leq v'$ if and only if:

- either $v = v'$,
- or $v = a_{uu'}^-$ and $v' = a_{uu'}^+$ for some u and u' .

One can easily check that it is indeed a (partial) order.

To every element $u \in U$, we now put into correspondence the following set

$$f(u) \stackrel{\text{def}}{=} \{a_{uu'}^- : uRu'\} \cup \{a_{u'u}^+ : uRu'\}. \quad (4)$$

From each element $v \in f(u)$, we can easily tell to which $u \in U$ this element corresponds. Indeed:

- either this element has the form $a_{uu'}^+$ with the desired u ,
- or this element has the form $a_{u'u}^-$ with the desired u .

Thus, if $u \neq u'$, the sets $f(u)$ and $f(u')$ do not have any common elements:

$$f(u) \cap f(u') = \emptyset. \quad (5)$$

Let us now prove the equivalence (2).

First, let us prove that if uRu' , then there exist $v \in f(u)$ and $v' \in f(u')$ for which $v \leq v'$. Indeed, if uRu' , then we can take $v \stackrel{\text{def}}{=} a_{uu'}^-$ and $v' \stackrel{\text{def}}{=} a_{uu'}^+$.

- By definition of the ordering relation, we have $v \leq v'$.
- By definition (4) of the set $f(u)$, we have $v \in f(u)$.
- By the same definition (4) of the set $f(u')$, we have $v' \in f(u')$.
- Thus, there exist value $v \in f(u)$ and $v' \in f(u')$ for which $v \leq v'$.

Vice versa, for $u \neq u'$, let $v \leq v'$ for some $v \in f(u)$ and $v' \in f(u')$. Let us prove that in this case, we have uRu' .

Indeed, since the sets $f(u)$ and $f(u')$ are disjoint, the values v and v' must differ, so $v \leq v'$ means $v < v'$. By definition of the ordering \leq , the condition $v < v'$ means that $v = a_{ab}^-$ and $v' = a_{ab}^+$ for some a and b .

By definition of the set $f(u)$, we have $v = a_{ab}^- \in f(u)$ if and only if $a = u$ and we have aRb . Similarly, we have $v' = a_{ab}^+ \in f(u')$ if and only if $b = u'$ and we have aRb . Thus, $a = u$, $b = u'$, and aRb means that uRu' .

The representation theorem is proven.

5 “Necessarily Larger” Relation: Formulation and Proof

“Necessarily larger” relation. In addition to the “possibly larger” relation, we can also defined a “necessarily larger” relation $\square \leq$, meaning that $v_1 \leq v_2$ for all values $v_1 \in S_1$ and $v_2 \in S_2$:

$$S_1 \square \leq S_2 \Leftrightarrow \forall v_1 \in S_1 \forall v_2 \in S_2 (v_1 \leq v_2). \quad (6)$$

This relation is transitive. One can easily check that the “necessarily larger” relation is transitive. Let us assume that $S_1 \square \leq S_2$ and $S_2 \square \leq S_3$. Let us prove that $S_1 \square \leq S_3$. Indeed:

- The condition $S_1 \square \leq S_2$ means that $v_1 \leq v_2$ for all $v_1 \in S_1$ and $v_2 \in S_2$.
- Similarly, the condition $S_2 \square \leq S_3$ means that $v_2 \leq v_3$ for all $v_2 \in S_2$ and $v_3 \in S_3$.
- Thus, for every $v_1 \in S_1$ and for every $v_3 \in S_3$, once we have picked any $v_2 \in S_2$, we get $v_1 \leq v_2$ and $v_2 \leq v_3$.
- Since the relation \leq on the set V is an ordering, it is transitive and thus, $v_1 \leq v_3$.
- So, indeed, for every $v_1 \in S_1$ and for every $v_3 \in S_3$, we have $v_1 \leq v_3$. By definition of the “necessarily larger” relation, this means that $S_1 \square \leq S_3$.

Transitivity is proven.

This relation is antisymmetric. Let us show that the “necessarily larger” relation is antisymmetric, i.e., $S_1 \square \leq S_2$ and $S_1 \square \leq S_2$ imply $S_1 = S_2$.

Indeed, let us assume that $S_1 \square \leq S_2$ and $S_1 \square \leq S_2$. Let us prove that $S_1 = S_2$. Indeed:

- The condition $S_1 \square \leq S_2$ means that $v_1 \leq v_2$ for all $v_1 \in S_1$ and $v_2 \in S_2$.
- Similarly, the condition $S_2 \square \leq S_1$ means that $v_2 \leq v_1$ for all $v_2 \in S_2$ and $v_1 \in S_1$.
- Thus, for every $v_1 \in S_1$ and for every $v_2 \in S_2$, we have $v_1 \leq v_2$ and $v_2 \leq v_1$.
- Since $v_1 \leq v_2$ is an ordering relation, we conclude that for every $v_1 \in S_1$ and $v_2 \in S_2$, we have $v_1 = v_2$.
- Thus, indeed, $S_1 = S_2$.

Example: this relation is not always reflexive. Let us show that the “necessarily larger” is not necessarily reflexive. Specifically, we will give an example of a set S for which the relation $S \square \leq S$ is not true.

Indeed, let us take $V = [0, 1]$ and $S = [0, 1]$. The relation $S \square \leq S$ would mean that for every $v_1 \in S$ and for every $v_2 \in S$, we would have $v_1 \leq v_2$. However, we have $v_1 = 1 \in S = [0, 1]$, $v_2 = 0 \in S = [0, 1]$, but $1 \not\leq 0$.

Theorem 5.1 *For every transitive antisymmetric relation uRu' on a set U , there exists an ordered set V and a mapping $f : U \rightarrow 2^V - \{\emptyset\}$ that maps each element of U into a non-empty set $f(u) \subseteq V$ in such a way that*

$$uRu' \Leftrightarrow \forall v \in f(u) \forall v' \in f(u') (v \leq v'). \quad (7)$$

Comment. In other words, uRu' if and only if $f(u) \square \leq f(u')$.

Proof. As the desired set V , let us take

$$V \stackrel{\text{def}}{=} \{u : uRu\} \cup \{u^- : \neg uRu\} \cup \{u^+ : \neg uRu\}, \quad (8)$$

where u^- and u^+ are different elements. For example, we can consider them as pairs $\langle \pm, u \rangle$, then $V \subseteq U \cup (\{+, -\} \times U)$.

The ordering relation \leq on this set V is defined as follows: $v \leq v'$ if and only if:

- either $v = v'$,
- or $v, v' \in U$ and vRv' ,
- or $v = u^\pm$ for some $u \in U$, $v' \in U$, and uRv' ;
- or $v \in U$, $v' = u^\pm$ for some $u \in U$, and vRu ;
- or $v = u^-$ and $v' = u^+$ for the same $u \in U$;
- or $v = u^\pm$ and $v' = (u')^\pm$ for some $u \in U$ and $u' \in U$ for which uRu' and $u \neq u'$.

One can easily check that it is indeed a (partial) order.

To every element $u \in U$ for which uRu , we now put into correspondence the set $f(u) = \{u\}$. To every other element $u \in U$, we put into correspondence the set $f(u) = \{u^-, u^+\}$.

Let us prove that uRu' if and only if $f(u) \square \leq f(u')$.

Let us first assume that uRu' , and let us show that in this case, for every $v \in f(u)$ and for every $v' \in f(u')$, we have $v \leq v'$. We will consider two possible cases:

- $u = u'$, and
- $u \neq u'$.

When $u = u'$, the condition uRu means that $f(u) = \{u\}$. Thus, $v \in f(u)$ implies that $v = u$, and similarly $v' \in f(u)$ implies that $v' = u$. In this case, we have $v = u \leq u = v'$, so indeed $v \leq v'$.

When $u \neq u'$, then, according to our definition of \leq , we also have $v \leq v'$ for all $v \in f(u)$ and for all $v' \in f(u')$. The first implication is proven.

To complete the proof, we need to show that if for some u and u' , we have $v \leq v'$ for every $v \in f(u)$ and for every $v' \in f(u')$, then uRu' . Indeed, let us assume that we have two values $u, u' \in U$ for which $v \leq v'$ for every $v \in f(u)$ and for every $v' \in f(u')$. We will also consider two possible cases:

- $u = u'$, and
- $u \neq u'$.

When $u = u'$ and $\neg uRu$, then $f(u) = \{u^-, u^+\}$. In this case, $u^+ \in f(u)$, $u^- \in f(u') = f(u)$, but $u^+ \not\leq u^-$ – which contradicts to our assumption that $v \leq v'$ for every $v \in f(u)$ and for every $v' \in f(u')$. Thus, in this case, we cannot have $\neg uRu$ – thus, we have uRu .

When $u \neq u'$, then for any $v \in f(u)$ and $v' \in f(u')$, from $v \leq v'$ and our definition of the relation \leq , we have uRu' .

The statement is proven.

6 Relation $\exists v_1 \in S_1 \forall v_2 \in S_2 (v_1 \leq v_2)$

New relation. In addition to the “possibly larger” and “necessarily larger” relations, we can also defined a relation $\leq_{\exists\forall}$ as follows:

$$S_1 \leq_{\exists\forall} S_2 \Leftrightarrow \exists v_1 \in S_1 \forall v_2 \in S_2 (v_1 \leq v_2). \quad (9)$$

This relation is transitive. One can easily check that the relation $\leq_{\exists\forall}$ is transitive. Let us assume that $S_1 \leq_{\exists\forall} S_2$ and $S_2 \leq_{\exists\forall} S_3$. Let us prove that $S_1 \leq_{\exists\forall} S_3$. Indeed:

- The condition $S_1 \leq_{\exists\forall} S_2$ means that there exists $v_1 \in S_1$ for which, for all $v_2 \in S_2$, we have $v_1 \leq v_2$.
- Similarly, the condition $S_2 \leq_{\exists\forall} S_3$ means that there exists $v_2 \in S_2$ for which, for all $v_3 \in S_3$, we have $v_2 \leq v_3$.
- Thus, for the fixed values $v_1 \in S_1$ and $v_2 \in S_2$, for every $v_3 \in S_3$, we have $v_1 \leq v_2$ and $v_2 \leq v_3$ and therefore, $v_1 \leq v_3$.

- So, there exists $v_1 \in S_1$ for which for every $v_3 \in S_3$, we have $v_1 \leq v_3$. In other words, we have $S_1 \leq_{\exists\forall} S_3$.

Transitivity is proven.

Example: this relation is not always antisymmetric. Let us give an example when $S_1 \leq_{\exists\forall} S_2$ and $S_2 \leq_{\exists\forall} S_1$ but $S_1 \neq S_2$.

Indeed, we can take $V = [0, 1]$, $S_1 = \{0\}$, and $S_2 = [0, 1]$. In this case, $S_1 \neq S_2$. Here:

- For $v_1 = 0 \in \{0\}$ we have $v_1 \leq v_2$ for all $v_2 \in [0, 1]$. Thus, $S_1 \leq_{\exists\forall} S_2$.
- For $v_2 = 0 \in [0, 1]$ we have $v_2 \leq v_1$ for all $v_2 \in \{0\}$. Thus, $S_2 \leq_{\exists\forall} S_1$.

Example: this relation is not always reflexive. Let us show that this relation is not necessarily reflexive. Specifically, we will give an example of a set S for which the relation $S \leq_{\exists\forall} S$ is not true.

Indeed, let us take $V = \{a, b\}$ and $S = \{a, b\}$ with $a \not\leq b$ and $b \not\leq a$. The relation $S \leq_{\exists\forall} S$ would mean that for one of the values $v_1 \in S$ and for every $v_2 \in S$, we have $v_1 \leq v_2$. Since the set S consists of two elements a and b , there are only two choices: $v_1 = a$ and $v_1 = b$.

- For $v_1 = a$, the inequality $v_1 \leq v_2$ is not true for $v_2 = b$.
- For $v_1 = b$, the inequality $v_1 \leq v_2$ is not true for $v_2 = a$.

Thus, the relation $S \leq_{\exists\forall} S$ is not satisfied.

Theorem 6.1 *For every transitive relation uRu' on a set U , there exists an ordered set V and a mapping $f : U \rightarrow 2^V - \{\emptyset\}$ that maps each element of U into a non-empty set $f(u) \subseteq V$ in such a way that*

$$uRu' \Leftrightarrow \exists v \in f(u) \forall v' \in f(u') (v \leq v'). \quad (10)$$

Comment. In other words, uRu' if and only if $f(u) \leq_{\exists\forall} f(u')$.

Proof. As the desired set V , let us take

$$V \stackrel{\text{def}}{=} \{u : uRu\} \cup \{u^- : \neg uRu\} \cup \{u^+ : \neg uRu\}, \quad (11)$$

where u^- and u^+ are different elements. For example, we can consider them as pairs $\langle \pm, u \rangle$, then $V \subseteq U \cup \{+, -\} \times U$.

The ordering relation \leq on this set V is defined as follows: $v \leq v'$ if and only if:

- either $v = v'$,
- or $v, v' \in U$ and vRv' ,
- or $v = u^\pm$ for some $u \in U, v' \in U$, and uRv' ;
- or $v \in U, v' = u^\pm$ for some $u \in U$, and vRu ;
- or $v = u^\pm$ and $v' = (u')^\pm$ for some $u \in U$ and $u' \in U$ for which uRu' and $u \neq u'$.

One can easily check that it is indeed a (partial) order.

To every element $u \in U$ for which uRu , we now put into correspondence the set $f(u) = \{u\}$. To every other element $u \in U$, we put into correspondence the set $f(u) = \{u^-, u^+\}$.

Let us prove that uRu' if and only if $f(u) \leq_{\exists\forall} f(u')$.

Let us first assume that uRu' , and let us show that in this case, there exists $v \in f(u)$ for which for every $v' \in f(u')$, we have $v \leq v'$. We will consider two possible cases:

- $u = u'$, and
- $u \neq u'$.

When $u = u'$, the condition uRu means that $f(u) = \{u\}$. Thus, $v \in f(u)$ implies that $v = u$, and similarly $v' \in f(u)$ implies that $v' = u$. In this case, we have $v = u \leq u = v'$, so indeed $v \leq v'$.

When $u \neq u'$, then, according to our definition of \leq , we also have $v \leq v'$ for all $v \in f(u)$ and for all $v' \in f(u')$. The first implication is proven.

To complete the proof, we need to show that if for some u and u' , we have $v \leq v'$ for some $v \in f(u)$ and for every $v' \in f(u')$, then uRu' . Indeed, let us assume that we have two values $u, u' \in U$ for which $v \leq v'$ for some $v \in f(u)$ and for every $v' \in f(u')$. We will also consider two possible cases:

- $u = u'$, and
- $u \neq u'$.

When $u = u'$ and $\neg uRu$, then $f(u) = \{u^-, u^+\}$. In this case, $u^+ \not\leq u^-$ and $u^- \not\leq u^+$. Thus, no matter which value $v_1 \in f(u)$ we take, we will not have $v \leq v'$ for all $v' \in f(u)$. This contradicts to our assumption that for some $v \in f(u)$, we have $v \leq v'$ for all $v' \in f(u)$. Thus, in this case, we cannot have $\neg uRu$ – thus, we have uRu .

When $u \neq u'$, then for any $v \in f(u)$ and $v' \in f(u')$, from $v \leq v'$ and our definition of the relation \leq , we have uRu' .

The statement is proven.

7 Relation $\exists v_2 \in S_2 \forall v_1 \in S_1 (v_1 \leq v_2)$

Similar results hold for the following relation $\leq'_{\exists\forall}$:

$$S_1 \leq'_{\exists\forall} S_2 \Leftrightarrow \exists v_2 \in S_2 \forall v_1 \in S_1 (v_1 \leq v_2). \quad (12)$$

This relation is transitive, not always antisymmetric, and not always reflexive.

Vice versa, every transitive relation can be thus represented. The proof comes from the fact that the relation $\leq'_{\exists\forall}$ is equivalent to relation $\leq_{\exists\forall}$ for the dual order $a \leq' b \stackrel{\text{def}}{=} b \leq a$.

8 Relation $\forall v_1 \in S_1 \exists v_2 \in S_2 (v_1 \leq v_2)$

New relation. We can also defined a relation $\leq_{\forall\exists}$ as follows:

$$S_1 \leq_{\forall\exists} S_2 \Leftrightarrow \forall v_1 \in S_1 \exists v_2 \in S_2 (v_1 \leq v_2). \quad (13)$$

This relation is transitive. One can easily check that the relation $\leq_{\forall\exists}$ is transitive. Let us assume that $S_1 \leq_{\forall\exists} S_2$ and $S_2 \leq_{\forall\exists} S_3$. Let us prove that $S_1 \leq_{\forall\exists} S_3$. Indeed:

- The condition $S_1 \leq_{\forall\exists} S_2$ means that for every $v_1 \in S_1$, there exists $v_2(v_1) \in S_2$ for which $v_1 \leq v_2$.
- Similarly, the condition $S_2 \leq_{\forall\exists} S_3$ means that for every $v_2 \in S_2$, there exists $v_3(v_2) \in S_3$ for which $v_2 \leq v_3$.
- Thus, for every $v_1 \in S_1$, for $v_3 = v_3(v_2(v_1)) \in S_3$, we have $v_1 \leq v_2(v_1)$ and $v_2(v_1) \leq v_3$ and therefore, $v_1 \leq v_3$.
- So, for every $v_1 \in S_1$, there exists $v_3 \in S_3$ for which $v_1 \leq v_3$. In other words, we have $S_1 \leq_{\forall\exists} S_3$.

Transitivity is proven.

This relation is reflexive. Indeed, for every $v \in S$, there exists a value $v' \in S$ for which $v \leq v'$: namely, the value $v' = v$. Thus, $S \leq_{\forall\exists} S$ for every set S .

Example: this relation is not always antisymmetric. Let us give an example when $S_1 \leq_{\forall\exists} S_2$ and $S_2 \leq_{\forall\exists} S_1$ but $S_1 \neq S_2$.

Indeed, we can take $V = [0, 1]$, $S_1 = \{1\}$, and $S_2 = [0, 1]$. In this case, $S_1 \leq_{\forall\exists} S_2$ and $S_2 \leq_{\forall\exists} S_1$ but $S_1 \neq S_2$. Here:

- For $v_1 = 1 \in \{1\}$ we have $v_1 \leq v_2$ for $v_2 = 1 \in [0, 1]$. Thus, $S_1 \leq_{\forall\exists} S_2$.
- For every $v_2 \in [0, 1]$, there exists $v_1 \in \{1\}$ for which $v_2 \leq v_1$: namely, $v_1 = 1$. Thus, $S_2 \leq_{\forall\exists} S_1$.

Theorem 8.1 *For every transitive reflexive relation uRu' on a set U , there exists an ordered set V and a mapping $f : U \rightarrow 2^V - \{\emptyset\}$ that maps each element of U into a non-empty set $f(u) \subseteq V$ in such a way that*

$$uRu' \Leftrightarrow \forall v \in f(u) \exists v' \in f(u') (v \leq v'). \quad (14)$$

Comment. In other words, uRu' if and only if $f(u) \leq_{\forall\exists} f(u')$.

Proof. Since R is transitive and reflexive, the relation

$$a \sim b \stackrel{\text{def}}{=} aRb \& bRa \quad (15)$$

is an equivalence relation. This equivalence relation divides the set U into disjoint equivalence classes. Let U/\sim be the set of all these equivalence classes. For every $u \in U$, let $\pi(u)$ denote an equivalence class containing u .

As the desired set V , let us take $V = U \cup (U/\sim)$. For each $u \in U$, we take $f(u) = \{u, \pi(u)\}$.

The ordering relation $v \leq v'$ on this set V is defined as follows:

- for every u , we have $u \leq \pi(u)$;
- if uRu' and $u \not\sim u'$, then $u \leq u'$, $u \leq \pi(u')$, $\pi(u) \leq u'$, and $\pi(u) \leq \pi(u')$;
- if $u \sim u'$, then $u \leq \pi(u')$, $u' \leq \pi(u)$, and $\pi(u) \leq \pi(u')$;
- for every u , we have $u \leq u$ and $\pi(u) \leq \pi(u)$.

One can easily check that it is indeed a (partial) order.

Let us prove that uRu' if and only if $f(u) \leq_{\forall\exists} f(u')$.

Let us first assume that uRu' , and let us show that in this case, for every $v \in f(u)$, there exists a $v' \in f(u')$ for which $v \leq v'$. Indeed, as v , we can take $\pi(u)$; then, for both elements of $f(v) = \{u, \pi(u)\}$, we have $u \leq \pi(u')$ and $\pi(u) \leq \pi(u')$.

To complete the proof, we need to show that if for some u and u' , for every $v \in f(u)$ there exists a $v' \in f(u')$ with $v \leq v'$, then uRu' . Indeed, let us assume that we have two values $u, u' \in U$ for for every $v \in f(u)$, there exists $v' \in f(u')$ for which $v \leq v'$. In particular, for $\pi(u) \in f(u)$, we have either $\pi(u) \leq u'$ or $\pi(u) \leq \pi(u')$. Since $u' \leq \pi(u')$, we thus have $\pi(u) \leq \pi(u')$. According to our definition of the ordering relation, this indeed means that uRu' .

The statement is proven.

9 Relation $\forall v_2 \in S_2 \exists v_1 \in S_1 (v_1 \leq v_2)$

Similar results hold for the following relation $\leq'_{\forall\exists}$:

$$S_1 \leq'_{\forall\exists} S_2 \Leftrightarrow \forall v_2 \in S_2 \exists v_1 \in S_1 (v_1 \leq v_2). \quad (16)$$

This relation is transitive and reflexive, but not always antisymmetric.

Every transitive reflexive relation can be thus represented. The proof comes from the fact that the relation $\leq'_{\forall\exists}$ is equivalent to relation $\leq_{\forall\exists}$ for the dual order $a \leq' b \stackrel{\text{def}}{=} b \leq a$.

10 Open Problems

Open problems: general formulation. In the previous sections, we considered representation theorems for single “ordering” relations. What if we have several such relations, e.g., what if we have all six different relations $\diamond \leq$, $\square \leq$, $\leq_{\exists\forall}$, $\leq'_{\exists\forall}$, $\leq_{\forall\exists}$, and $\leq'_{\forall\exists}$?

Natural questions appear:

- What are the connections between these relations?
- What are the conditions under which six relation on a set can be obtained from an ordering \leq as the corresponding relations?

Most of these questions are open. Let us describe what is known.

Connections between six relations. One connection immediately follows from the definitions:

- if $S \square \leq S_2$, then $S_1 \leq_{\exists\forall} S_2$ and $S_1 \leq'_{\exists\forall} S_2$;
- if $S_1 \leq_{\exists\forall} S_2$, then $S_1 \leq'_{\forall\exists} S_2$;
- if $S_1 \leq'_{\exists\forall} S_2$, then $S_1 \leq_{\forall\exists} S_2$;
- if $S_1 \leq_{\forall\exists} S_2$ or $S_1 \leq'_{\forall\exists} S_2$, then $S_1 \diamond \leq S_2$.

In other words,

$$(\square \leq) \subseteq \leq_{\exists\forall} \subseteq \leq'_{\exists\forall} \subseteq (\diamond \leq); \quad (17)$$

$$(\square \leq) \subseteq \leq'_{\exists\forall} \subseteq \leq_{\forall\exists} \subseteq (\diamond \leq). \quad (18)$$

Transitivity between “possible” and “necessary” relations. One can prove that there is “transitivity” between “possibly larger” and “necessarily larger” relations, e.g.,

$$(S_1 \diamond \leq S_2) \& (S_2 \square \leq S_3) \rightarrow (S_1 \leq_{\exists\forall} S_3). \quad (19)$$

Indeed:

- The condition $S_1 \diamond \leq S_2$ means that $v_1 \leq v_2$ for some $v_1 \in S_1$ and $v_2 \in S_2$.
- Similarly, the condition $S_2 \square \leq S_3$ means that $v_2 \leq v_3$ for all $v_2 \in S_2$ and $v_3 \in S_3$.
- Thus, for the original $v_1 \in S_1$ and $v_2 \in S_2$, and for all $v_3 \in S_3$, we have $v_1 \leq v_2$ and $v_2 \leq v_3$ and thus, by transitivity, $v_1 \leq v_3$.
- So, indeed, there exist $v_1 \in S_1$ for which, for all $v_3 \in S_3$, we have $v_1 \leq v_3$. By definition, this means that $S_1 \leq_{\exists\forall} S_3$.

Similarly, we can prove that

$$(S_1 \square \leq S_2) \& (S_2 \diamond \leq S_3) \rightarrow (S_1 \leq'_{\exists\forall} S_3). \quad (20)$$

Indeed:

- The condition $S_2 \diamond \leq S_3$ means that $v_2 \leq v_3$ for some $v_2 \in S_1$ and $v_3 \in S_2$.
- Similarly, the condition $S_1 \square \leq S_2$ means that $v_1 \leq v_2$ for all $v_1 \in S_1$ and $v_2 \in S_2$.
- Thus, for the original $v_2 \in S_2$ and $v_3 \in S_3$, and for all $v_1 \in S_1$, we have $v_1 \leq v_2$ and $v_2 \leq v_3$ and thus, by transitivity, $v_1 \leq v_3$.
- So, indeed, there exist $v_3 \in S_3$ for which, for all $v_1 \in S_1$, we have $v_1 \leq v_3$. By definition, this means that $S_1 \leq'_{\exists\forall} S_3$.

References

- [1] G. Klir and B. Yuan, *Fuzzy sets and fuzzy logic: theory and applications*. Prentice Hall, Upper Saddle River, New Jersey, 1995.
- [2] J. M. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*, Prentice-Hall, 2001.

- [3] H. T. Nguyen and E. A. Walker, *A first course in fuzzy logic*, CRC Press, Boca Raton, Florida, 2005.
- [4] S. Rabinovich, *Measurement Errors and Uncertainties: Theory and Practice*, Springer Verlag, New York, 2005.

Received: August, 2009