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Application of Sumudu Transform in Reaction-Diffusion Systems and Nonlinear Waves

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Abstract

In this paper we investigate the solution of a nonlinear reaction-diffusion equation connected with nonlinear waves by the application of Sumudu transform. The results presented here are in compact and elegant form expressed in terms of Mittag-Leffler function and generalized Mittag-Liffler function which are suitable for numerical computation. On account of the most general character of our derived result, a large number of solutions obtained earlier by several authors of fractional reaction, fractional diffusion, anomalous diffusion problem and fractional telegraph equations is derived as special cases of our result.

Mathematics Subject Classification: 26A33

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1 Introduction :

Reaction-diffusion models have found numerous applications in pattern formation in biology, chemistry, and physics, see Smoller (1983), Grindrod (1991), Gilding and Kersner (2004), and Wilhelmsson and Lazzaro (2001). These systems show that diffusion can produce the spontaneous formation spatio-temporal patterns. For details, refer to the work of Nicolis and Pri-gogine

(1977), and Haken (2004). A general model for reaction diffusion systems is discussed by Henry and Wearne (2000, 2002), and Henry, Langlands, and Wearne (2005). A piecewise linear approach in connection with the diffusive processes has been developed by Strier, Zanette, and Wio (1995) which leads to analytic results in reaction-diffusion systems.

A similar approach was recently used by Manne, Hurd, and Kenkre (2000) to investigate effects on the propagation of nonlinear wave fronts. The simplest reaction-diffusion models can be described by an equation.

$$\frac{\partial N}{\partial t} = D\frac{\partial^2 N}{\partial x^2} + \gamma F(N) \tag{1}$$

where D is the diffusion coefficient and $F(N)$ is a nonlinear function representing reaction kinetics. For $F(N) = \gamma N(1-N)$, eq.(l) reduces to Fisher-Kolmogorov equation and for $F(N) = \gamma N(1-N^2)$, it reduces to the real Ginsburg-Landau equation.

A generalization of (1) has been considered by Manne, Hurd, and Kenkre (2000) in the form

$$\frac{\partial^2 N}{\partial t^2} + a\frac{\partial N}{\partial t} = \nu^2\frac{\partial^2 N}{\partial x^2} + \xi^2 N(x,t) \tag{2}$$

where ξ indicates the strength of the nonlinearity of the system. Recently R.K.Saxena,Mathai and Haubold [30] generalize this equation in terms of fractional derivative in the following form

$${}_0D_t^\alpha N(x,t) + a\,{}_0D_t^\beta N(x,t) = \nu^2\,{}_{-\infty}D_t^\gamma N(x,t) + \xi^2 N(x,t) + \psi(x,t) \tag{3}$$

Where $\psi(x,t)$ describes the nonlinearity in the system. ξ indicates the strength of the nonlinearity of the system.

2 Mathematical prerequisites :

A generalization of the Mittag-Leffler function (Mittag-Leffler 1903, 1905)

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}, \qquad \alpha \in C,\ \mathrm{Re}(\alpha) > 0 \tag{4}$$

was introduced by Wiman (1905) in the general form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)}, \qquad \alpha,\ \beta \in C,\ \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \tag{5}$$

The main results of these functions are available in the handbook of Erdelyi, Magnus, Oberhettinger, and Tricomi (1955, Section 18.1) and the monographs by Dzherbashyan (1966, 1993). Prabhakar (1971) introduced a generalization of (5) in the form

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(n\alpha+\beta)!n}, \qquad \alpha,\ \beta \in C,\ \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \quad (6)$$

where $(\gamma)_n$ is Pochhammer's symbol

The Riemann-Liouville fractional integral of order ν is defined by Miller and Ross (1993, p.45).

$$_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} f(u)du,\ \mathrm{Re}(\nu) > 0 \qquad (7)$$

Here we define the fractional derivative for $Re(\alpha) > 0$ in the form

$$_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(u)du}{(t-u)^{\alpha-n+1}}; n = [\alpha] + 1 \qquad (8)$$

where $[\alpha]$ means the integral part of the number α. In particular, if $0 < \alpha < 1$,

$$_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(u)du}{(t-u)^{\alpha}} \qquad (9)$$

and if $\alpha = n,\ n \in N = \{1, 2, ...\}$, then

$$_0D_t^n f(t) = D^n f(t), \qquad D \equiv \frac{d}{dt}$$

Caputo [4] introduced fractional derivative in the following form

$$\begin{aligned} {}_0D_t^{\alpha} f(t) &= \tfrac{1}{\Gamma(m-\alpha)} \int_0^t \tfrac{f^m(\tau)}{(t-\tau)^{\alpha-m+1}}\, d\tau, \qquad m-1 < \alpha \le m, \quad \mathrm{Re}(\alpha) > 0. \\ &= \tfrac{d^m f}{dt^m}, \qquad if\ \alpha = m. \end{aligned} \qquad (10)$$

we also need the Weyl fractional operator defined by

$$_{-\infty}D_x^{\mu} f(t) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dt^n} \int_{-\infty}^t \frac{f(u)}{(t-u)^{\mu-n+1}}\, du, \qquad (11)$$

where $n = [\mu]$ is an integral part of $\mu > 0$. Its Fourier transform is given by Metzler and Klafter [19].

$$F\{_{-\infty}D_t^{\mu} f(t)\} = (i\,k)^{\mu} f^*(k) \qquad (12)$$

Where $f^*(k)$ denote the Fourier transform of the function $f(x)$.

Further modification of result (12) is given by Metzler and Klafter [20],

$$F\{_{-\infty}D_x^{\mu} f(x)\} = -|\,k|^{\mu} f^*(k) \tag{13}$$

Definition :- Sumudu Transform

Over the set of functions

$$A = \{f(t)|\exists\, M, \tau_1, \tau_2 > 0, |f(t)| < Me^{|t|/\tau_j},\ \ if\, t \in (-1)^j \times [0, \infty)\}$$

The Sumudu transform is defined by

$$G(u) = S[f(t)] = \int_0^{\infty} f(ut)\, e^{-t} dt, \qquad u \in (-\tau_1, \tau_2). \tag{14}$$

For further detail and properties of this transform (see [1], [2] and [3]).

We will establish the following results which are directly applicable in the analysis of reaction-diffusion systems.

(i)

$$S^{-1}[u^{\gamma-1}(1-\omega u^{\beta})^{-\delta}] = t^{\gamma-1} E^{\delta}_{\beta,\gamma}(\omega t^{\beta}) \tag{15}$$

where $S^{-1}(.)$ denote the inverse Sumudu transform.

We can prove this result in another way

$$S[t^{\gamma-1}E^{\delta}_{\beta,\gamma}(\omega t^{\beta})] \ = \int_0^{\infty} e^{-t}(ut)^{\gamma-1} E^{\delta}_{\beta,\gamma}(\omega (ut)^{\beta})\, dt \tag{16}$$

By using eq. (6)

$$u^{\gamma-1}\sum_{n=0}^{\infty}\frac{(\delta)_n(\omega u^{\beta})^n}{n\,!} = u^{\gamma-1}(1-\omega u^{\beta})^{-\delta} \tag{17}$$

By applying inverse Summudu transform, we get our required result.

(ii)

$$S^{-1}\left[\frac{1}{u(u^{-\alpha}+au^{-\beta}+b)}\right] = \sum_{r=0}^{\infty}(-b)^r\, t^{\alpha(r+1)-1} E^{r+1}_{\alpha-\beta,\alpha(r+1)}\left[-at^{\alpha-\beta}\right] \tag{18}$$

To find inverse Sumudu transform of this function we will use result (15)

$$\frac{1}{u(u^{-\alpha}+au^{-\beta}+b)} = \frac{1}{u(u^{-\alpha}+au^{-\beta})[1+\frac{b}{(u^{-\alpha}+au^{-\beta})}]} = \sum_{r=0}^{\infty}\frac{(-b)^r}{u(u^{-\alpha}+au^{-\beta})^{r+1}}$$

$$= \sum_{r=0}^{\infty} u^{\alpha(r+1)-1}(-b)^r(1+au^{\alpha-\beta})^{-(r+1)} \tag{19}$$

then by using result (15)

$$S^{-1}\left[\frac{1}{u(u^{-\alpha}+au^{-\beta}+b}\right]=\sum_{r=0}^{\infty}(-b)^{r}t^{\alpha(r+1)-1}E^{r+1}_{\alpha-\beta,\alpha(r+1)}(-at^{\alpha-\beta}) \tag{20}$$

(iii)

$$S^{-1}\left[\frac{u^{-\alpha}+au^{-\beta}}{u^{-\alpha}+au^{-\beta}+b}\right]=\sum_{r=0}^{\infty}(-b)^{r}t^{\alpha r}E^{r}_{\alpha-\beta,\alpha r+1}(-at^{\alpha-\beta}) \tag{21}$$

(iv)

$$S^{-1}\left[\frac{u^{-2\alpha}+au^{-\alpha}}{u^{-2\alpha}+au^{-\alpha}+b}\right]=\frac{1}{\sqrt{a^2-4b}}\left[(\lambda+a)E_{\alpha}(\lambda t^{\alpha})-(\mu+a)E_{\alpha}(\mu t^{\alpha})\right] \tag{22}$$

$$Re(\alpha)>0,\rightharpoondown \quad Re(\beta)>0$$

where $a^2-4b>0$ and $E_{\alpha}(z)$ is Mittag-Leffler function defined in equation (4)and λ and μ are the real and distinct roots of the quadratic equation $x^2+ax+b=0$,

Proof : We have

$$\frac{u^{-2\alpha}+au^{-\alpha}}{u^{-2\alpha}+au^{-\alpha}+b}=\frac{1}{\lambda-\mu}\left[\frac{(\lambda+a)u^{-\alpha}}{u^{-\alpha}-\lambda}-\frac{(\mu+a)u^{-\alpha}}{u^{-\alpha}-\mu}\right] \tag{23}$$

Taking the inverse Sumudu transform on both side and using result (14), we have

$$\begin{aligned}&S^{-1}\left[\frac{u^{-2\alpha}+au^{-\alpha}}{u^{-2\alpha}+au^{-\alpha}+b}\right]\\&=\frac{1}{\sqrt{a^2-4b}}\left[(\lambda+a)E_{\alpha}(\lambda t^{\alpha})-(\mu+a)E_{\alpha}(\mu t^{\alpha})\right]\end{aligned} \tag{24}$$

Similerly we can prove that (v)

$$\begin{aligned}&S^{-1}\left[\frac{1}{u(u^{-2\alpha}+au^{-\alpha}+b})\right]\\&=\frac{1}{\sqrt{a^2-4b}}\left[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})-E_{\alpha,\alpha}(\mu t^{\alpha})\right]\end{aligned} \tag{25}$$

To solve fractional reaction diffusion equation the following Lemma of Sumudu transform are required.

Lemma 1. The Sumudu transform of the fractional derivative is given by

$$S\left[{}_0\mathrm{D}_{\mathrm{t}}^{\alpha}f(t)\right]=S\left[\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t\frac{f(u)du}{(t-u)^{\alpha-n+1}}\right] \tag{26}$$

by using equation (8)

$$\mathrm{S}\left[{}_0\mathrm{D}_{\mathrm{t}}^{\alpha}f(t)\right]=u^{-\alpha}F(u)-\sum_{k=0}^{n-1}\frac{{}_0D_{\alpha}^{n-k}(0)}{u^{\alpha-k}} \tag{27}$$

by using the properties of Sumudu transform (see [3], Theorem 4.1 and 4.2) in particular if $0 < \alpha < 1$ we have

$$ {}_0\mathrm{D}_\mathrm{t}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(u)du}{(t-u)^\alpha} \tag{28} $$

$$ S\left[{}_0\mathrm{D}_\mathrm{t}^\alpha f(t)\right] = S\left[\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(u)du}{(t-u)^\alpha}\right] = u^{-\alpha} F(u) \tag{29} $$

Where $F(u) = S[f(t)]$.
Lemma 2. Now we derive Sumudu transform of the fractional derivative introduced by Caputo [4].

$$ S\left[\mathrm{D}_\mathrm{t}^\alpha f(t)\right] = S\left[\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(\tau)d\tau}{(t-\tau)^{\alpha-m+1}}\right] \tag{30} $$

By using convolution theorem of Sumudu transform (see [3]).

$$ = \frac{u}{\Gamma(m-\alpha)} S\left[f^m(t)\right] S\left[t^{-\alpha+m-1}\right] \tag{31} $$

By using Sumudu transform of multiple differentiation. We obtain

$$ S\left[\mathrm{D}_\mathrm{t}^\alpha f(t)\right] = u^{m-\alpha}\left[\frac{G(u)}{u^m} - \sum_{k=0}^{m-1} \frac{f^k(0)}{u^{m-k}}\right] $$

$$ = \left[\frac{G(u)}{u^\alpha} - \sum_{k=0}^{m-1} \frac{f^k(0)}{u^{\alpha-k}}\right] \tag{32} $$

Where $G(u) = S\left[f(t)\right]$.

In the next section we derive solution of a nonlinear reaction diffusion equation connected with nonlinear waves by application of Sumudu transform.

3 Solution of fractional reaction-diffusion equation

Consider the fractional reaction diffusion equation

$$ {}_0D_t^\alpha N(x,t) + a_0 D_t^\beta N(x,t) = \upsilon^2_{-\infty} D_x^\gamma N(x,t) + \xi^2 N(x,t) + \psi(x,t) $$

$$ 0 \le \alpha \le 1, 0 \le \beta \le 1 \tag{33} $$

With initial conditions

$$N(x,0) = f(x), for\ x\ \in R \tag{34}$$

Where ν *is a diffusion coefficient,* ψ *is constant which describes the nonlinearity in the system, and is a nonlinear function for reaction kinetics then there holds the following formula for the solution of (21).*

$$\begin{aligned} N(x,t) &= \sum_{r=0}^{\infty} \frac{(-b)^r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{\alpha r} f^*(k) \exp(-ikx) \\ &\times E^r_{\alpha-\beta,1+\alpha r}(-at^{\alpha-\beta})dk + \sum_{r=0}^{\infty} \frac{(-b)^r}{\sqrt{2\pi}} \int_0^t \tau^{\alpha(r+1)-1} \int_{-\infty}^{\infty} \psi^*(k,t-\tau)\exp(-ikx) \\ &\times E^{r+1}_{\alpha-\beta,\alpha(r+1)}(-a\tau^{\alpha-\beta})dkd\tau \end{aligned} \tag{35}$$

Where $\alpha > \beta$ *and* $E^{\delta}_{\beta,\gamma}(z)$ *is the generalized Mittag-Leffler function, defined in (6) and* $b = \upsilon^2 |k|^{\gamma} - \xi^2$.

Proof : Applying the Sumudu transform with respect to the time variable t and using the boundary condition in eq. (32) , we find,

$$\begin{aligned} &u^{-\alpha}\overline{N}(x,u) - u^{-\alpha}f(x) + au^{-\beta}\overline{N}(x,u) - au^{-\beta}f(x) \\ &= \upsilon^2 {}_{-\infty}D^{\gamma}_x N(x,u) + \xi^2 N(x,u) + \overline{\psi}(x,u) \end{aligned} \tag{36}$$

Taking the Fourier transform of above equation

$$\begin{aligned} &u^{-\alpha}\overline{N}^*(k,u) - u^{-\alpha}f^*(k) + au^{-\beta}\overline{N}^*(k,u) - au^{-\beta}f^*(k) \\ &= -\upsilon^2 |k|^{\gamma}\, \overline{N}^*(k,u) + \xi^2\overline{N}^*(k,u) + \overline{\psi^*}(k,u) \end{aligned} \tag{37}$$

Solving for $\overline{N^*}(k,u)$,

$$\overline{N}^*(k,u) = \frac{(u^{-\alpha} + au^{-\beta})f^*(k)}{(u^{-\alpha} + au^{-\beta} + b)} + \frac{u\overline{\psi^*}(k,u)}{u(u^{-\alpha} + au^{-\beta} + b)} \tag{38}$$

Where $b = \upsilon^2\ |k|^{\gamma} -\ \xi^2$, Inverting the Sumudu transform with the help of equation (18) and (21).

$$\begin{aligned} N^*(k,t) &= \sum_{r=0}^{\infty} (-b)^r t^{\alpha r} E^r_{\alpha-\beta,1+\alpha r}(-at^{\alpha-\beta})f^*(k) + \sum_{r=0}^{\infty} (-b)^r \int_0^t \psi^*(k,t-\tau)\tau^{\alpha(r+1)-1} \\ &\times E^{r+1}_{\alpha-\beta,\alpha(r+1)}(-a\tau^{\alpha-\beta})d\tau \end{aligned} \tag{39}$$

Using the convolution theorem of Sumudu transform (see [3]). Now by applying inverse Fourier transform we get the required result of reaction diffusion equation in terms of generalized Mittag-Leffler function.

$$\begin{aligned} N(x,t) &= \sum_{r=0}^{\infty} \frac{(-b)^r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{\alpha r} f^*(k) \exp(-ikx) \times E^r_{\alpha-\beta,1+\alpha r}(-at^{\alpha-\beta})dk \\ &+ \sum_{r=0}^{\infty} \frac{(-b)^r}{\sqrt{2\pi}} \int_0^t \tau^{\alpha(r+1)-1} \int_{-\infty}^{\infty} \psi^*(k,t-\tau)\exp(-ikx) \\ &\times E^{r+1}_{\alpha-\beta,\alpha(r+1)+1}(-a\tau^{\alpha-\beta})dkd\tau \end{aligned} \tag{40}$$

4. Special cases

When $f(x) = \delta(x)$, where $\delta(x)$ is Dirac delta function. The Theorem reduces to the following.

Corollary (i). *Consider the fractional reaction-diffusion system*

$$ {}_0D_t^{\alpha}N(x,t) + a_0D_t^{\beta}N(x,t) = \upsilon^2{}_{-\infty}D_x^{\gamma}N(x,t) + \xi^2 N(x,t) + \psi(x,t) \qquad (41)$$

Subject to the initial condition $N(x,0) = \delta(x)$ *for* $0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$. *Where* $\delta(x)$ *is the dirac delta function. Here* ξ *is a constant that describes the nonlinearity of the system, and* $\psi(x,t)$ *is a nonlinear function which belongs to the reaction kinetics. Then there exists the following equation for the solution of (32) subject to the initial condition (33).*

$$N(x,t) = \sum_{r=0}^{\infty} \frac{(-b)^r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{\alpha r} \exp(-ikx) \times E^{r}_{\alpha-\beta,1+\alpha r}(-at^{\alpha-\beta})dk$$
$$+ \sum_{r=0}^{\infty} \frac{(-b)^r}{\sqrt{2\pi}} \int_{0}^{t} \tau^{\alpha(r+1)-1} \int_{-\infty}^{\infty} \psi^*(k,t-\tau)\exp(-ikx) \times E^{r+1}_{\alpha-\beta,\alpha(r+1)}(-a\tau^{\alpha-\beta})dkd\tau \qquad (42)$$

where $b = \upsilon^2 |k|^{\gamma} - \xi^2$.

Now if we set $f(x) = \delta(x)$, $\gamma = 2$, α *is replaced by 2* α, *and* β *by* α *in eq.(33). The following results obtained*

Corollary (ii). *Consider the following reaction diffusion system*

$$\frac{\partial^{2\alpha}N(x,t)}{\partial t^{2\alpha}} + \alpha\frac{\partial^{\alpha}N(x,t)}{\partial t^{\alpha}} = \upsilon^2\frac{\partial^2 N(x,t)}{\partial x^2} + \xi^2 N(x,t) + \psi(x,t) \qquad (43)$$

With initial conditions

$$N(x,0) = \delta(x),\ N_t(x,0) = 0, 0 \leq \alpha \leq 1, \qquad (44)$$

where $\psi(x,t)$ *is a nonlinear function belonging to the reaction kinetics. Then for the solution of (43) subject to the initial condition (44) there hold the formula.*

$$N(x,t) = \frac{1}{\sqrt{2\pi}\sqrt{(a^2-4b)}}[\int_{-\infty}^{\infty} \exp(-ikx) \times \{(\lambda+a)E_{\alpha}(\lambda t^{\alpha}) - (\mu+a)E_{\alpha}(\mu t^{\alpha})\}dk$$
$$+ \frac{1}{\sqrt{2\pi}} \int_{0}^{t} \tau^{\alpha-1} \int_{-\infty}^{\infty} \exp(-ikx)\psi^*(k,t-\tau)$$
$$\times [E_{\alpha,\alpha}(\lambda\tau^{\alpha}) - E_{\alpha,\alpha}(\mu\tau^{\alpha})]dkd\tau \qquad (45)$$

where λ *and* μ *are the real and distinct roots of the quadratic equation* $x^2 + ax + b = 0$ *which are given by* $\lambda = \frac{1}{2}(-a + \sqrt{a^2-4b})$ *and* $\mu = \frac{1}{2}(-a - \sqrt{a^2-4b})$ *where* $b = \upsilon^2 |k|^{\gamma} - \xi^2$.

Proof : In order to solve (43), equation (24) reduces

$$\overline{N}^*(k,u) = \frac{u^{-2\alpha} + au^{-\alpha} + \overline{\psi}^*(k,u)}{u^{-2\alpha} + au^{-\alpha} + b} \qquad (46)$$

Taking the inverse Sumudu transform by using equation (15) and (18) for special case replace α by 2α and β by α.

$$\begin{aligned} N^*(k,t) &= \tfrac{1}{\sqrt{(a^2-4b)}}\left[(\lambda+a)E_\alpha(\lambda t^\alpha)-(\mu+a)E_\alpha(\mu t^\alpha)\right] \\ &+\int_0^t \psi^*(k,t-\tau)\tau^{\alpha-1}\left[E_{\alpha,\alpha}(\lambda\tau^\alpha)-E_{\alpha,\alpha}(\mu\tau^\alpha)\right]d\tau \end{aligned} \quad ;\lambda\neq\mu \qquad (47)$$

Where λ and μ are given by eq. (45). Now taking inverse Fourier transform of equation (47), we get the required result (43). Next, if we set $\psi(x,t) = 0, \gamma = 2$, replace α by 2α and β by α in equation (33) we obtain the following result which include many known result on the fractional telegraph equation including the one recently given by Orsingher and Beghin (2004)([26]).

Corollary (iii). *Consider the following reaction diffusion system*

$$\frac{\partial^{2\alpha}N(x,t)}{\partial t^{2\alpha}}+\alpha\frac{\partial^{\alpha}N(x,t)}{\partial t^{\alpha}}=\upsilon^2\frac{\partial^2N(x,t)}{\partial x^2}+\xi^2N(x,t) \qquad (48)$$

With initial condition

$$N(x,0)=\delta(x), N_t(x,0)=0, 0\leq\alpha\leq 1, \qquad (49)$$

Then for the solution of (47) subject to initial condition (48), there hold the formula

$$N(x,t)=\tfrac{1}{\sqrt{2\pi}\sqrt{(a^2-4b)}}\times\int_{-\infty}^{\infty}\exp(-ikx)\left[(\lambda+a)E_\alpha(\lambda t^\alpha)-(\mu+a)E_\alpha(\mu t^\alpha)\right]dk \qquad (50)$$

Where λ and μ are defined in (45),$b=\upsilon^2k^2-\xi^2$ and $E_\alpha(t)$is the Mittag Leffler function defined by (3).

If we set $\xi^2 = 0$ in Corollary (iii) reduces to the result, which states that the reaction diffusion system

$$\frac{\partial^{2\alpha}N(x,t)}{\partial t^{2\alpha}}+\alpha\frac{\partial^{\alpha}N(x,t)}{\partial t^{\alpha}}=\upsilon^2\frac{\partial^2N(x,t)}{\partial x^2} \qquad (51)$$

with initial conditions,

$$N(x,0)=\delta(x),\quad N_t(x,0)=0,\ \ 0\leq\alpha\leq 1,$$

has the solution , given by

$$N(x,t)=\tfrac{1}{\sqrt{2\pi}\sqrt{(a^2-4b)}}\times\int_{-\infty}^{\infty}\exp(ikx)\left[(\lambda+a)E_\alpha(\lambda t^\alpha)-(\mu+a)E_\alpha(\mu t^\alpha)\right]dk \qquad (52)$$

Where λ and μ are defined in (45),$b=\upsilon^2k^2-\xi^2$ and $E_\alpha(t)$is the Mittag Leffler function defined by (3). Equation (49) can be rewritten in the form.

$$N(x,t)=\frac{1}{2\sqrt{2\pi}}\times\int_{-\infty}^{\infty}\exp(ikx)\left[(1+\frac{a}{\sqrt{a^2-4\upsilon^2k^2}})E_\alpha(\lambda t^\alpha)+(1-\frac{a}{\sqrt{a^2-4\upsilon^2k^2}})E_\alpha(\mu t^\alpha)\right]dk \qquad (53)$$

Above equation (53) represent the solution of the time fractional telegraph equation (48).

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