Pareto Solutions of an Inconsistent System

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Abstract

By using the definition of the "Pareto minimum solution" of an inconsistent system, we will find in the paper a connection between this notion and the so-called "infrasolution" of a such kind of system. The results can be used in the optimisation theory, with possible applications in economy and engineering.

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1 Introduction

In 1973 ([2]), I. A. Marusciac defined a class of extremal approximate solution of a linear inconsistent system that contains as particular cases the least square solution and the Tschebychev's best approximation solution of the system, the two main methods used to obtain an approximate solution of an inconsistent system. The least squares method was applied by M. Fekete and J.M Walsh in 1951 ([1]) in order to obtain an approximate solution of an inconsistent system, whereas the Tschebychev's best approximation method was used for the same reason by R.L. Remez in 1969 ([3]). In this paper we obtain a connection between the "Pareto minimum solutions" of an inconsistent system and the "infrasolutions" of this system. First of all we will start with some definitions and known results.

2   Preliminaries

Let consider the following system of m equations and n unknowns:

$$f_k(z) = \sum_{j=1}^{n} a_{kj}z - b_j = 0, \quad k \in M = \{1, 2, \ldots, m\} \tag{1}$$

or, equivalently,

$$Az - b = 0$$

where the notations are obvious:
$A = (a^k)_{k=1,2,\ldots,m} := (a_{k1}, a_{k2} \ldots, a_{kn})_{k=1,2,\ldots,m} \in \mathcal{M}_{m,n}(\mathbb{C}), b = (b_1, b_2, \ldots b_m) \in \mathcal{M}_{m,1}(\mathbb{C})$ and $z = (z_1, z_2, \ldots, z_n) \in \mathcal{M}_{n,1}(\mathbb{C})$ (here $a^k := (a_{k1}, a_{k2}, \ldots, a_{kn})$)

Definition 2.1 *$z \in \mathbb{C}^n$ is an infrasolution of the system (1) if there is no $u \in \mathbb{C}^n$ so that:*

- *$Au \neq Az$*

- *If, for $k \in M$, $f_k(z) = 0$, then $f_k(u) = 0$*

- *If, for $k \in M$, $f_k(z) \neq 0$, then $|f_k(u)| < |f_k(z)|$*

Let denote the set of all infrasolutions of (1) by $\mathrm{IS}(A, b)$

Directly from the Definition 2.1 we have:

Lemma 2.1 *The system (1) is consistent if and only if every solution z of the system is also an infrasolution, **i.e.** $\mathrm{IS}(A, b)$ coincides with the set of <u>all</u> solutions of (1).*

Definition 2.2 *([2]) $z \in \mathbb{C}^n$ is a Pareto minimum solution or Pareto minimum point of the system (1) if there is no $u \in \mathbb{C}^n$ such that:*

- *$|f_k(u)| \leq |f_k(z)|$ for all $k \in M$.*

- *There is a $k_0 \in M$ so that $|f_{k_0}(u)| < |f_{k_0}(z)|$*

Definition 2.3 *([2]) $z \in \mathbb{C}^n$ is called a weak Pareto minimum solution of the system (1) if there is no $u \in \mathbb{C}^n$ such that $|f_k(u)| < |f_k(z)|$ for all $k \in M$.*

We denote by $PA(A, b)$, and, respectively by $PA^*(A, b)$ the sets of all Pareto minimum solutions, respectively of all weak Pareto minimum solutions of the system (1).

Definition 2.4 *An approximate solution $z_0 \in \mathbb{C}^n$ of the system (1) is called Tschebychev uniform best approximation solution of (1), or a Tschebychev's point for the system (1) if*

$$\max_{k \in M}\{|f_k(z_0)|\} = \inf_{z \in \mathbb{C}^n} \max_{k \in M}\{|f_k(z)|\} \tag{2}$$

Definition 2.5 *Let now $(\lambda_k)^n$ be a system of weights, so that $\lambda_k > 0$, $\sum_{k=1}^n \lambda_k = 1$ and let also $p > 0$. An approximate solution $z^* \in \mathbb{C}^n$ of the system (1) is called solution of the least deviaton from 0 in weighted mean of order p of (1), if*

$$\left(\sum_{k=1}^n \lambda_k |f_k(z^*)|^p\right)^{1/p} = \inf_{z \in \mathbb{C}^n} \left(\sum_{k=1}^n \lambda_k |f_k(z)|^p\right)^{1/p} \tag{3}$$

In the particular case $p = 2$ and $\lambda_k = 1/n$ for all $k \in M$, the solution of the least deviation from 0 of the system (1) is called the least squares solution of the system (1).

Definition 2.6 *A matrix $A \in \mathcal{M}_{m,n}(\mathbb{C})$, $m \geq n$ is said to have the "H-property" (Haar property) if all quadratic submatrices of A of order n have the rank exactly n.*

Lemma 2.2 *If $z_0 \in \mathbb{C}^n$ and $A \in \mathcal{M}_{m,n}$, $m \geq n$ and A has the H-property and if there exist $l \geq n$ and $k_1, k_2, \ldots, k_l \in M$ so that*

$$f_{k_j}(z_0) = 0 \ \ for \ \ j = 1, 2, \ldots, l$$

then $z_0 \in \mathrm{P}A(A, b)$

<u>Proof</u>. If z_0 satisfies the assertion we can suppose first that $z_0 \notin \mathrm{P}A(A, b)$. In this case we can find $v \in \mathbb{C}^n$ so that: $|f_k(v)| \leq |f_k(z_0)|$ for all $k \in M$ and there is a $k_0 \in M$ so that $|f_{k_0}(v)| < |f_{k_0}(z_0)|$. Because $f_{k_j} = 0$ for $j = 1, 2, \ldots, l$, it follows that: $a^{k_j}v - b_{k_j} = 0$ for $j = 1, 2, \ldots, l$ and v is considered as a column vector. Thus, if we put $u = z_0 - v \in \mathbb{C}^n$, we deduce that $a_{k_j}(u) = 0$ for $j = 1, 2, \ldots, l$. Because A has the H-property, it follows that the rank of the matrix ${}^t(a_{k_1}, a_{k_2}, \ldots a_{k_l})$ is exactly n and hence the last system has only the trivial solution $u = 0$. It follows that $v = z_0$, which contradicts the hypotesis.

Definition 2.7 *Let denote by $\mathrm{I}S_0(A, b)$, $\mathrm{P}A_0(A, b)$ and, respectively $\mathrm{P}A_0^*(A, b)$ the subsets of infrasolutions, Pareto minimum solutions, respectively weak Pareto minimum solutions z of the system (1), for which $f_k(z) \neq 0$ for all $k \in M$.*

Example 1 *Let consider the system:*

$$\begin{cases} y - 1 = 0 \\ y + 1 = 0 \\ x = 0 \end{cases}$$

Obviously, here

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

A simple calculation shows that:

- $\mathrm{I}S(A, b) = \{z \in \mathbb{R}^2 : x \in \mathbb{R}, y \in [-1, 1]\}$

- $\mathrm{P}A(A, b) = \{z \in \mathbb{R}^2 : x = 0, y \in [-1, 1]\}$

- $\mathrm{P}A^*(A, b) = \{z \in \mathbb{R}^2 : x \in \mathbb{R}, y \in [-1, 1]\} \bigcup \{z \in \mathbb{R}^2 : x = 0, y \in \mathbb{R}\}$

From the example it follows that $\mathrm{I}S(A, b) \neq \mathrm{P}A(A, b) \neq \mathrm{P}A^*(A, b)$. Also, for example, $z_0 = (1, 0)$ is a Tschebychev point of the system, whereas it is not a Pareto minimum point (because $u = (0, 0)$ is a point that satisfies the two conditions in the Definition 2.2, that should be **not** satisfied by any point in $\mathbb{C}^n$)

3 Main results

Theorem 3.1 *Between the three classes defined in the previous section, we have the following inclusions:*

$$\mathrm{P}A(A, b) \subset \mathrm{I}S(A, b) \subset \mathrm{P}A^*(A, b)$$

<u>Proof</u>. Let $z \in \mathrm{P}A(A, b)$ If there is an $u \in \mathbb{C}^n$ satisfying the conditions in Definition 2.1, then, from the first two conditions it follows that $a^{k_0} u \neq a^{k_0} z$ for $k \in M$. That means that $f_{k_0}(u) < f_{k_0}(z)$. That implies the fact that for every $k \in M$ we have: $|f_k(u)| \leq |f_k(z)|$ and it exists a $k_0 \in M$ so that $|f_{k_0}(u)| < |f_{k_0}(z)|$, which is a contradiction that shows the first inclusion. For

the second one, we assume that $z \in \mathrm{IS}(A, b)$ and $z \notin \mathrm{P}A^*(A, b)$. Then, we can find an $u \in \mathbb{C}^n$ such that for every $k \in M$ we have $|f_k(u)| < |f_k(z)|$, which implies that $Au \neq Az$. Since the other two conditions in Definition 2.1 hold, it follows that $z \notin \mathrm{IS}(A, b)$, which is a contradiction that shows the second inclusion.

Remark 3.1 *As it was shown in Example 1, the above inclusions are strict.*

Definition 3.1 *Let $X \subset \mathbb{C}^n$ and $f : X \to R^m$, $g : X \to R^p$ and let $\Omega = \{x \in X : g(x) \leq 0\} \neq \Phi$. x_0 is called a Pareto minimum point of f on Ω (or Pareto minimum solution) of the problem:*

$$f(x) \to \min$$

under the condition:

$$g(x) \leq 0$$

if there is no $x \in \Omega$ such that

$$f(x) \leq f(x_0), \quad f(x) \neq f(x_0)$$

*(Here, the inequalities mean inequalities between the similar **real** components of f, respectively g)*

Let now consider the following minimization problem:

$$(*) \quad (u_1, u_2, \ldots, u_m) \to \min$$

under the conditions:

$$|f_k(z)| \leq u_k, \quad k \in M, \quad z \in \mathbb{C}^n, \quad u \in \mathbb{R}_+^m$$

Using this problem, we have the following characterization of the Pareto minimum solutions of system (1):

Theorem 3.2 $z_0 \in \mathrm{P}A(A, b)$ *if and only if $z_0, u_0)$ is a Pareto minimum solution to the problem defined above, where $u_0 = (|f_1(z_0)|, |f_2(z_0)|, \ldots, |f_m(z_0)|) \in \mathbb{R}_+^m$.*

<u>Proof</u> Let $z_0 \in \mathrm{P}A(A, b)$. Assume that (z_0, u_0) (with u_0 defined above) is not a Pareto minimum solution for the defined problem. Then, there exists

$(z, u) \in \mathbb{C}^n \times \mathbb{R}^m_+$ so that $|f_k(z)| \leq u_k$ for all $k \in M$ and $u < u_0$ (**i.e.** $u \neq u_0$). It follows tat there is a $k_0 \in M$ such that $u_{k_0} < u_{0k_0}$ and, therefore:

$$|f_k(z)| \leq u_k \leq u_{0k} = |f_k(z_0)| \quad for \ all \ \ k \in M$$

$$|f_{k_0}(z)| \leq u_{k_0} < u_{0k_0}$$

which is a contradiction. Hence (z_0, u_0) is a Pareto minimum solution of the problem $(*)$.

Conversely, assume that $(z^*, u^*) \in \mathbb{C}^n \times \mathbb{R}^m_+$ is a Pareto minimum solution of the problem $(*)$. If $z_0 \notin \mathrm{P}A(A, b)$, then we can find $z_0 \in \mathbb{C}^n$ such that for all $k \in M$:

$$|f_k(z_0)| \leq |f_k(z^*)|$$

and there exists $k_0 \in M$ so that:

$$|f_{k_0}(z_0)| < |f_{k_0}(z^*)|$$

Let now $u_0 = (|f_1(z_0)|, f_2(z_0)|, \ldots, f_m(z_0)|) \in \mathbb{R}^m_+$ Then, we have for all $k \in M$

$$|f_k(z_0)| = u_{0k} \leq u^*_k$$

and there is a $k_0 \in M$ so that:

$$|f_{k_0}(z_0)| \leq u_{0k_0} < u^*_{k_0}$$

Therefore, for all $k \in M$ we have $|f_k(z_0)| \leq u_{0k}$ and $u_0 \leq u^*$, $u_0 \neq u^*$. The contradiction shows that $z^* \in \mathrm{P}A(A, b)$

Remark 3.2 *It is obvious that in the real case the corresponding problem is linear. The problem of finding a Pareto minimum solution for the nonlinear vector-minimization problem $(*)$ is equivalent, by Theorem 2, with the problem of finding a Pareto minimum solution of a complex linear inconsistent system.*

Theorem 3.3 *If the system (1) is consistent, then all its solutions are Pareto minimum solutions.*

<u>Proof</u> Let z_0 be a solution of the system (1). That means that $Az_0 = b$ and then, for all $k \in M$, $f_k(z_0) = a^k z_0 - b = 0$, and, consequently, $|f_k(z_0)| = 0$. It follows that there is no $u \in \mathbb{C}^n$ such that for every $k \in M$, $|f_k(u)| < |f_k(z_0)| = 0$. Thus, $z_0 \in \mathrm{P}A(A, b)$.

Theorem 3.4 *If $z^* \in \mathbb{C}_n$ is a solution of the least deviation from 0 in weighted mean of order p of the system (1), then z^* is a Pareto minimum solution of (1).*

<u>Proof</u> Let $z^* \in \mathbb{C}^n$ be a solution of the least deviation from 0 of the system (1) and assume that $z^* \notin \mathrm{P}A(A, b)$. Then, there is an $u \in \mathbb{C}^n$ satisfying the the two conditions in Definition 2.2. That means that for every $k \in M$ we have $|f_k(u)| \leq |f_k(z^*)|$ and that there is a $k_0 \in M$ so that $|f_{k_0}(u) < |f_{k_0}(z^*)|$. But, in this case,

$$\left(\sum_{k=1}^{m} \lambda_k |f_k(u)|^p \right)^{1/p} < \left(\sum_{k=1}^{m} \lambda_k |f_k(z^*)|^p \right)^{1/p}$$

which is a contradiction.

Corolarry 3.1 *If $z_0 \in \mathbb{C}^n$ is a least square solution of the system (1), then $z_0 \in \mathrm{P}A(A, b)$.*

<u>Proof</u> The proof follows immediately from Theorem 3.4 and Definition 2.5.

Remark 3.3 *In Example 1, the least square solution of the system is $z_0 = (0, 0) \in \mathrm{P}A(A, b)$.*

Theorem 3.5 *If $z_0 \in \mathbb{C}_n$ is a Tschebychev point point of the system (1), then z_0 is also an infrasolution of the system (1)*

<u>Proof</u> Let $z_0 \in \mathbb{C}_n$ be a Tschebychev point point of the system (1) and assume that $z_0 \notin \mathrm{I}S(A, b)$. Then, we can find an $u \in \mathbb{C}^n$ satisfying the three conditions in Definition 2.1:

$$Au \neq Az_0$$

$$For \;\; all \;\; k \in M \;\; f_k(z_0) = 0 \;\; implies \;\; that \;\; f_k(u) = 0$$

$$For \;\; all \;\; k \in M \;\; f_k(z_0) \neq 0 \;\; implies \;\; that \;\; |f_k(u)| < |f_k(z_0)|$$

But, in this case, we will have:

$$\max_{k \in M} \{|f_k(u)|\} < \max_{k \in M} \{|f_k(z_0)|\}$$

which is a contradiction. Thus, $z_0 \in \mathrm{I}S(A, b)$.

Remark 3.4 *As it was seen in Example 1, a Tschebychev point of a system is not necessary a Pareto minimum solution.*

The last result gives a condition when the three classes defined in the second paragraph coincide.

Theorem 3.6 *If the matrix $A \in \mathcal{M}_{mn}(\mathbb{C})$ has the H-property, then:*

$$\mathrm{P}A_0(A, b) = \mathrm{I}S_0(A, b) = \mathrm{P}A_0{}^*(A, b)$$

<u>Proof</u> From Theorem 3.1 and Definition 2.7 we have:

$$PA_0(A,b) \subset IS_0(A,b) \subset PA_0{}^*(A,b)$$

Hence, we have only to prove the inclusion:

$$PA_0{}^*(A,b) \subset PA_0(A,b)$$

Assume that $z_0 \in PA_0{}^*(A,b)$, but $z_0 \notin PA_0(A,b)$. Then, we can find an $u \in \mathbb{C}^n$ such that for all $k \in M$ $|a^k u - b_k| \le |a^k z_0 - b_k|$ and there is a $k_0 \in M$ so that $|a^{k_0} u - b_{k_0}| < |a^{k_0} z_0 - b_{k_0}|$. Let $v := (u - z_0)/2$. Hence, from above, we deduce:

$$|f_k(v)| = |a^k v - b_k| = \left| \frac{a^k u + a^k z_0}{2} - b_k \right| = \frac{1}{2} \left| (a^k u - b_k) + (a^k z_0 - b_k) \right|$$

$$\le \frac{1}{2} \left(|a^k u - b_k| + |a^k z_0 - b_k| \right) \le \le \frac{1}{2} 2 |a^k z_0 - b_k| = f_k(z_0) \tag{4}$$

and also:

$$|f_{k_0}(v) = |f_{k_0}(z_0)| \tag{5}$$

If $|f_k(u) = |f_k(z_0)|$, by applying (4), we have that $|f_k(v) = |f_k(z_0)|$ and this means that $f_k(u) = |f_k(z_0)$, or, $a^k v = a^k z_0$. Also, from (5) it follows that $v \ne z_0$.

Let now $k_j \in M$ for which $a^{k_j} v = a^{k_j} z_0$ $(j = 1, 2, \ldots, l, l \in \mathbb{N})$. If the matrix A has the H-property, then $l \le n - 1$, because, otherwise, the system

$$a^{k_j}(v - z_0) = 0 \quad j = 1, 2, \ldots, l$$

has only the trivial solution $v - z_0 = 0$, which is a contradiction. Therefore:

$$f_k(v) = f_k(z_0) \quad if \quad k \in L = \{k_1, k_2, \ldots, k_l\} \quad and \quad l \le n - 1 \tag{6}$$

$$|f_k(v)| < |f_k(z_0)| \quad for \quad all \quad k \in M \setminus L \tag{7}$$

Let now $\mu \in \mathbb{C}^n$ be a solution of the system

$$a_k \mu = f_k(z_0), \quad k \in M$$

$$a^{k_p} \mu = \gamma_0 > 0, \quad p = l + 1, l + 2, \ldots n$$

It is obvious that the existence of such a solution is assured by the fact that A has the H-property and $l \leq n - 1$. From this system and from (6) it follows that for $k \in M$ we have:

$$0 = |f_k(v) - a^k \mu| = |a^k z_0 - b_k - a^k \mu| = |a^k z_0 - b_k - a^k z_0 + b_k| < |f_k(z_0)| \tag{8}$$

Let $\epsilon \in (0, 1]$ and $u_\epsilon := v - \epsilon\mu \in \mathbb{C}^n$. From (8), we have ($k \in M$):

$$|f_k(u_\epsilon)| = |a^k v - b^k - \epsilon a^k \mu| = |\epsilon(a^k v - b_k - a^k \mu) + (1 - \epsilon)(a^k v - b_k)| \leq$$

$$\leq \epsilon|f_k(v) - f_k(z_0)| + (1 - \epsilon)|f_k(v)| < |f_k(z_0)|$$

We have thus, for all $k \in L$,

$$|f_k(u_\epsilon)| < |f_k(z_0)| \tag{9}$$

Let denote:

$$\delta = \min_{k \in M \setminus L} \{|f_k(z_0)| - |f_k(v)|\} > 0$$

$$\lambda = \max_{k \in M \setminus L} \{|a^k \mu|\} \geq \gamma > 0$$

$$0 < \epsilon_0 < \min\{\frac{\delta}{\lambda}, 1\}$$

For $k \in M \setminus L$ we have:

$$|f_k(u_\epsilon)| = |a^k v - b_k - \epsilon a^k \mu| \leq |a^k v - b_k| + \epsilon_0 |a^k \mu| <$$

$$< |f_k(v)| + \frac{\delta}{\lambda}|a^k \mu| \leq |f_k(v)| + \frac{\delta}{\lambda}\lambda \leq |f_k(v)| + \delta \leq$$

$$\leq |f_k(v)| + |f_k(z_0)| - |f_k(v)| = |f_k(z_0|$$

Hence, for every $k \in M \setminus L$, we have:

$$|f_k(u_\epsilon)| < |f_k(z_0)|$$

From this relation and from (9) we conclude that for all $k \in M$ the inequality $|f_k(u_\epsilon)| < f_k(z_0)|$ is true. This contradicts the fact that $z_0 \in \mathrm{P}A^*(A, b)$ and the theorem is proved.

References

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