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On Eigenvalues of Differential Equations with Singularities, Discontinuity Conditions and Turning Points

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Abstract

We consider the Sturm-Liouville equation on the bounded interval with two singularities in end points. This equation contains one turning point together with discontinuity conditions, moreover the turning point lies before the jump point. In this paper, by using the spectral characteristic function, we study eigenvalues.

1 Introduction

We consider the boundary value problem L

$$-(p(x)y'(x))' + q(x)y(x) = \lambda s(x)y(x), \quad -1 < x < 1, \tag{1}$$

$$y(x) = O(1), \quad x \to \pm 1, \tag{2}$$

with the jump conditions

$$y(a+0,\lambda) = \alpha y(a-0,\lambda), \qquad y'(a+0,\lambda) = -\alpha y'(a-0,\lambda), \tag{3}$$

in an interior point $a \in (-1,1)$. Here, put $p(x) = (1-x^2)p_0(x)$, $s(x) = (x-x_0)s_0(x)$, $x_0 \in (-1,1)$, $x_0 < a$, $P_0(x)$, $S_0(x) \in C^2[-1,1]$, $q(x) \in C[-1,1]$, $P_0(x)S_0(x) \neq 0$ for all $x \in [-1,1]$, and λ is the spectral parameter. Put

$$r(x) := \frac{s(x)}{p(x)} = \frac{(x-x_0)s_0(x)}{(1-x^2)p_0(x)}, \quad -1 < x < 1.$$

The sign of the potential function $r(x)$ changes in x_0 (x_0 is called turning point). Let for definiteness, $s_0(x) > 0, p_0(x) > 0$ and let $r(x) = R^2(x)$, where $R(x) > 0$ for $x > x_0$ and $-iR(x) > 0$ for $x < x_0$. Denote

$$R_- = \int_{-1}^{x_0} |R(s)|ds, \qquad R_+ = \int_{x_0}^{1} |R(s)|ds. \tag{4}$$

Boundary value problems with singularities inside the interval have been studied in [1]-[2]. We note direct and inverse problems of spectral analysis for various classes of differential equations with singularities and turning points were studied in many works (see[3-9]). Freiling, Rykhlov and Yurko in [10], investigated boundary value problems with singularities and turning points. Now, in this paper, we add discontinuity conditions to previous conditions and calculate the asymptotic behavior of solutions and eigenvalues.

2 properties of the spectrum

We transform (1), (2) by $z(x) = \sqrt{p(x)}y(x)$ to the boundary value problem L_1 of the form

$$-z''(x) + \chi(x)z(x) = \lambda r(x)z(x), \quad -1 < x < 1 \tag{5}$$

$$z(x) = O(\sqrt{1 \pm x}) \quad as \;\; x \to \mp 1, \tag{6}$$

where

$$\chi(x) = \frac{-1}{(1-x^2)^2} - \frac{xh_0(x)}{1-x^2} + \frac{q(x)}{(1-x^2)p_0(x)} + h_0'(x) + h_o^2(x),$$

$$h_0(x) := p_0'(x)/2p_0(x).$$

Clearly, the spectrum of L coincides with the spectrum of L_1. Let $\lambda = \rho^2$, and let for definiteness $\Re\rho \geq 0$. Moreover, we assume $\rho \in \overline{S_0 \cup S_{-1}}$, where

$$S_j = \{\rho : \arg(\rho) \in (\frac{\pi j}{2}, \frac{\pi(j+1)}{2})\}, \quad j = -1, 0.$$

Denote

$$\xi_{-1} = \int_{-1}^{x} |R(s)|ds, \;\; \xi_1 = \int_{x}^{1} |R(s)|ds, \;\; \xi_0 = \int_{x_0}^{x} |R(s)|ds, \;\; \xi = \int_{x_0}^{a} |R(s)|ds,$$

$$J_j = \{x : |\rho\xi_j| \leq 1\}, \quad J = J_{-1} \cup J_0 \cup J_1, \quad I_{-1} = (-1, x_0)\backslash J, \quad I_1 = (x_0, 1)\backslash J, \quad I = I_{-1} \cup I_1,$$
$$I_{a^-} = (x_0, a], \quad I_{a^+} = (a, 1).$$

Here we consider the dependence of ξ_j and J_j, I_j, J, I and ρ, respectively. Fix $\epsilon > 0$ and consider the intervals $\theta_{-1,\epsilon} := (-1, x_0 - \epsilon]$, $\theta_{0,\epsilon} := [-1 + \epsilon, 1 - \epsilon]$, $\theta_{1,\epsilon} := [x_0 + \epsilon, 1]$. Let $[1] = 1+O(\rho^{-1})$, $[1]_j = 1+O((\rho\xi_j)^{-1})$ for $|\rho\xi_j| \geq 1$, $x \in \theta_{j,\epsilon}$ (*i.e.* $f(x, \rho) = [1]_j$ means that $|f(x, \rho) - 1| \leq C_\epsilon |\rho\xi_j|^{-1}$ for $|\rho\xi_j| \geq 1, x \in \theta_{j,\epsilon}$). Denote $[\tilde{1}] = [1]_j$ for $|\rho\xi_j| \geq 1$, $x \in \theta_{j,\epsilon}$. Let for definiteness $\rho \in \overline{S_0}$ (the arguments are similar for $\rho \in \overline{S_{-1}}$). According to [11](see also [12]), for $x \in \theta_{0,\epsilon}$ there exist solutions $w_j(x, \rho)$, j=1,2, of Eq.(5) such that $w_j^m(x, \rho)$, m=0,1, are absolutely continuous on $\theta_{0,\epsilon}$, and furthermore, for $x \in \theta_{0,\epsilon}\backslash J_0$,

$$\begin{cases} w_1^{(m)}(x, \rho) = \rho^m |R(x)|^{m-\frac{1}{2}} \exp(\rho\xi_0)[1]_0, & x < x_0, \\ w_2^{(m)}(x, \rho) = \rho^m |R(x)|^{m-\frac{1}{2}} ((-1)^m \exp(-\rho\xi_0)[1]_0 + i \exp(\rho\xi_0)[1]_0), & x < x_0, \end{cases} \tag{7}$$

$$\begin{cases} w_1^{(m)}(x, \rho) = (i\rho)^m |R(x)|^{m-\frac{1}{2}} \exp(i\frac{\pi}{4})((-1)^m \exp(-i\rho\xi_0)[1]_0 - i \exp(i\rho\xi_0)[1]_0), & x > x_0, \\ w_2^{(m)}(x, \rho) = (i\rho)^m |R(x)|^{m-\frac{1}{2}} \exp(i\frac{\pi}{4}) \exp(i\rho\xi_0)[1]_0), & x > x_0. \end{cases} \tag{8}$$

Moreover,

$$\det[w_1^{(m)}(x, \rho)] \equiv -2\rho[1]. \tag{9}$$

Analogously, one can construct fundamental systems of solutions in $\theta_{-1,\epsilon}$ and $\theta_{1,\epsilon}$. For $x \in \theta_{-1,\epsilon}$, there exist solutions $u_j(x, \lambda)$, j=1,2, of Eq.(5) such that the function $u_1^{(m)}(x, \lambda)$, m=0,1, is entire in λ, and

$$\begin{cases} u_1^{(m)}(x, \lambda) = (\rho|R(x)|)^{m-\frac{1}{2}} (\exp(\rho\xi_{-1})[1]_{-1} + i(-1)^m \exp(-\rho\xi_{-1})[1]_{-1}), \ x \in \theta_{-1,\epsilon}\backslash J_{-1}, \\ u_2^{(m)}(x, \lambda) = (-\rho)^m |R(x)|^{m-\frac{1}{2}} \exp(-\rho\xi_{-1})[1]_{-1}, \ x \in \theta_{-1,\epsilon}\backslash J_{-1}. \end{cases} \tag{10}$$

For $x \in \theta_{1,\epsilon}$, there exist solutions $v_j(x, \lambda)$, j=1,2 of Eq.(5) such that the function $v_1^{(m)}(x, \lambda)$, m=0,1, is entire in λ, and

$$\begin{cases} v_1^{(m)}(x, \lambda) = (\rho|R(x)|)^{m-\frac{1}{2}} i^m ((-1)^m \exp(i\rho\xi_1)[1]_1 + i \exp(-i\rho\xi_1)[1]_1), \ x \in \theta_{1,\epsilon}\backslash J_1, \\ v_2^{(m)}(x, \lambda) = (-i\rho)^m |R(x)|^{m-\frac{1}{2}} \exp(i\rho\xi_1)[1]_1, \ x \in \theta_{1,\epsilon}\backslash J_1. \end{cases} \tag{11}$$

We extend $u_j(x, \lambda)$, $v_j(x, \lambda)$ to the whole interval $(-1, 1)$ as smooth solutions of (5) and put $u(x, \lambda) := u_1(x, \lambda)$. Using the fundamental system of solutions $w_j(x, \rho)$ one can write

$$u(x, \lambda) = \bar{a_1}(\rho) w_1(x, \rho) + \bar{a_2}(\rho) w_2(x, \rho). \tag{12}$$

Fix $x_* \in (-1, x_0)$, in view of (9), according to cramer's rule and using (7), (10) in (12), we obtain

$$\begin{cases} \bar{a_1}(\rho) = \rho^{-1/2}(\exp(\rho R_-)[1] + \exp(-\rho R_-)[1]), \\ \bar{a_2}(\rho) = \rho^{-1/2}(i\exp(-\rho R_-)[1] + o(\rho^{-1})\exp(\rho R_-)\exp(-2\rho\int_{x_*}^{x_0}|R(s)|ds)). \end{cases} \tag{13}$$

On the interval $\theta_{0,\varepsilon}\backslash J_0$, by substituting (7), (8),(13), into (12), we obtain

$$u^m(x,\lambda) = (\rho|R(x)|)^{m-\frac{1}{2}}(\exp(\rho\xi_{-1})[\tilde{1}] + i(-1)^{j+m}\exp(\rho\xi_{-1})[\tilde{1}]), \quad \rho \in \bar{S}_j, \quad x \in I_{-1}, \tag{14}$$

$$u^m(x,\lambda) = (\rho|R(x)|)^{m-\frac{1}{2}} i^m \exp(i\pi/4)(\exp(\rho R_-)[1] + \exp(-\rho R_-)[1])$$

$$\times((-1)^m\exp(-i\rho\xi_0)[\tilde{1}] - i\exp(i\rho\xi_0)[\tilde{1}]), \quad x \in I_{a^-}, \tag{15}$$

and by using the fundamental systems $\{w_j(x,\lambda)\}_{j=1,2}$ for $x > a$, we have

$$u(x,\lambda) = a_1^+(\rho)w_1(x,\rho) + a_2^+(\rho)w_2(x,\rho), \quad x \in I_{a^+}. \tag{16}$$

In order hand, $u(x,\lambda)$ satisfies the matching conditions (3), thus, according to cramer's rule in $x = a$, we get

$$\begin{cases} a_1^+(\rho) = -i\alpha\rho^{-\frac{1}{2}}\exp(2i\rho\xi)(\exp(\rho R_-)[1] + \exp(-\rho R_-)[1]), \\ a_2^+(\rho) = \alpha\rho^{-\frac{1}{2}}(\exp(2i\rho\xi)[1] + \exp(-2i\rho\xi)[1])(\exp(\rho R_-)[1] + \exp(-\rho R_-)[1]). \end{cases} \tag{17}$$

Substituting (17), (8), into (16), we get

$$u^m(x,\lambda) = i^m\alpha(\rho R(x))^{m-\frac{1}{2}}\exp(i\pi/4)(\exp(\rho R_-)[1] + \exp(-\rho R_-)[1])$$

$$\times(\exp(i\rho(\xi_0 - 2\xi))[\tilde{1}] + i(-1)^{m+1}\exp(-i\rho(\xi_0 - 2\xi))[\tilde{1}]), \quad x \in I_{a^+}. \tag{18}$$

We extend (18) by using the fundamental system of solutions $v_j(x,\lambda)$ to the whole interval I_1. According to [10], $u_j^m(x,\lambda), v_j^m(x,\lambda)$, $w_j^m(x,\lambda)$, in the given interval and also $u^m(x,\lambda)$ in J_{-1}, J_0 are bounded. In the interval J_1 we have

$$|u^m(x,\lambda)| \le C(1-x)^{\frac{1}{2}-m}|\ln|\rho\xi_1/2||\exp((R_+ + 2\xi)|\Im\rho|)\exp(|\Re\rho|R_-).$$

Denote the characteristic function

$$\Delta(\lambda) =< u(x,\lambda), v_1(x,\lambda) > . \tag{19}$$

Substituting (11) and (18) in (19), we obtain

$$\Delta(\lambda) = i\exp(i\pi/4)(\exp(\rho R_-)[1] + \exp(-\rho R_-)[1])$$

$$\times(\exp(i\rho(R_+ - 2\xi)[1] - \exp(-i\rho(R_+ - 2\xi)[1]), \quad \rho \in \overline{S_0 \cup S_{-1}}, \quad |\rho| \to \infty, \qquad (20)$$

where $< y, z >:= yz' - y'z$.
The function $\Delta(\lambda)$ has a countable set of zeros $\lambda_n = \rho_n^2$, $n \in Z$ such that

$$\rho_n = \frac{i\pi}{R_-}(n + \frac{1}{2}) + O(\frac{1}{n}), \quad n \to \infty,$$

$$\rho_n = \frac{n\pi}{R_+ - 2\xi} + O(\frac{1}{n}), \quad n \to -\infty.$$

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