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Diffuse Element Kansa Method

Mehrzad Ghorbani

Department of Mathematics
Iranshahr University, Iranshahr, Iran
ghorbany@hamoon.usb.ac.ir
mehrzadghorbany@yahoo.com

Abstract

Diffuse Element Kansa Method (DEKM) is a new method for numerical solution of PDEs that is more exact than Kansa's method. In this method, a radial base function, a weight function and two group of nodes, one category for building our approximation space in the artificial domain and the other category for finding the coefficient functions in the domain is needed. The coefficient functions are found by moving least square (MLS) method.

Keywords: moving least square (MLS) method, diffuse element method (DEM), Kansa's method, radial base functions (RBFs), meshless methods, nodal distribution

1 Introduction

In the last fourteen years ago, meshless methods were a relatively new numerical approach that absorbed many researchers, specially finite element people. These methods approximate PDEs using scattered data particles, and also have ability of simulating unsteady problems with shocks. Two main advantages of meshless methods are: computational efficiency by avoiding mesh generation or remeshing, and high smoothing degree of approximation. These methods inherit some of the finite element properties such as locality and reduce computational effort for problems with complex domain and moving fronts. But these methods have two important disadvantages: the high computational volume of finding an inverse of a matrix, and the lack of the interpolation property which enters difficulties in enforcing essential boundary conditions.

In 1968, Shepard [10] presented a meshless interpolation for irregularly-spaced data points. After introducing the MLS method by Lancaster and Salkauskas in 1981 [8], Nayroles et. al. [9], employed a local form of this approximation for numerical solution of some PDE's using nonsingular weight

functions with compact support. In spite of disadvantage of having to find complete derivatives of their basis functions, they obtained acceptable results and named their method diffuse element method (DEM). The DEM has the following properties: (1) locality of the finite element method; (2) increases degree of smoothing in the approximation; (3) avoids using time-consuming mesh generation process; (4) the basis function derivatives are not complete; (5) essential boundary conditions cannot be satisfied exactly and need another methods; (6) approximation is based on an irregular distribution of nodes; (7) smoothing degree of the approximation is directly depend on the smoothing degree of the weights.

In 1994, Belytschko et. al., [1] generalized the DEM and introduced element-free Galerkin (EFG) method. Some of properties of the EFG method are: (1) high accuracy; (2) use of complete derivatives; and (3) relative to the DEM, the EFG method is computationally more expensive.

Approximation of scattered data based on radial base functions (RBFs), is another meshless method. These type of approximations similar to the others, can be applied for approximating PDEs. In these methods, a radial function is core for approximation space and this space is made by translating a standard radial function with zero as its center (core), to all of the space particles. Here, we present an interesting approximation space using the nodes that most of them are selected out of real domain and the others, are selected in the domain.

One of the best advantages of meshless methods based on RBFs is decrease of computational volume that arises from changing multidimensions to one dimension. Kansa, [7] is the first person that applied an approximation by RBFs (pseudo interpolation) to the PDEs.

In this paper, we introduce a generalized Kansa method based on both the MLS method and incomplete derivatives that used in the DEM by Nayroles et. al. [9]. This method is very simple and need not any high previous knowledge of meshless methods, it has a short algorithm and the result of this method is a direct approximation of the related model problem. This means that enforcement of boundary conditions is done in the body of our new method. This task is very important, because all of the meshless methods don't have interpolation property, therefore they need to a slack method for enforcing their boundary conditions and completing these methods.

The rest of our paper is structured as follows: Section 1, introduces the MLS method and in Section 2 the DEM is reviewed. In Section 3, we explains the RBFs and Kansa's method. Section 4, introduces our new method i.e., *Diffuse Element Kansa method* (DEKM). In Section 5, we present two 2–D steady numerical examples of Laplace and Poisson's equation on the unit square. Section 6, presents our concluding remarks.

2 Moving Least Square Method (MLS)

Let $u : \Omega \longrightarrow I\!R$, where $\Omega \subset I\!R^d$, $d = 1, 2, 3$ be an unknown continuous function that we try to approximate it by having some data point of it. Given $\mathbf{x}_j \in \Omega$, $j = 1, 2, \ldots, n$, an irregular distribution of nodes in the domain and $u_j = u(\mathbf{x}_j)$. Let $\mathbf{P}(\mathbf{x})$ be a given m dimensional vector, for example in 1–D case, let $\mathbf{P}^T(x) = \{1, x, \ldots, x^{m-1}\}$. Define the following local approximation

$$\tilde{u}_{\mathbf{y}}(\mathbf{x}) = \mathbf{P}^T(\mathbf{x})\ \mathbf{a}(\mathbf{y}), \tag{1}$$

where $\mathbf{y} \in \Omega$ is an arbitrary fixed point and its coefficient vector $\mathbf{a}(\mathbf{y}) = [a_1(\mathbf{y}), a_2(\mathbf{y}),\ \ldots,\ a_m(\mathbf{y})]^T$ should be found. Let $w_j(\mathbf{x})$, $j = 1, 2, \ldots, n$ be the weights are made by a standard form and translated on the points. By minimizing the following weighted discrete square of local error functional

$$\begin{aligned} \mathbf{J}(\mathbf{a}(\mathbf{y})) &= \|u(\cdot) - \tilde{u}_{\mathbf{y}}(\cdot)\|_w^2, \\ &= \sum_{j=1}^{n} w(\mathbf{y} - \mathbf{x}_j)\,(u_j - \tilde{u}_{\mathbf{y}}(\mathbf{x}_j))^2, \\ &= \sum_{j=1}^{n} w(\mathbf{y} - \mathbf{x}_j)\,(u_j - \mathbf{P}^T(\mathbf{x}_j)\,\mathbf{a}(\mathbf{y}))^2, \end{aligned} \tag{2}$$

with respect to the coefficient vector $\mathbf{a}(\mathbf{y})$, we will have the following system:

$$\mathbf{A}(\mathbf{y})\,\mathbf{a}(\mathbf{y}) = \mathbf{F}(\mathbf{y})\,\mathbf{u}, \tag{3}$$

where

$$\begin{aligned} \mathbf{A}(\mathbf{y}) &= \mathbf{B}\,\mathbf{W}(\mathbf{y})\,\mathbf{B}^T, \\ \mathbf{F}(\mathbf{y}) &= \mathbf{B}\,\mathbf{W}(\mathbf{y}), \\ \mathbf{B} &= \{\mathbf{x}_j^{i-1}\}, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \\ \mathbf{W}(\mathbf{y}) &= diag(w(\mathbf{y} - \mathbf{x}_1), w(\mathbf{y} - \mathbf{x}_2), \ldots, w(\mathbf{y} - \mathbf{x}_n)), \\ \mathbf{u} &= [u(\mathbf{x}_1), u(\mathbf{x}_2), \ldots, u(\mathbf{x}_n)]^T. \end{aligned}$$

Then the local approximation (1) becomes

$$\tilde{u}_{\mathbf{y}}(\mathbf{x}) = \Phi_{\mathbf{y}}^T(\mathbf{x})\,\mathbf{u}, \tag{4}$$

and its related global approximation will be

$$\tilde{u}(\mathbf{x}) = \Phi^T(\mathbf{x})\,\mathbf{u} = \sum_{j=1}^{n} \phi_j(\mathbf{x})\,u_j, \tag{5}$$

where in it the vector base function in the global form with components $\{\phi_j(\mathbf{x})\}_{j=1}^n$ is

$$\Phi^T(\mathbf{x}) = \mathbf{P}^T(\mathbf{x})\,\mathbf{A}^{-1}(\mathbf{x})\,\mathbf{F}(\mathbf{x}). \tag{6}$$

3 Diffuse Element Method (DEM)

All of the meshless fittings such as the MLS method can be applied to the numerical solution of PDEs. Lancaster & Salkauskas [8] developed the MLS method as the generalization of the Taylor series method. Nayroles et. al., [9] introduced diffuse element approximation (DEA) as a smart change of the polynomials with constant coefficients that are defined on the elements in the finite element literature to the polynomials with variable coefficients defined on overlapped neighborhoods centered on some finite number of points named particle. The elements such as the triangles are changed to the neighborhoods that have supports for the weights. The size of these supports can be constant or variable and the bases built by the MLS method are completely meshless. Nayroles et. al., [9] used the following local and incomplete derivatives

$$\frac{\partial \Phi^T}{\partial x_i}(\mathbf{x}) = \frac{\partial \mathbf{P}^T}{\partial x_i}(\mathbf{x})\, \mathbf{A}^{-1}(\mathbf{x})\ \mathbf{F}(\mathbf{x}), \quad i = 1, 2, \ldots, d, \tag{7}$$

which are local x_i th derivatives of the base functions that are in the approximation (1). For large number of particles the DEM, gives relatively satisfactory approximations. The smoothing degree of the DEM approximation is directly depend on the weights. In the Element Free Galerkin method (EFG), Belytschko et. al., [1, 2], used the complete derivatives of the MLS bases or differentiation of the global form of the approximation (5) in the following form,

$$\begin{aligned} \frac{\partial \Phi}{\partial x_i}(\mathbf{x}) &= \frac{\partial \mathbf{P}^T}{\partial x_i}(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\, \mathbf{F}(\mathbf{x}) + \mathbf{P}^T(\mathbf{x})\, \mathbf{A}^{-1}(\mathbf{x})\, (\frac{\partial \mathbf{F}}{\partial x_i}(\mathbf{x}) - \\ &\quad \frac{\partial \mathbf{A}}{\partial x_i}(\mathbf{x})\, \mathbf{A}^{-1}(\mathbf{x})\, \mathbf{F}(\mathbf{x})), \quad i = 1, 2, \ldots, d. \end{aligned} \tag{8}$$

The use of the complete derivatives in numerical solution of PDEs gives an increasing accuracy, specially for problems that have high active subregion such as crack and shock and also raises the computational volume and complexity of the systems. One can employ and mix both the DEM and the EFGM for approximating a problem, this will have an intermediate computational volume.

4 Approximation based on the radial base functions and the Kansa's method

The radial base functions (RBFs) are used to approximate scattered data and construct another powerful meshless approximation. In 1971, Hardy [6] for the first time employed a typical RBF named multi-quadric RBF (MQ-RBF)

for approximating scattered data. These type of approximations, similar to another, can be applied for approximating PDEs. In these methods, approximation space is constructed by translating one standard radial function that its center is zero, to all of the other particles.

The approximation by RBFs is as follows:

Let $\{\mathbf{x}_j, u_j\}_{j=1}^n$ be a set of given disjoint scattered points and their values distributed in a domain. Then a RBF must be selected experimentally suitable for the model problem. The most famous RBFs are:

- Polyharmonic Spline: $\phi(r) = r^{2k-1}$.
- Direct Multiquadric: $\phi(r) = (c^2 + r^2)^{1/2}$.
- Reciprocal Multiquadric: $\phi(r) = (c^2 + r^2)^{-1/2}$.
- Gaussian: $\phi(r) = \exp(-(c\, r)^2)$.
- Thin Plate Spline: $\phi(r) = r^{2k}\, \ln(r)$.
- Wendland: $\phi(r) = (1-r)_+^m\, p(r)$. The cut-off function $(\cdot)_+$ ensures the compact support of these functions. In fact, these type of RBFs are piecewise polynomials.

All of these RBFs, are globally supported, except the Wendland RBF, which has local compact support. In these functions, $r = \| \mathbf{x} \| /\rho,\ c > 0$ which is a smoothing parameter and its suitable amount is an open problem, ρ is radius of a circular support, $k,\ m \in I\!N$ and $p(r)$ is a polynomial of r. For example, Wendland $C^2(\Omega)$ locally compact support RBF is $\phi(r) = (1-r)_+^4\ (1+4\,r)$. All of the RBFs have the following properties:

1. Univariate and insensitive to space dimension.
2. They are meshless.
3. The RBFs are usually symmetric.
4. $\phi : [0, \infty) \longrightarrow I\!R$.
5. The Kronecker delta property doesn't satisfy or $\phi(r_{ij}) \neq \delta_{i,j}$.

The approximation space based on the RBFs is $\{\phi(r_j)\}_{j=1}^n$ in which $r_j = \| \mathbf{x} - \mathbf{x}_j \| /\rho$. Given the data $\{\mathbf{x}_j, u_j\}_{j=1}^n$ and by using this information, the RBF approximation will be in the following form,

$$\tilde{u}(\mathbf{x}) = \sum_{j=1}^{n} \phi(r_j)\, u_j. \tag{9}$$

One of the best advantages of meshless methods based on the RBFs with respect to another, is high decrease of computations from multidimensions to one dimension. The use of the globally supported RBFs, reaches to the large linear systems, poorly condition number, full and diagonally dominant matrix that means uniqueness of the solution.

Kansa, [7] was the first, who applied the RBFs for approximating PDE's. By substituting the approximation equation (9), into a model problem and collocating the residual on the particles, one can approximate the model problem by this method. It is important to know that choosing and finding the RBFs proportional to a problem, increases accuracy of the method. By appending a space such as $\{p_i(\mathbf{x})\}_{i=1}^m$, to the approximation space $\{\phi(r_j)\}_{j=1}^n$, an improved form of this method based on an enrichment space is found. Therefore, the new approximation space will be $\{\phi(r_j)\}_{j=1}^n \bigcup \{p_i(\mathbf{x})\}_{i=1}^m$. Usually, enrichment part of this space is constructed by polynomials, i. e. $p_i(\mathbf{x}) = \mathbf{x}^\alpha, \quad i = 1, 2, \ldots, m$, and α is a multi-index, but this space can be extended to singular functions and functions containing PDE activities. Based on the enriched space the new and generalized approximation will be in the following form,

$$\tilde{u}(\mathbf{x}) = \sum_{j=1}^{n} c_j \, \phi(r_j) + \sum_{|\alpha|=1}^{m} d_\alpha \, \mathbf{x}^{\alpha-1} \tag{10}$$

In 1–D cases, the number of unknowns are $n + m + 1$. By collocating the approximation (10) on the particles, a system with $n + 1$ equations will be found (see [3, 4]).

5 Diffuse Element Kansa Method (DEKM)

Let Ω be an open domain of a model problem, Γ be its boundary, $\hat{\Omega}$ be an artificial domain greater than real domain Ω and, $\{\mathbf{x}_j\}_{j=1}^n$ be given or selected points that are chosen out of the real domain and are in the artificial domain.

Define the following local approximation

$$\tilde{u}_{\mathbf{y}}(\mathbf{x}) = \sum_{j=1}^{n} c_j(\mathbf{y})\phi(r_j), \tag{11}$$

where $\mathbf{y} \in \bar{\Omega} = \Omega \bigcup \Gamma$ is an arbitrary fixed point and $r_j = \| \mathbf{x} - \mathbf{x}_j \|$, $j = 1, 2, \ldots, n$. We consider the following model problem

$$\begin{aligned} \mathfrak{L}\, u(\mathbf{x}) &= f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ u(\mathbf{x}) &= g(\mathbf{x}), \quad \mathbf{x} \in \Gamma. \end{aligned} \tag{12}$$

By applying the differential operator $\mathfrak{L}$ of the model problem (12) to the local approximation (11), we will have

$$\mathfrak{L}\tilde{u}_{\mathbf{y}}(\mathbf{x}) = \sum_{j=1}^{n} c_j(\mathbf{y})\, \mathfrak{L}\phi(r_j), \tag{13}$$

where the operator $\mathfrak{L}$ are affected on the variable $\mathbf{x}$. After substituting the approximation (11) in the model problem (12), the following two residuals can be defined and constructed

$$\begin{aligned} R_1(\mathbf{x},\mathbf{y}) &= \mathfrak{L}\,\tilde{u}_{\mathbf{y}}(\mathbf{x}) - f(\mathbf{x}), \\ &= \sum_{j=1}^{n} c_j(\mathbf{y})\, \mathfrak{L}\phi(r_j) - f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \end{aligned} \tag{14}$$

$$\begin{aligned} R_2(\mathbf{x},\mathbf{y}) &= \tilde{u}_{\mathbf{y}}(\mathbf{x}) - g(\mathbf{x}), \\ &= \sum_{j=1}^{n} c_j(\mathbf{y})\, \phi(r_j) - g(\mathbf{x}), \quad \mathbf{x} \in \Gamma. \end{aligned} \tag{15}$$

As it can be seen, the residual R_1 is related to interior of the problem domain and the residual R_2 belong to its boundary. By selecting two category of points $\{\xi_{\mathbf{k}}\}_{k=1}^{n_I} \subset \Omega$ and $\{\eta_{\mathbf{k}}\}_{k=1}^{n_B} \subset \Gamma$, where $n_I,\, n_B \gg n$, we can use of the MLS method for finding the coefficient functions $c_j(\mathbf{x}), \quad j = 1,\, 2, \ldots,\, n$. Therefore, we must select a suitable weight function, such as cubic spline weight function

$$w(r) = \begin{cases} 2/3 - 4r^2 + 4r^3 & r \le 1/2, \\ 4/3 - 4r + 4r^2 - 4/3r^3 & 1/2 < r \le 1, \\ 0 & r > 1. \end{cases} \tag{16}$$

Now, we can define the following residual functional

$$\begin{aligned} \mathbf{J}(c_1(\mathbf{y}), c_2(\mathbf{y}), \ldots, c_n(\mathbf{y})) &= \|R_1(\cdot,\mathbf{y})\|_{w,\Omega}^2 + \|R_2(\cdot,\mathbf{y})\|_{w,\Gamma}^2, \\ &= \|\mathfrak{L}\tilde{u}_{\mathbf{y}}(\cdot) - f(\cdot)\|_{w,\Omega}^2 + \|\tilde{u}(\cdot) - g(\cdot)\|_{w,\Gamma}^2, \\ &= \sum_{k=1}^{n_I} w(\mathbf{y} - \xi_k)\, (\mathfrak{L}\tilde{u}_{\mathbf{y}}(\xi_k) - f(\xi_k))^2 \\ &+ \sum_{k=1}^{n_B} w(\mathbf{y} - \eta_k)\, (\tilde{u}_{\mathbf{y}}(\eta_k) - g(\eta_k))^2. \end{aligned} \tag{17}$$

By minimizing the above functional with respect to the coefficient functions $c_j(\mathbf{y}), \quad j = 1,\, 2, \ldots,\, n$, the following system will be found

$$(\mathbf{A}_1(\mathbf{x}) + \mathbf{A}_2(\mathbf{x}))\, \mathbf{c}(\mathbf{x}) = \mathbf{F}_1(\mathbf{x})\, \mathbf{f} + \mathbf{F}_2(\mathbf{x})\, \mathbf{g}, \tag{18}$$

where

$$\begin{aligned}
\mathbf{A}_1(\mathbf{x}) &= \mathbf{B}_1\,\mathbf{W}_1(\mathbf{x})\,\mathbf{B}_1^T,\\
\mathbf{A}_2(\mathbf{x}) &= \mathbf{B}_2\,\mathbf{W}_2(\mathbf{x})\,\mathbf{B}_2^T,\\
\mathbf{F}_1(\mathbf{x}) &= \mathbf{B}_1\,\mathbf{W}_1(\mathbf{x}),\\
\mathbf{F}_2(\mathbf{x}) &= \mathbf{B}_2\,\mathbf{W}_2(\mathbf{x}),\\
\mathbf{B}_1 &= \{\phi(r_{ij})\}, \quad i=1,2,\ldots,n, \quad j=1,2,\ldots,n_I, \quad r_{ij}=\|\mathbf{x}_i-\xi_j\|,\\
\mathbf{B}_2 &= \{\phi(r_{ij})\}, \quad i=1,2,\ldots,n, \quad j=1,2,\ldots,n_B, \quad r_{ij}=\|\mathbf{x}_i-\eta_j\|,\\
\mathbf{W}_1(\mathbf{x}) &= diag(w(\mathbf{x}-\xi_1), w(\mathbf{x}-\xi_2),\ldots,w(\mathbf{x}-\xi_{n_I})),\\
\mathbf{W}_2(\mathbf{x}) &= diag(w(\mathbf{x}-\eta_1), w(\mathbf{x}-\eta_2),\ldots,w(\mathbf{x}-\eta_{n_B})),\\
\mathbf{f} &= [f(\xi_1), f(\xi_2),\ldots,f(\xi_{n_I})]^T,\\
\mathbf{g} &= [g(\eta_1), g(\eta_2),\ldots,g(\eta_{n_B})]^T.
\end{aligned}$$

Because of selecting a suitable weight, the coefficient matrix will be diagonally dominant, and so the final system (18) will have a unique solution. Finally, the numerical solution of the model problem (12) can be shown in the following global form

$$\tilde{u}(\mathbf{x}) = \sum_{j=1}^{n} c_j(\mathbf{x})\phi(r_j). \tag{19}$$

The smoothing degree of this new approximation depends on the weight w and its selected RBF. This means that if $w \in C^{\ell_1}(I\!R^d)$ and $\phi \in C^{\ell_2}(I\!R^d)$, then $\tilde{u} \in C^{min\{\ell_1,\ell_2\}}(I\!R^d)$.

We called our new method Diffuse Element Kansa Method (DEKM). This method uses of advantages of both the Kansa's method and the DEM. It also uses of the MLS method and the artificial boundary for improving accuracy of the Kansa's method [7]. Here, we didn't use of the Element Free Galerkin (EFG) method [1, 2] and complete or global derivatives in constructing the DEKM, because high prices of computations can degenerate our new method.

6 Numerical Examples

Let $\overline{\Omega} = [0,1]\times[0,1]$ be closure of the real domain and $\overline{\widetilde{\Omega}}$ be closure of the artificial domain which is greater than unit square domain. In this section, we approximate a Laplace problem and a Poisson problem, and we use of a real and artificial rectangular domain.

Example 6.1 *We consider the following Laplace 2–D model problem:*

$$\begin{aligned}
\Delta\, u(x,y) &= 0, \qquad (x,y)\in\Omega,\\
u(0,y) &= u(1,y) = u(x,1) = 0,
\end{aligned}$$

$$u(x,0) \;=\; \sin(\pi x), \quad x \in [\,0,\,1], \tag{20}$$

Exact solution of this problem is $u(x,y) = \sin(\pi x) \quad (\cosh(\pi\, y) - \coth(\pi)$ $sinh(\pi\, y))$, *where* $(x,y) \in \overline{\Omega}$. *The artificial domain is* $\widehat{\Omega} = [-1,2] \times [-1,2]$, *and the MLS weight function is the inverse square singular weight function* $w(x,y,\xi,\eta) = 1/((x-\xi)^2+(y-\eta)^2)$ *which have global support and is decreasing radially from its center* (ξ,η). *The RBF that in this example we used is the reciprocal multiquadric (RMQ)* $\phi(r) = 1/\sqrt{r^2+c^2}$ $c = 0.9$ *as its smoother parameter. We selected* 11 *number of points that distributed uniformly on each line of the rectangular artificial boundary and therefore, the number of base functions that we used as the approximation space is* 40 *number. The number of points that we used on the real boundary for minimizing the discrete moving least square or MLS functional (17) is* 80 *number, that is* 21 *number of points on each line of the unit square domain. Figure 1 shows approximation of the Laplace problem (20) by the DEKM and Figure 2 accuracy of the two linear errors* $\tilde{u}(x,0.1) - u(x,0.1)$ $\tilde{u}(x,0.5) - u(x,0.5)$ *for* $x \in [\,0,\,1]$.

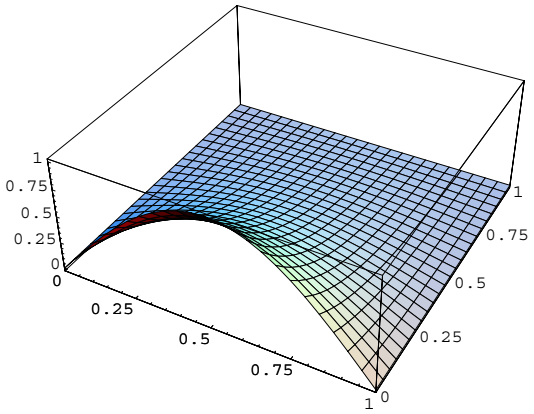

Figure 1: Approximation of the Laplace equation under Dirichlet boundary conditions of example 1, by the DEKM.

Example 6.2 *we consider the following Poisson model problem*

$$\begin{aligned} \Delta\, u(x,y) &= -2\ \pi^2\ \cos(\pi x)\ \sin(\pi y), \qquad (x,y) \in \Omega, \\ u(x,0) &= u(x,1) = 0,\ x \in [\,0,\,1], \\ u_x(0,y) &= u_x(1,y) = 0, \quad y \in [\,0,\,1]. \end{aligned} \tag{21}$$

The analytical solution of this problem is $u(x,y) = \cos(\pi x)\ \sin(\pi y)$. *In this example, we used an artificial boundary* $\widehat{\Omega} = [-0.5, 1.5] \times [-0.5, 1.5]$, *and the*

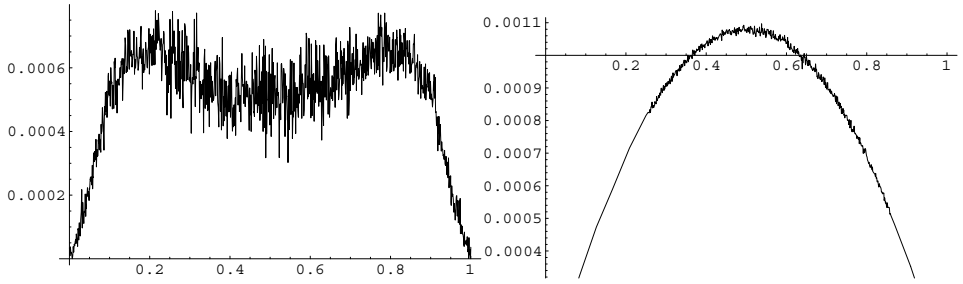

Figure 2: Linear error of the DEKM approximation at $x = 0.1$ and $x = 0.5$.

weight function $w(x, y, \xi, \eta) = 1/((x-\xi)^2 + (y-\eta)^2)$. *There are* 11 *uniformly spaced points on each line of the rectangular boundary and the total number of points that were used on the artificial boundary are* 40 *and is equal to the approximation space dimension. The total number uniformly spaced points used on the real boundary for using in the MLS method are* 140. *The RBF of this example is the Thin Plate Spline (TPS)* $\phi(r) = r^2 \ln(r)$. *This RBF have not any parameter. Figure 3 shows an approximation of this example and Figure 4 shows its linear error for the two constant* $y = 0.2$ *and* $y = 0.5$.

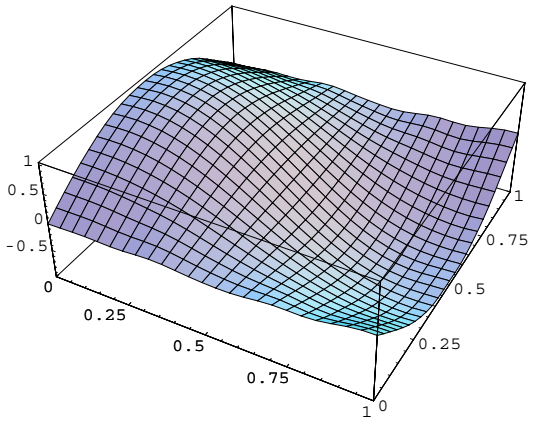

Figure 3: Approximation of the Poisson equation of Example 2 by the DEKM.

7 Concluding Remarks

- Computational task of the DEKM is more than the Kansa's method. Because, the inverse of a $n \times n$ matrix in the system (18) must be calculated for each evaluation point.

- The DEKM require neither domain nor boundary discretization, so, it is a meshless method. The weight function or RBF or both, may have

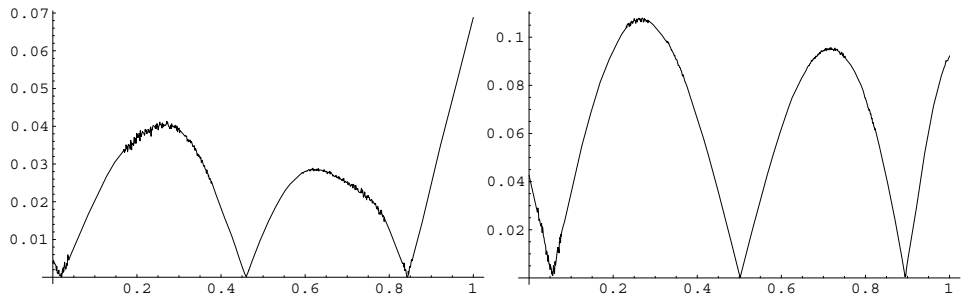

Figure 4: Linear error of the approximation at $y = 0.2$, $y = 0.5$ under conditions of Example 2.

local support. This localization reflexes into the approximation and the size of the supports must be selected carefully.

- Radial form of the DEKM is intensive to dimensionality of the problem and thus is very attractive to high dimensional problems.

- The DEKM need not to any numerical quadrature. In most of the meshless methods, integration of the inner products need a numerical quadrature rule and consequently this numerical quadrature need quadrature points or discretization.

- Relation between a real domain and its artificial domain and type of the artificial boundary have not a strict definition and it is experimental.

- In this paper, we applied this method to the static problems and can be extended to the time dependent problems.

- Computational cost of the DEKM is relatively inexpensive with respect to domain or mesh methods.

- The solution can be extended outside of its real domain and be defined in its artificial domain. This is similar to the space that is needed for a tent and its useable space.

- During minimizing the functional (17), one can select two type of the weights, one for minimizing L^2-norm of the domain residual (14) and the other for minimizing L^2-norm of the boundary residual (15).

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