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Uniform Convergence of Schwarz Method for Variational Inequalities

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Abstract

In this paper we study variational inequalities using the Schwarz method. The main idea of this method consists in decomposing the domain in two subdomains. We give a simple proof for the main result concerning error estimates in uniform norm, using S. Zhou [14] geometrical convergence and Courtey-Dumont [6] uniform convergence.

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1 Introduction

Schwarz method has more than one century of history. Since its invention by Herman Amandus Schwarz in 1890, to answer merely theoretical requirements. Sixty years later, we find it in the work of Keith Miller, like an effective tool of simulation of the problems to the limits.
With parallel calculators, this rediscovery of the method as algorithm of calculation, combined to the survey, based on an modern variational approach Pierre-Louis Lions, was the starting point of an intense research activity to develop this tool of calculation.
In this work, we are interested in the analysis of error estimates in uniform norm for the variational inequality (V.I). G.H. Meyer [13] discussed their numerical solution, J. Zeng and S. Zhou [14] obtained the geometrical convergence and the convergence rate of the algorithm. M. Boulbrachene and S. Saadi [1] showed that the discretization on each subdomain converges in uniform norm. Ph Cortey-Dumont [6] proved uniform convergence of finite element approximation for variational inequalities and M. Haiour and E. Hadidi [8] proved uniform convergence of noncoercive variational inequalities. In section 2, we

give our continuous V.I problem, we study the existence and the uniqueness of the solution. In section 3, we consider the discrete problem and we introduce the discrete Schwarz method. In section 4, we give a simple proof for the main result concerning error estimates in the L^∞ norm for the problem studied, taking into account the combination of S. Zhou [14] geometrical convergence and Cortey-Dumont [6] uniform convergence of finite element approximation for variational inequalities.

2 The Continuous Problem

2.1 Notations and Assumptions

Let Ω be a convex domain in $\mathbb{R}^2$ with sufficiently regular boundary $\partial\Omega$.
Let's consider the bilinear form as follows

$$a(u,v) = \int_\Omega \nabla u \nabla v dx \tag{1}$$

The linear form

$$(f,v) = \int_\Omega f v dx \tag{2}$$

Such that

$$f \in L^\infty(\Omega), f \geq 0 \tag{3}$$

Where on Ω

$$K_{(\Psi,g)} = \left\{ v \in H^1(\Omega) / v - g \in H_0^1(\Omega), 0 \leq v \leq \Psi \right\} \tag{4}$$

With the obstacle Ψ and g is a regular function defined on $\partial\Omega$.

$$\Psi \in W^{2,\infty}(\Omega), p > 2; o \leq g \leq \Psi \tag{5}$$

2.2 Elliptic Variational Inequalities

Find $u \in K_{(\Psi,g)}$ the solution of

$$a(u, v-u) \geq (f, v-u), \forall v \in K_{(\Psi,g)} \tag{6}$$

J. Hannouzet in [9] gives the result of existence, unicity and the regularity of solution.

Theorem 2.1 *([9]) Under the conditions (1) to (5), the problem (6) has an unique solution $u \in K_{(\Psi,g)}$. Moreover we have*

$$u \in W^{2,p}(\Omega), 2 < p < \infty \tag{7}$$

3 The Discrete Problem

3.1 Discretization

Let $V_{h_i} = V_{h_i}(\Omega_i)$ be the space of continuous piecewise linear functions on τ^{h_i} which vanish on $\partial\Omega \cap \Omega_i$.
For $w \in C\left(\overline{\Lambda}_i\right)$, we define the following space

$$V_{h_i}^{(w)} = \{v \in V_{h_i}/v = 0 \ \texttt{on}\partial\Omega \cap \Omega_i; v = \pi_{h_i}(w) \ \texttt{on}\Lambda_i\} \tag{8}$$

where π_{h_i} denotes the interpolation operator on Λ_i. For $i = 1, 2$, let τ^{h_i} be a standard regular finite element triangulation in Ω_i, h_i being the meshsize. We suppose that the two triangulations are mutually independent on $\Omega_1 \cup \Omega_2$ a triangle belonging to one triangulation does not necessarily belong to the other. We assume that the corresponding matrices resulting from the discretization of problem are M-matrices ([14]).

3.2 Position of The Discrete Problem

The discrete problem is find $u_h \in H_0^1(\Omega)$ the solution of

$$\left\{ \begin{array}{c} a(u_h, v_h - u_h) \geq (f, v_h - u_h) \\ u_h \leq r_h\Psi, v_h \leq r_h\Psi \end{array} \right. \tag{9}$$

Let $\bar{u}_h$ be the solution of

$$a_h(\bar{u}_h, v_h) = a(u, v_h) \tag{10}$$

where u is the solution of the continuous variational inequality.
We suppose that

$$\|u - \bar{u}_h\| \leq Ch^2|lnh|^2 \tag{11}$$

and

$$\|u - r_h u\| \leq Ch^2|lnh|^2 \tag{12}$$

We give an assumption related to (5), we take $\rho = \psi_{|B(x_0;Ch)}$.
Thus $\forall x \in B(x_0; Ch)$ such that $u(x_0) = \psi(x_0)$ then

$$|u(x) - \rho(x)| \leq Ch^2|lnh|^2 \tag{13}$$

Theorem 3.1 *([6]) Under the conditions in (1) to (5),(10) to (13) and discrete maximum principle, there exists a constant C_1 independent of h such that*

$$\|u - u_h\|_{L^\infty(\Omega)} \leq C_1 h^2 |\ln h|^2 \tag{14}$$

Remark 3.2 *([4])The nature of our problem is the existence of the free boundary between*

$$\Omega_0 = \{x \in \Omega / u(x) = \psi(x)\} \tag{15}$$

and $C\Omega_0$ *the complementary set of* Ω_0 *in* Ω*. Let the following set, that is the discrete approximation of the coincidence set*

$$\Omega_h = \{x \in \cup T_h / T_h \cap \Omega_0 \neq \phi\} \tag{16}$$

The lemma that follows is given in ([6], pages 51 and 52), we take the demonstration to enrich this work and to have a good enlightenment.

Lemma 3.3 *Under the conditions in (1) to (5) and (12),(13) we have the following estimates*

$$\|u - \Psi\| \leq Ch^2 |\ln h|^2 \tag{17}$$

and

$$\|\Psi - r_h \Psi\| \leq Ch^2 |\ln h|^2 \tag{18}$$

Proof. Given T_h in Ω_h, there exists x_0 belonging to T_h such that

$$u(x_0) = \psi(x_0)$$

Moreover

$$T_h \subset B(x_0; Ch)$$

so for every x in T_h

$$u(x) \leq \psi(x)$$

because u is the solution of the variational inequality and by assumption (5) and (13), we know

$$u(x) \leq \rho(x)$$

so that

$$u(x) \leq \psi(x) \leq u(x) + Ch^2 |lnh|^2$$

thus

$$\|u - \Psi\| \leq Ch^2 |\ln h|^2$$

The seconde estimate follows from (13) as

$$\|u - r_h u\|_{L^\infty(\Omega)} \leq Ch^2 |lnh|^2$$

and

$$r_h u \leq r_h \psi \leq r_h u + Ch^2 |lnh|^2$$

Of what precedes, one can pull to the lemma that plays an essential role in our result.

Lemma 3.4 *Under the conditions in (1) to (5), (10) to (13) and discrete maximum principle, there exists a constant C_2 independent of h such that*

$$\|u_h - r_h\Psi\|_{L^\infty(\Omega_i)} \leq C_2 h^2 \left|\ln h\right|^2 \tag{19}$$

Proof. We use estimations (14), (17) and (18)
Thus

$$\begin{aligned} \|u_h - r_h\Psi\| &\leq \|u_h - u + u - \Psi + \Psi - r_h\Psi\| \\ &\leq \|u_h - u\| + \|u - \Psi\| + \|\Psi - r_h\Psi\| \\ &\leq C_2 h^2 \left|\ln h\right|^2 \end{aligned}$$

3.3 Domain Decomposition Method

We decompose Ω into two overlapping polygonal subdomains Ω_1 and Ω_2 such that

$$\Omega = \Omega_1 \cup \Omega_2 \tag{20}$$

In theorem 2.1, the solution u satisfies the condition of the following local regularity

$$u \;/\; \Omega_i \in W^{2,p}\left(\Omega_i\right), 2 \leq p < \infty \tag{21}$$

We denote $\partial\Omega_i$ the boundary of Ω_i and

$$\Lambda_1 = \partial\Omega_1 \cap \Omega_2, \Lambda_2 = \partial\Omega_2 \cap \Omega_1 \tag{22}$$

We assume that

$$\overline{\Lambda}_1 \cap \overline{\Lambda}_2 = \emptyset \tag{23}$$

where $f_i = f/\Omega_i, i = 1, 2$ and $u_i = u/\Omega_i, i = 1, 2$.
and

$$a_i\left(u, v\right) = \int_{\Omega_i} \nabla u \nabla v dx \tag{24}$$

3.4 The Discrete Schwarz Method

We give the discrete Schwarz method as follows.
Starting from

$$u_h^0 = r_h\Psi \tag{25}$$

We define the discrete sequence of Schwarz $(u_h^n)_{n\in\mathbb{N}}$ such that

$$\left\{ \begin{array}{c} u_{1h}^{n+1} \in V_{h_1}^{(u_{2h}^n)} \texttt{ is a solution of} \\ a_1\left(u_{1h}^{n+1}, v - u_{1h}^{n+1}\right) \geq \left(f_1, v - u_{1h}^{n+1}\right), \forall v \in V_{h_1}^{(u_{2h}^n)} \\ u_{1h}^{n+1} \leq r_h\Psi, v \leq r_h\Psi \end{array} \right. \tag{26}$$

and

$$\left\{ \begin{array}{c} u_{2h}^{n+1} \in V_{h_2}^{(u_{1h}^{n+1})} \texttt{ is a solution of} \\ a_2\left(u_{2h}^{n+1}, v - u_{2h}^{n+1}\right) \geq \left(f_2, v - u_{2h}^{n+1}\right), \forall v \in V_{h_2}^{(u_{1h}^{n+1})} \\ u_{2h}^{n+1} \leq r_h \Psi, v \leq r_h \Psi \end{array} \right. \tag{27}$$

S. Zhou in [14] gives the algebraic form of the discrete algorithm and the geometrical convergence of the sequences.

Theorem 3.5 *([14]) Under the conditions in (1) to (5), the sequences*

$$\left(u_{1h}^{n+1}\right) \textit{and} \left(u_{2h}^{n+1}\right); n \geq 0$$

converge geometrically to the unique solution u of the discrete problem, such that $\exists \theta \in \left]0,1\right[, \forall n \geq 0$

$$\left\| u_{ih} - u_{ih}^{n+1} \right\|_{L^\infty(\Omega_i)} \leq (\theta)^n \left\| u_h - u_h^0 \right\|_{L^\infty(\Lambda_i)}; i = 1, 2. \tag{28}$$

4 $L^\infty-$Error Estimate

4.1 $L^\infty-$Error Estimate

We finish by the main result.

Theorem 4.1 *There exists a constant C independent of h such that*

$$\left\| u_i - u_{ih}^{n+1} \right\|_{L^\infty(\Omega_i)} \leq Ch^2 \left|\ln h\right|^2; i = 1, 2. \tag{29}$$

Proof. We have

$$\left\| u_i - u_{ih}^{n+1} \right\|_{L^\infty(\Omega_i)} \leq \left\| u_i - u_{ih} \right\|_{L^\infty(\Omega_i)} + \left\| u_{ih} - u_{ih}^{n+1} \right\|_{L^\infty(\Omega_i)}$$

We use theorems 3.1 and 3.5

$$\leq C_1 h^2 \left|\ln h\right|^2 + (\theta)^n \left\| u_h - u_h^0 \right\|_{L^\infty(\Lambda_i)}$$

$$\leq C_1 h^2 \left|\ln h\right|^2 + (\theta)^n \left\| u_h - r_h \Psi \right\|_{L^\infty(\Lambda_i)}$$

and the lemma 3.4

$$\leq C_1 h^2 \left|\ln h\right|^2 + (\theta)^n C_2 h^2 \left|\ln h\right|^2$$

$$\leq (C_1 + (\theta)^n C_2) h^2 \left|\ln h\right|^2$$

Therefore one gets

$$\left\| u_i - u_{ih}^{n+1} \right\|_{L^\infty(\Omega_i)} \leq Ch^2 \left|\ln h\right|^2 \quad \square$$

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