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Bi-Frobenius Algebras in Left Yetter-Drinfeld Module Categories*

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Abstract: Bi-Frobenius algebras (A, ϕ, t, ψ) in the left Yetter-Drinfeld module category ${}_H^H\mathcal{YD}$ are considered. The dual algebra of bi-Frobenius algebras in ${}_H^H\mathcal{YD}$ are also bi-Frobenius algebras in ${}_H^H\mathcal{YD}$. The module and comodule structure of $\phi \in \int_{A^*}^r$, $t \in \int_A^r$, modular function α and modular element g are given. The Radford's antipode ψ^4 -formula for bi-Frobenius algebras in ${}_H^H\mathcal{YD}$ is also given.

Key words: Hopf algebra; Frobenius algebra; Yetter-Drinfeld module category

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0 Introduction

Throughout this paper, k denotes a fixed field. Let H be a Hopf algebra over k with a bijective antipode S, $\overline{S}$ be the inverse of S. We use Sweedler's notations in Hopf algebra[1].

By a Frobenius algebra we mean a finite dimensional associative algebra A which has a non-degenerate linear function. Equivalently, A and its dual A^* are isomorphic as left A -module or right A -module. There is an interesting connection between Frobenius algebras and Hopf subalgebras, solution of the Yang-Baxter Equation, the Jones polynomials, and 2-dimensional topological quantum field theories[2].

A double Frobenius algebra is a vector space with two Frobenius algebra structure such that they are coupled in a certain way, as was recently introduced by Koppinen. Double Frobenius algebras are a common generalization of finite-dimensional Hopf algebras, adjacency algebras of (possibly non-commutative) association schemes and character-algebras. See [3] for details.

Recently, Doi and Takeuchi introduced bi-Frobenius algebras in [4]. Bi-Frobenius algebras

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are some essential subclasses of double Frobenius algebras. Roughly speaking, bi-Frobenius algebras are equivalent to double Frobenius algebras modulo some simplication. A bi-Frobenius algebra (A, ϕ, t, ψ) is a Frobenius algebra and Frobenius coalgebra with a bijective antipode ψ, see [4] 2.2. But a bi-Frobenius algebra need not be a bialgebra and the antipode need not be the inverse of id under the convolution product. Finite dimensional Hopf algebras are basic examples of bi-Frobenius algebras.

In recent years, with the development of braided Hopf algebras, many finite dimensional Hopf algebras are studied in braided tensor categories. Doi studied the Hopf modules in the left Yetter-Drinfeld module category in [5]. If L is a finite dimensional Hopf algebra in ${}_H^H\mathcal{YD}$, he proved that the dual algebra L^* has a right L-Hopf module structure which is not analogous to the usual one. Braided bi-Frobenius algebras are also introduced in [4]. Doi and Takeuchi gave two expressions of the Nakayama automorphism and proved the Radford's S^4-formula by braided graphes.

Since bi-Frobenius algebras are the generalization of finite dimensional Hopf algebras and ${}_H^H\mathcal{YD}$ is a braided tensor category, motivated by [4] and using the technique in [1] and [5], we study bi-Frobenius algebras in ${}_H^H\mathcal{YD}$. In this paper, some results in Hopf algebras in ${}_H^H\mathcal{YD}$ are generalized to bi-Frobenius algebras (A, ϕ, t, ψ) in ${}_H^H\mathcal{YD}$ and give another way to prove some results in [4]. In particular, if (A, ϕ, t, ψ) is a bi-Frobenius algebra in ${}_H^H\mathcal{YD}$, we prove that the dual algebra is also a bi-Frobenius algebra in ${}_H^H\mathcal{YD}$. But it does not hold in a k-linear abelian rigid monoidal category. We give the ψ^4-formula which does not involve the double ribbon map as Doi and Takeuchi's 16.3 in [4]. Our method is easier to be understood for those who are not familiar with braided graphes.

This paper is organized as follows:

In Section 1, we give the definition and examples of bi-Frobenius algebras in ${}_H^H\mathcal{YD}$.

In Section 2, we prove that the dual algebras (A^*, t, ϕ, ψ^*) are also bi-Frobenius algebras in ${}_H^H\mathcal{YD}$ with distinct multiplication and comultiplication from usual dual algeboas. We also get the module and comodule structure of $\phi \in \int_{A^*}^r$, $t \in \int_A^r$.

In Section 3, we give several automorphisms of bi-Frobenius algebras in ${}_H^H\mathcal{YD}$.

In Section 4, by comparing two expressions of Nakayama automorphism, we get the ψ^4-formula.

1 Preliminaries

We denote the left module action as $\rightarrow$ and the left H-comodule structure map as $\rho: V \rightarrow H \otimes V$, $\rho(v) = \sum v^{-1} \otimes v^0$.

Definition 1[6] A left Yetter-Drinfeld module category ${}_H^H\mathcal{YD}$ is a category whose objects are both left H-module and left H-comodule and satisfy the compatibility condition:

$$\sum h_1 v^{-1} \otimes h_2 \rightarrow v^0 = \sum (h_1 \rightarrow v)^{-1} h_2 \otimes (h_1 \rightarrow v)^0, \tag{1}$$

which is equivalent to

$$\sum (h \rightarrow v)^{-1} \otimes (h \rightarrow v)^0 = \sum h_1 v^{-1} S(h_3) \otimes h_2 \rightarrow v^0, \tag{2}$$

for all $V \in {}_H^H\mathcal{YD}$ v $\in$ V, h $\in$ H.

By [6], ${}_H^H\mathcal{YD}$ has a braiding on objects V, W as follows:

$$\tau: V \otimes W \rightarrow W \otimes V;\ v \otimes w \mapsto \sum v^{-1} \rightarrow w \otimes v^0,\ v \in V,\ w \in W.$$

$$\tau^{-1}: W \otimes V \rightarrow V \otimes W;\ w \otimes v \mapsto \sum v^0 \otimes \bar{S}(v^{-1}) \rightarrow w.$$

Lemma 1 A is an algebra in ${}_H^H\mathcal{YD}$ if and only if A is a left H -module algebra and left H -comodule algebra, i. e. (3) and (4) hold :

$$h \rightarrow (ab) = \sum (h_1 \rightarrow a)(h_2 \rightarrow b),\ h \rightarrow 1_A = \epsilon(h) 1_A. \tag{3}$$

$$\rho(ab) = \sum (ab)^{-1} \otimes (ab)^0 = \sum a^{-1} b^{-1} \otimes a^0 b^0,\ \rho(1_A) = 1_H \otimes 1_A. \tag{4}$$

A is a coalgebra in ${}_H^H\mathcal{YD}$ iff A is a left H -module coalgebra and left H -comodule coalgebra, i. e. (5) and (6) hold:

$$\Delta(h \rightarrow a) = \sum (h_1 \rightarrow a_1) \otimes (h_2 \rightarrow a_2),\ \epsilon(h \rightarrow a) = \epsilon_H(h) \epsilon_A(a). \tag{5}$$

$$\sum a^{-1} \otimes (a^0)_1 \otimes (a^0)_2 = \sum {a_1}^{-1} {a_2}^{-1} \otimes {a_1}^0 \otimes {a_2}^0,\ \sum a^{-1} \epsilon_A(a^0) = \epsilon_A(a) 1_H. \tag{6}$$

Proof It is easy to get from [6].

Definition 2 Let A be a finite dimensional algebra and coalgebra in ${}_H^H\mathcal{YD}$, t $\in$ A and $\phi \in A^*$. Suppose that

i) ϵ_A: $A \rightarrow k$ and $\epsilon_A(ab) = \epsilon_A(a)\epsilon_A(b)$.

ii) u_A: $k \rightarrow A$ and $\Delta(u_A(1)) = u_A(1) \otimes u_A(1)$.

iii) (A, ϕ) is a Frobenius algebra in ${}_H^H\mathcal{YD}$.

iv) (A, t) is a Frobenius coalgebra in ${}_H^H\mathcal{YD}$.

Define a map ψ: $A \rightarrow A$ by

$$\psi(a) = \sum \phi(t_1({t_2}^{-1} \rightarrow a)) {t_2}^0,\ \forall a \in A, \tag{7}$$

such that

v) ψ is an anti-algebra map in ${}_H^H\mathcal{YD}$.

vi) ψ is an anti-coalgebra map in ${}_H^H\mathcal{YD}$.

We call (A, ϕ, t, ψ) a bi-Frobenius algebra in ${}_H^H\mathcal{YD}$, ψ the antipode of A.

Remark 1) That ψ is an anti-algebra map and anti-coalgebra map in ${}_H^H\mathcal{YD}$ means that:

$$\psi(ab) = \sum \psi(a^{-1} \rightarrow b) \psi(a^0) = \sum (a^{-1} \rightarrow \psi(b)) \psi(a^0). \tag{8}$$

$$\Delta\psi(a) = \sum \psi({a_1}^{-1} \rightarrow a_2) \otimes \psi({a_1}^0) = \sum ({a_1}^{-1} \rightarrow \psi(a_2)) \otimes \psi({a_1}^0). \tag{9}$$

2) From [4] ψ is bijective, so $\psi(1) = 1$, $\epsilon \circ \psi = \epsilon$. We can show $\phi \in \int_{A^*}^r$, $t \in \int_A^r$. Indeed, by the definition of ψ ,

$$1=\psi(1)=\sum\phi(t_1(t_2{}^{-1}\rightarrow 1))t_2{}^0=$$
$$\sum\phi(t_1\epsilon(t_2{}^{-1}))t_2{}^0=\sum\phi(t_1)t_2=t\leftharpoonup\phi,$$

and hence $\phi(t_1)t_2=1$, i. e. $\phi\in\int_{A^*}^r$.

$$\epsilon(a)=\epsilon\circ\psi(a)=\sum\phi(t_1(t_2{}^{-1}\rightarrow a))\epsilon(t_2{}^0)=$$
$$\sum\phi(t_1\epsilon(t_2)a)=\phi(ta)=(\phi\leftharpoonup t)(a),$$

so $\forall a\in A$, $\phi(ta)=\epsilon(a)\phi(t)=\epsilon(a)$, and hence $t\in\int_A^r$.

3) $\dim\int_{A^*}^r=1$ and $\dim\int_A^r=1$ by [4] or [6].

Example 1 Finite dimensional Hopf algebras are bi-Frobenius algebras, so all finite dimensional Hopf algebras in ${}_H^H\mathcal{YD}$ are bi-Frobenius algebras in ${}_H^H\mathcal{YD}$.

Example 2 A Hopf algebra H is a left Yetter-Drinfeld module by considering H as a left H-module *via* the left adjoint action and as a left H-comodule *via* Δ. Let H be a finite dimensional commutative Hopf algebra in ${}_H^H\mathcal{YD}$ and A be a bi-Frobenius algebra in ${}_H^H\mathcal{YD}$, then $A\otimes H$ is a bi-Frobenius algebra with multiplication $(a\otimes k)(b\otimes l)=(ab\otimes kl)$ and comultiplication $\Delta(a\otimes k)=\sum(a_1\otimes k_1)\otimes(a_2\otimes k_2)$ for $a, b\in A$, $k, l\in H$. If we define the module by $h\rightarrow(a\otimes l)=\sum(h_1\rightarrow a)\otimes(h_2\rightarrow l)$ and comodule by $\rho(a\otimes l)=\sum a^{-1}l^{-1}\otimes a^0\otimes l^0$, then $(A\otimes H, \phi_A\otimes\phi_H, t_A\otimes t_H, \psi\otimes S)$ is a bi-Frobenius algebra in ${}_H^H\mathcal{YD}$.

2 The dual algebras

In [4], Doi and Takeuchi has pointed that if A is an object in a k-linear abelian rigid monoidal category, the dual A^* has no structure as an object in the category. But, if A is a bi-Frobenius algebra in ${}_H^H\mathcal{YD}$, we prove that A^* is also a bi-Frobenius algebra in ${}_H^H\mathcal{YD}$.

Firstly, A^* has the contragredient left H-module structure [7], i. e.

$$(h\rightarrow f)(a)=f(S(h)\rightarrow a),\ \forall h\in H, a\in A, f\in A^*. \qquad (10)$$

Since A is a left H-module, A^* has the transposed right H-comodule structure by [7] and it becomes a left H-comodule *via* $\overline{S}$. $\rho: A^*\rightarrow H\otimes A^*$, $\rho(f)=\sum f^{-1}\otimes f^0$, where

$$\sum f^{-1}f^0(a)=\sum\overline{S}(a^{-1})f(a^0),\ \forall a\in A. \qquad (11)$$

Lemma 2[5] For any left H-comodule V, define $\theta^{(2)}: A^*\otimes A^*\rightarrow(A\otimes A)^*$ by

$$\theta^{(2)}(f\otimes j)(x\otimes y)=\sum f(\overline{S}(y^{-1})\rightarrow x)j(y^0),\ \forall f, j\in A^*, x, y\in A.$$

Then $\theta^{(2)}$ is bijective.

Theorem 1 If (A, ϕ, t, ψ) is a bi-Frobenius algebra in ${}_H^H\mathcal{YD}$, then (A^*, t, ϕ, ψ^*) is also a bi-Frobenius algebra in ${}_H^H\mathcal{YD}$, with multiplication $m_{A^*}=(\Delta_A)^*\theta^{(2)}$, unit $1_{A^*}=\epsilon_A$, comultiplication $\Delta_{A^*}=(\theta^{(2)})^{-1}(m_A)^*$, counit $\epsilon_{A^*}: f\mapsto f(1_A)$ and antipode ψ^*. Explicitly, $\forall f$,

$g \in A^*$, x, $y \in A$ multiplication is given by

$$(fg)(x) = \sum f(g^{-1} \rightarrow x_1)g^0(x_2) = \sum f(\overline{S}(x_2{}^{-1}) \rightarrow x_1)g(x_2{}^0) \tag{12}$$

comultiplication $\Delta_{A^*}(f) = \sum f_1 \otimes f_2$ is given by

$$f(xy) = \sum f_1(f_2{}^{-1} \rightarrow x)f_2{}^0(y) = \sum f_1(\overline{S}(y^{-1}) \rightarrow x)f_2(y^0). \tag{13}$$

Proof In [4], Doi and Takeuchi pointed out that if (A, ϕ, t, ψ) is a bi-Frobenius algebra, then the dual algebra (A^*, t, ϕ, ψ^*) is also a bi-Frobenius algebra. Here we only need to prove that $A^* \in {}_H^H\mathcal{YD}$ and A^* is an algebra and a coalgebra in ${}_H^H\mathcal{YD}$.

1) Since

$$\begin{aligned}
\sum h_1 f^{-1}(h_2 \rightarrow f^0)(a) &= \sum h_1 f^{-1} f^0(S(h_2) \rightarrow a) \text{ by } (10) \\
&= \sum h_1 \overline{S}((S(h_2) \rightarrow a)^{-1}) f((S(h_2) \rightarrow a)^0) \text{ by } (11) \\
&= \sum h_1 \overline{S}(S(h_4)a^{-1}S^2(h_2)) f(S(h_3) \rightarrow a^0) \text{ by } (2) \\
&= \sum h_1 S(h_2)\overline{S}(a^{-1})h_4 f(S(h_3) \rightarrow a^0) \\
&= \sum \overline{S}(a^{-1})h_2 f(S(h_1) \rightarrow a^0) \\
&= \sum \overline{S}(a^{-1})h_2(h_1 \rightarrow f)(a^0) \text{ by } (10) \\
&= \sum (h_1 \rightarrow f)^{-1} h_2 (h_1 \rightarrow f)^0(a) \text{ by } (11).
\end{aligned}$$

So (2) holds, i. e. $A^* \in {}_H^H\mathcal{YD}$.

2) A^* is a left H-module algebra, i. e. (3) holds. In fact, we have

$$\begin{aligned}
(h \rightarrow (fg))(x) &= (fg)(S(h) \rightarrow x) \text{ by } (10) \\
&= \sum f(g^{-1} \rightarrow (S(h) \rightarrow x)_1)g^0((S(h) \rightarrow x)_2) \text{ by } (12) \\
&= \sum f(g^{-1}S(h_2) \rightarrow x_1)g^0(S(h_1) \rightarrow x_2).
\end{aligned}$$

and

$$\begin{aligned}
&\sum ((h_1 \rightarrow f)(h_2 \rightarrow g))(x) \\
&= \sum (h_1 \rightarrow f)((h_2 \rightarrow g)^{-1} \rightarrow x_1)(h_2 \rightarrow g)^0(x_2) \text{ by } (12) \\
&= \sum f(S(h_1)h_2 g^{-1} S(h_4) \rightarrow x_1)(h_3 \rightarrow g^0)(x_2) \text{ by } (2),(10) \\
&= \sum f(g^{-1}S(h_2) \rightarrow x_1)g^0(S(h_1) \rightarrow x_2).
\end{aligned}$$

$$(h \rightarrow 1_{A^*})(a) = 1_{A^*}(S(h) \rightarrow a) = \epsilon(S(h) \rightarrow a) = \epsilon(h)\epsilon(a) = \epsilon(h)1_{A^*}(a)$$

3) A^* is a left H-comodule algebra, i. e. (4) holds. Indeed,

$$\begin{aligned}
\sum (f^0 g^0)(x) f^{-1} g^{-1} &= \sum f^0(\overline{S}(x_2{}^{-1}) \rightarrow x_1)g^0(x_2{}^0) f^{-1} g^{-1} \text{ by } (12) \\
&= \sum f^0(\overline{S}(x_2{}^{-2}) \rightarrow x_1) f^{-1} g(x_2{}^0)\overline{S}(x_2{}^{-1}) \text{ by } (11) \\
&= \sum f(\overline{S}(x_2{}^{-3}) \rightarrow x_1{}^0)\overline{S}(\overline{S}(x_2{}^{-2})x_1{}^{-1}x_2{}^{-4})\overline{S}(x_2{}^{-1})g(x_2{}^0) \text{ by } (11) \\
&= \sum f(\overline{S}(x_2{}^{-1}) \rightarrow x_1{}^0)g(x_2{}^0)\overline{S}(x_1{}^{-1}x_2{}^{-2})
\end{aligned}$$

$$= \sum (fg)(x^0)\overline{S}(x^{-1}) \text{ by } (6),(12)$$

$$= \sum (fg)^0(x)(fg)^{-1} \text{ by } (11)$$

and $\rho(1_{A^*}) = \rho(\epsilon_A) = 1_H \otimes \epsilon_A = 1_H \otimes 1_{A^*}$.

4) A^* is a left H-module coalgebra, i.e. (5) holds.

$$(\theta^{(2)}\Delta_{A^*}(h \rightharpoonup f))(a \otimes b) = (\theta^{(2)}(\theta^{(2)})^{-1}(m_A)^*(h \rightharpoonup f))(a \otimes b)$$

$$= (h \rightharpoonup f)(ab)$$

$$= \sum f((S(h_2) \rightharpoonup a)(S(h_1) \rightharpoonup b)) \text{ by } (10),(3)$$

$$= \sum f_1(\overline{S}(S(h_3)b^{-1}S^2(h_1))S(h_4) \rightharpoonup a)f_2(S(h_2) \rightharpoonup b^0) \text{ by } (12)$$

$$= \sum f_1(S(h_1)\overline{S}(b^{-1}) \rightharpoonup a)f_2(S(h_2) \rightharpoonup b^0).$$

$$= \sum (h_1 \rightharpoonup f_1)(\overline{S}(b^{-1}) \rightharpoonup a)(h_2 \rightharpoonup f_2)(b^0)$$

$$= \theta^{(2)}(\sum (h_1 \rightharpoonup f_1) \otimes (h_2 \rightharpoonup f_2))(a \otimes b)$$

and $\epsilon_{A^*}(h \rightharpoonup f) = (h \rightharpoonup f)(1) = \epsilon(h)f(1)$.

5) A^* is a left H-comodule coalgebra, i.e. (6) holds, since

$$\sum f_1{}^{-1}f_2{}^{-1}\theta^{(2)}(f_1{}^0 \otimes f_2{}^0)(x \otimes y)$$

$$= \sum f_1{}^{-1}f_2{}^{-1}f_1{}^0(\overline{S}(y^{-1}) \rightharpoonup x)f_2{}^0(y^0)$$

$$= \sum \overline{S}((\overline{S}(y^{-2}) \rightharpoonup x)^{-1})\overline{S}(y^{-1})f_1((\overline{S}(y^{-2}) \rightharpoonup x)^0)f_2(y^0) \text{ by } (11)$$

$$= \sum \overline{S}(\overline{S}(y^{-2})x^{-1}y^{-4})\overline{S}(y^{-1})f_1(\overline{S}(y^{-3}) \rightharpoonup x^0)f_2(y^0) \text{ by } (11)$$

$$= \sum \overline{S}(x^{-1}y^{-2})f_1(\overline{S}(y^{-1}) \rightharpoonup x^0)f_2(y^0)$$

$$= \sum \overline{S}(x^{-1}y^{-1})f(x^0y^0) \text{ by } (13)$$

$$= \sum f^{-1}f^0(xy) \text{ by } (11)$$

$$= \sum f^{-1}(f^0)_1(\overline{S}(y^{-1}) \rightharpoonup x)(f^0)_2(y^0)$$

$$= \sum f^{-1}\theta'((f^0)_1 \otimes (f^0)_2)(x \otimes y)$$

and $\sum f^{-1}\epsilon_{A^*}(f^0) = \epsilon_{A^*}(f)1_H = f(1)1_H$. Hence A^* is a bi-Frobenius algebra in ${}_H^H\mathcal{YD}$.

Proposition 1 If (A, ϕ, t, ψ) is a bi-Frobenius algebra in ${}_H^H\mathcal{YD}$, $t \in \int_A^r$, $\phi \in \int_{A^*}^r$, there exists $\beta \in Alg(H, k)$ and a group-like element $n \in H$ such that $\forall a \in A$, $h \in H$

$$\phi(h \rightharpoonup a) = \beta(h)\phi(a) \tag{14}$$

$$n\phi(a) = \sum \overline{S}(a^{-1})\phi(a^0) \tag{15}$$

$$h \rightharpoonup t = \beta(h)t \tag{16}$$

$$\sum t^{-1} \otimes t^0 = S(n) \otimes t. \tag{17}$$

Proof For $\forall h \in H$, $\phi \in \int_{A^*}^r$,

$$(h \rightarrow \phi)f = \sum h_1 \rightarrow (\phi(S(h_2) \rightarrow f)) = \sum h_1 \rightarrow (\phi(S(h_2) \rightarrow f)(1)) =$$
$$\sum h_1 \rightarrow (\phi f(S^2(h_2) \rightarrow 1)) = (h \rightarrow \phi)f(1),$$

so $h \rightarrow \phi \in \int_{A^*}^{r}$. This implies $h \rightarrow \phi = \beta(S(h))\phi$ for $\beta \in Alg(H, k)$ and then $\phi(h \rightarrow a) = \beta(h)\phi(a)$. The others can be likewise deduced.

By §1, $t \in \int_A^r$, $\phi \in \int_{A^*}^r$. Let $I = kt$ then $\dim I = 1$. For any $A \in {}_H^H\mathcal{YD}$, we have that $Hom(I \otimes A, I \otimes A) \rightarrow Hom(A, A)$ which assigns $id \otimes f$ to f is bijective. It can be applied to the isomorphism $\tau \circ \tau: I \otimes A \rightarrow A \otimes I \rightarrow I \otimes A$, and so there exists a natural automorphism $\Omega: A \rightarrow A$ such that

$$\Omega(a) = \sum S(n) \rightarrow a^0 \beta(a^{-1}). \tag{18}$$

Each Ω induces a functor $\Omega: {}_H^H\mathcal{YD} \rightarrow {}_H^H\mathcal{YD}$[8]. That is $f \circ \Omega_A = \Omega_B \circ f$ for any $f \in Hom(A, B)$, $A, B \in {}_H^H\mathcal{YD}$. Let

$$\Omega_1(a) = S(n) \rightarrow a \quad \text{and} \quad \Omega_2(a) = \sum \beta(a^{-1})a^0.$$

So $\Omega = \Omega_2 \circ \Omega_1$. By Proposition 1, we have $\Omega_2(t) = \Omega_1(t) = \beta(S(n))t$.

Proposition 2　Ω is an algebra and coalgebra map in ${}_H^H\mathcal{YD}$.

Proof　Since $t \in A$ satisfies (1), by (16) and (17), we have

$$h_1 S(n)\beta(h_2) = \beta(h_1)S(n)h_2. \tag{*}$$

We prove that Ω is a left H-module and left H-comodule map. In fact, $\forall h \in H$, $a \in A$,

$$\begin{aligned}\Omega(h \rightarrow a) &= S(n) \rightarrow (h \rightarrow a)^0 \beta((h \rightarrow a)^{-1}) \\ &= \sum S(n)h_2 \rightarrow a^0 \beta(h_1)\beta(a^{-1})\beta(S(h_3)) \text{ by } (2) \\ &= \sum \beta(S(h_3))\beta(h_2)h_1 S(n) \rightarrow a^0\beta(a^{-1}) \text{ by } (*) \\ &= \sum hS(n) \rightarrow a^0 \beta(a^{-1}) = h \rightarrow \Omega(a)\end{aligned}$$

and

$$\begin{aligned}\sum \Omega(a)^{-1} \otimes \Omega(a)^0 &= \sum (S(n) \rightarrow a^0\beta(a^{-1}))^{-1} \otimes (S(n) \rightarrow a^0 \beta(a^{-1}))^0 \\ &= \sum S(n)a^{-1}n \otimes S(n) \rightarrow a^0 \beta(a^{-2}) \text{ by } (2) \\ &= \sum a^{-2} \otimes S(n) \rightarrow a^0 \beta(a^{-1}) \text{ by } (*) \\ &= \sum a^{-1} \otimes \Omega(a^0).\end{aligned}$$

It is easy to prove that Ω is an algebra and coalgebra map. So the assertion follows.

Let $\bar{\psi}$ be the inverse of ψ, then $\forall a, b \in A$,

$$\bar{\psi}(ab) = \sum \bar{\psi}(b^0)(\bar{S}(b^{-1}) \rightarrow \bar{\psi}(a)). \tag{19}$$

$$\Delta(\bar{\psi}(a)) = \sum \bar{\psi}(a_2{}^0) \otimes (\bar{S}(a_2{}^{-1}) \rightarrow \bar{\psi}(a_1)). \tag{20}$$

Dual basis plays an important role in Frobenius algebras. In the following, we will consider the dual basis of bi-Frobenius algebras in ${}_H^H\mathcal{YD}$.

Proposition 3 $(\sum \Omega_2(\bar{\psi}(t_2{}^0)), \bar{S}(t_2{}^{-1}) \rightarrow t_1)$ is the dual basis of bi-Frobenius algebra (A, ϕ, t, ψ) in ${}_H^H\mathcal{YD}$.

Proof By $a = \bar{\psi} \circ \psi(a) = \sum \phi(t_1(t_2{}^{-1} \rightarrow a))\bar{\psi}(t_2{}^0)$, we have

$$\begin{aligned}\Omega_2{}^{-1}(a) &= \sum a^0 \beta(\bar{S}(a^{-1}))\\ &= \sum \bar{\psi}(t_2{}^0)\beta(\bar{S}(t_2{}^{-1}))\phi(t_1(t_2{}^{-2} \rightarrow a)) \text{ by}(7)\\ &= \sum \bar{\psi}(t_2{}^0)\phi((\bar{S}(t_2{}^{-1}) \rightarrow t_1)a).\end{aligned}$$

So $a = \sum \Omega_2(\bar{\psi}(t_2{}^0))\phi((\bar{S}(t_2{}^{-1}) \rightarrow t_1)a)$. By $\Omega_2(a) = \sum \beta(a^{-1})a^0$, we also have $(\sum \bar{\psi}(t_2{}^0), \beta(t_2{}^{-1})\bar{S}(t_2{}^{-2}) \rightarrow t_1)$ is the dual basis of (A, ϕ, t, ψ) in ${}_H^H\mathcal{YD}$.

Take $t\phi \in Hom(A, A)$, then $\Omega \circ (t\phi) = (t\phi) \circ \Omega$. By Proposition 3, we have

$$\begin{aligned}&\sum t \otimes \beta(a^{-1})a^0 = t \otimes \Omega_2(a) =\\ &\sum t\phi(\Omega_2(a)\Omega_2(\bar{\psi}(t_2{}^0))) \otimes \bar{S}(t_2{}^{-1}) \rightarrow t_1 =\\ &\sum \Omega_2(t)\phi(a\bar{\psi}(t_2{}^0)) \otimes \bar{S}(t_2{}^{-1}) \rightarrow t_1 =\\ &\sum t \otimes \beta(S(n))\phi(a\bar{\psi}(t_2{}^0))\bar{S}(t_2{}^{-1}) \rightarrow t_1. \qquad (21)\end{aligned}$$

We recall the definition of antipode ψ: $\psi(a) = \sum \phi(t_1(t_2{}^{-1} \rightarrow a))t_2{}^0$. It follows that $\sum \phi(t_1(t_2{}^{-1} \rightarrow a))t_2{}^0 = \sum \phi(t)\epsilon(a_1)\psi(a_2) = \sum \phi(ta_1)\psi(a_2)$ for $t \in \int_A^r$, $\phi \in \int_{A^*}^r$, $\forall a \in A$. We can replace t with $a \in A$.

Proposition 4 If (A, ϕ, t, ψ) is a bi-Frobenius algebra in ${}_H^H\mathcal{YD}$, then

$$\sum \phi(a_1(a_2{}^{-1} \rightarrow b))a_2{}^0 = \sum \phi(ab_1)\psi(b_2), \ \forall\ a, b \in A. \qquad (22)$$

Proof $b = \bar{\psi} \circ \psi(b) = \sum \phi(t_1(t_2{}^{-1} \rightarrow b))\bar{\psi}(t_2{}^0)$, so

$$\begin{aligned}\Delta(b) &= \sum b_1 \otimes b_2\\ &= \sum \phi(t_1(t_2{}^{-1} \rightarrow b))\bar{\psi}((t_2{}^0)_2{}^0) \otimes \bar{S}((t_2{}^0)_2{}^{-1}) \rightarrow \bar{\psi}((t_2{}^0)_1) \text{ by } (20)\\ &= \sum \phi(t_1(t_2{}^{-1}t_3{}^{-2} \rightarrow b))\bar{\psi}(t_3{}^0) \otimes \bar{S}(t_3{}^{-1}) \rightarrow \bar{\psi}(t_2{}^0) \text{ by } (6).\end{aligned}$$

We only need to show that $\sum\beta(a^{-1})\phi((a^0)_1(((a^0)_2)^{-1} \rightarrow b))((a^0)_2)^0 = \sum\beta(a^{-1})\phi(a^0 b_1)\psi(b_2)$. In fact, we have

$$\begin{aligned}&\sum \beta(a^{-1})\phi((a^0)_1(((a^0)_2)^{-1} \rightarrow b))((a^0)_2)^0\\ &= \sum \beta(S(n))\phi(a\bar{\psi}(t_3{}^0))\phi((\bar{S}(t_3{}^{-1}) \rightarrow t_1)(\bar{S}(t_3{}^{-2})t_2{}^{-1}t_3{}^{-4} \rightarrow b))\\ &\quad \cdot \bar{S}(t_3{}^{-3}) \rightarrow t_2{}^0 \text{ by } (21), (2)\\ &= \sum \beta(S(n))\phi(a\bar{\psi}(t_3{}^0))\beta(\bar{S}(t_3{}^{-1}))\phi(t_1(t_2{}^{-1}t_3{}^{-3} \rightarrow b))\bar{S}(t_3{}^{-2}) \rightarrow t_2{}^0\\ &= \sum \beta(S(n))\phi(ab_1{}^0)\beta(\bar{S}(b_1{}^{-1}))\psi(b_2)\\ &= \sum \beta(S(n)\bar{S}(b_1{}^{-1}))\phi(ab_1{}^0)\psi(b_2)\end{aligned}$$

$$= \sum \beta(a^{-1}b_1{}^{-1}\overline{S}(b_1{}^{-2}))\phi(a^0 b_1{}^0)\psi(b_2) \text{ by } (15)$$
$$= \sum \beta(a^{-1})\phi(a^0 b_1)\psi(b_2).$$

3 Nakayama automorphisms

In this section, we give the module and comodule structures of the modular element g and the modular function α of A. We also give several automorphisms of bi-Frobenius algebras in ${}_H^H\mathcal{YD}$.

If $t \in \int_A^r$, $at \in \int_A^r$ for $\forall a \in A$. Since $\int_A^r$ is one dimensional, it follows that $at = \alpha(a)t$ for some $\alpha \in Alg(A, k)$. That is $\phi(at) = \alpha(a)$, so $t \rightharpoonup \phi = \alpha$. Dually, there exists a group-like element $g \in A$ satisfying $f * \phi = f(g)\phi$ for $\forall f \in A^*$. $(f*\phi)(t) = f(t_1)\phi(t_2) = f(g)\phi(t)$, so $g = \phi \rightharpoonup t$. We call α the modular function of A, g the modular element of A, see [2].

For $a, b \in A$, define the Nakayama automorphism $N: A \to A$ of the bi-Frobenius algebra (A, ϕ, t, ψ) in ${}_H^H\mathcal{YD}$ by

$$\phi(aN(b)) = \sum \phi((a^{-1} \to b)a^0). \tag{23}$$

For $t \in A$, define the coNakayama automorphism ${}^cN: A \to A$ of the bi-Frobenius algebra (A, ϕ, t, ψ) in ${}_H^H\mathcal{YD}$ by

$$\sum t_1 \otimes {}^cN(t_2) = \sum t_1{}^{-1} \to t_2 \otimes t_1{}^0. \tag{24}$$

Proposition 5 If (A, ϕ, t, ψ) is a bi-Frobenius algebra in ${}_H^H\mathcal{YD}$, N and ${}^cN$ are the Nakayama automorphism and coNakayama automorphism respectively, then

1) N and ${}^cN$ are automorphisms in ${}_H^H\mathcal{YD}$.

2) N: $A^{(op)2} \to A$ is an algebra map in ${}_H^H\mathcal{YD}$.

3) ${}^cN$: $A^{(cop)2} \to A$ is a coalgebra map in ${}_H^H\mathcal{YD}$.

Proof 1) $\forall a, b \in A$, we have

$$\begin{aligned}
\phi(a(h \to N(b))) &= \sum \phi(h_2 \to [(\overline{S}(h_1) \to a)N(b)]) \\
&= \sum \beta(h_2)\phi((\overline{S}(h_1) \to a)N(b)) \text{ by } (14) \\
&= \sum \beta(h_1)\phi([(\overline{S}(h_1) \to a)^{-1} \to b][\overline{S}(h_1) \to a]^0) \text{ by } (23) \\
&= \sum \beta(h_4)\phi((\overline{S}(h_3)a^{-1}h_1 \to b)(\overline{S}(h_2) \to a^0)) \text{ by } (2) \\
&= \sum \phi((h_4\overline{S}(h_3)a^{-1}h_1 \to b)(h_5\overline{S}(h_2) \to a^0)) \text{ by } (14) \\
&= \sum \phi((a^{-1}h_1 \to b)(h_3\overline{S}(h_2) \to a^0)) \\
&= \sum \phi((a^{-1}h \to b)a^0) \\
&= \phi(aN(h \to b)).
\end{aligned}$$

So N is a left H-module map.

Next we show N is a left H-comodule map. Firstly, it is easy to check $A^* \otimes A \to A^*: f \otimes a \mapsto f \leftharpoonup a$ is a left H-comodule map. Since $A^* = \phi \leftharpoonup A$, there exists a $b \in A$ such that $f = \phi \leftharpoonup b$.

We have

$$\sum (N(a))^{-1}f((N(a))^{0}) = \sum S\circ\overline{S}((N(a))^{-1})f((N(a))^{0})$$
$$= \sum S(f^{-1})f^{0}(N(a)) = \sum S((\phi\leftharpoonup b)^{-1})(\phi\leftharpoonup b)^{0}(N(a))$$
$$= \sum S(\phi^{-1}b^{-1})\phi^{0}(b^{0}N(a)) = \sum S(b^{-2})S(n)\phi((b^{-1}\rightarrow a)b^{0}) \text{ by (15),(23)}$$
$$= \sum S(b^{-2})((b^{-1}\rightarrow a)b^{0})^{-1}\phi(((b^{-1}\rightarrow a)b^{0})^{0}) \text{ by (15)}$$
$$= \sum S(b^{-5})b^{-4}a^{-1}S(b^{-2})b^{-1}\phi((b^{-3}\rightarrow a^{0})b^{0}) \text{ by (2)}$$
$$= \sum a^{-1}\phi((b^{-1}\rightarrow a^{0})b^{0}) = \sum a^{-1}\phi(bN(a^{0}))$$
$$= \sum a^{-1}f(N(a^{0})).$$

So N is a left H-comodule map. Hence N is an automorphism in ${}^{H}_{H}\mathcal{YD}$. Next we show that ${}^{c}N$ is also a map in ${}^{H}_{H}\mathcal{YD}$. We have

$$\sum t_1\otimes h\rightarrow{}^{c}N(t_2) = \sum h_2\rightarrow[\overline{S}(h_1)\rightarrow t_1\otimes{}^{c}N(t_2)]$$
$$= \sum h_4\rightarrow(\overline{S}(h_3)t_1{}^{-1}h_1\rightarrow t_2\otimes\overline{S}(h_2)\rightarrow t_1{}^{0}) \text{ by (24)}$$
$$= \sum h_4\overline{S}(h_3)t_1{}^{-1}h_1\rightarrow t_2\otimes h_5\overline{S}(h_2)\rightarrow t_1{}^{0}$$
$$= \sum t_1{}^{-1}h\rightarrow t_2\otimes t_1{}^{0} = \sum t_1\otimes{}^{c}N(h\rightarrow t_2)$$

and

$$\sum {}^{c}N(t)^{-1}\phi({}^{c}N(t)^{0}) = \sum S(\phi^{-1})\phi^{0}({}^{c}N(t)) = S(n)\phi({}^{c}N(t)) = t^{-1}\phi({}^{c}N(t^{0})).$$

So ${}^{c}N$ is an automorphism in ${}^{H}_{H}\mathcal{YD}$.

2) We only need to check that $\sum(\overline{S}(c^{-2})\rightarrow N(b^{0}))(\overline{S}(c^{-3})\overline{S}(b^{-1})\overline{S}^{2}(c^{-1})\rightarrow N(c^{0})) = N(bc)$ for $\forall a, b, c\in A$ and $N(1)=1$.

$$\sum \phi(a(\overline{S}(c^{-2})\rightarrow N(b^{0}))(\overline{S}(c^{-3})\overline{S}(b^{-1})\overline{S}^{2}(c^{-1})\rightarrow N(c^{0})))$$
$$= \sum \phi([(a(\overline{S}(c^{-2})\rightarrow N(b^{0})))^{-1}\overline{S}(c^{-3})\overline{S}(b^{-1})\overline{S}^{2}(c^{-1})\rightarrow c^{0}]$$
$$\cdot[a(\overline{S}(c^{-2})\rightarrow N(b^{0}))]^{0}) \text{ by (23)}$$
$$= \sum \phi((a^{-1}\overline{S}(c^{-2})b^{-1}c^{-4}\overline{S}(c^{-5})\overline{S}(b^{-2})\overline{S}^{2}(c^{-1})\rightarrow c^{0})a^{0}(\overline{S}(c^{-3})\rightarrow N(b^{0})))$$
$$\text{by (5) and (2)}$$
$$= \sum \phi((a^{-1}\rightarrow c^{0})a^{0}(\overline{S}(c^{-1})\rightarrow N(b)))$$
$$= \sum \phi([(a^{-2}\rightarrow c^{0})^{-1}a^{-1}\rightarrow(\overline{S}(c^{-1})\rightarrow b)](a^{-2}\rightarrow c^{0})^{0}a^{0}) \text{ by (23) and (5)}$$
$$= \sum \phi((a^{-4}c^{-1}S(a^{-2})a^{-1}\overline{S}(c^{-2})\rightarrow b)(a^{-3}\rightarrow c^{0})a^{0}) \text{ by (2)}$$
$$= \sum \phi((a^{-2}\rightarrow b)(a^{-1}\rightarrow c)a^{0}) = \sum \phi((a^{-1}\rightarrow(bc))a^{0}) \text{ by (3)}$$
$$= \sum \phi(aN(bc)) \text{ by (23)}.$$

By $A^{*}=\phi\leftharpoonup A$, $\sum(\overline{S}(c^{-2})\rightarrow N(b^{0}))(\overline{S}(c^{-3})\overline{S}(b^{-1})\overline{S}^{2}(c^{-1})\rightarrow N(c^{0})) = N(bc)$. It is easy to see that $N(1)=1$. Hence $N: A^{(op)^2}\to A$ is an algebra map in ${}^H_H\mathcal{YD}$.

3) Dualizing the proof of 2).

Proposition 6 Let α be the modular function, g be the modular element of A, then

$$h\to g=\epsilon(h)g$$

$$\sum g^{-1}\otimes g^{0}=1\otimes g$$

$$\alpha(h\to a)=h\to\alpha(a)=\epsilon(h)\alpha(a)$$

$$\sum a^{-1}\alpha(a^{0})=\alpha(a)1_H.$$

Proof 1) We have

$$\begin{aligned}&\sum(h_1\to g)\beta(h_2)=\sum(h_1\to t_1\phi(t_2))\beta(h_2)\\&=\sum(h_1\to t_1)\beta(h_2)\phi(t_2)=\sum(h_1\to t_1)\phi(h_2\to t_2)\text{ by }(14)\\&=\sum(h\to t)_1\phi((h\to t)_2)=\sum\beta(h)t_1\phi(t_2)\text{ by }(16)\\&=\beta(h)g.\end{aligned}$$

So

$$\begin{aligned}h\to g&=\sum(h_1\to g)\beta(h_2)\beta(S(h_3))=\sum\beta(h_1)g\beta(S(h_2))=\\&\sum\beta(h_1)\beta(S(h_2))g=\beta(\epsilon(h))g=\epsilon(h)g.\end{aligned}$$

2) In fact, we have

$$\begin{aligned}\sum g^{-1}S(n)\otimes g^{0}&=\sum(\phi\rightharpoonup t)^{-1}S(n)\otimes(\phi\rightharpoonup t)^{0}\\&=\sum t_1{}^{-1}S(n)\otimes t_1{}^{0}\phi(t_2)=\sum t_1{}^{-1}t_2{}^{-1}\otimes t_1{}^{0}\phi(t_2{}^{0})\text{ by }(15)\\&=\sum t^{-1}\otimes(t^{0})_1\phi((t^{0})_2)=\sum S(n)\otimes t_1\phi(t_2)\text{ by }(17)\\&=S(n)\otimes g.\end{aligned}$$

Since $S(n)$ is the inverse of n, $\sum g^{-1}\otimes g^{0}=1\otimes g$.

3) We have

$$\begin{aligned}\alpha(a)S(n)&=(t\rightharpoonup\phi)(a)S(n)=\phi(at)S(n)=\\&\sum\phi(a^{0}t^{0})a^{-1}t^{-1}=\sum\phi(a^{0}t)a^{-1}S(n)=\\&\sum\alpha(a^{0})a^{-1}S(n).\end{aligned}$$

So $\alpha(a)1_H=\sum a^{-1}\alpha(a^{0})$.

4) Indeed,

$$\begin{aligned}\alpha(h\to a)t&=(h\to a)t=\sum h_1\to(a(S(h_2)\to t))\\&=\sum h_1\to(a\beta(S(h_2))t)=\sum h_1\to t\alpha(a)\beta((S(h_2)))\\&=\sum\beta(h_1)\beta(S(h_2))\alpha(a)t\text{ by}(16)\\&=\epsilon(h)\alpha(a)t=(h\to\alpha(a))t.\end{aligned}$$

Proposition 7 Let N be the Nakayama automorphism, Ω be the natural automorphism and ψ

be the antipode of the bi-Frobenius algebra (A, ϕ, t, ψ) in ${}^H_H\mathcal{YD}$, then

$$N = \bar{\psi}^2 \circ \Omega \circ (\leftharpoonup \alpha) = \Omega \circ \bar{\psi}^2 \circ (\leftharpoonup \alpha)$$

Proof $\forall a \in A$, we have

$$\begin{aligned}
N(a) &= \bar{\psi} \circ \psi \circ N(a) \\
&= \sum \phi(t_1(t_2{}^{-1} \rightarrow N(a)))\bar{\psi}(t_2{}^0) \text{ by } (7) \\
&= \sum \phi((t_1{}^{-1}t_2{}^{-1} \rightarrow a)t_1{}^0)\bar{\psi}(t_2{}^0) \text{ by } (23) \\
&= \sum \phi((t^{-1} \rightarrow a)(t^0)_1)\bar{\psi}((t^0)_2) \text{ by } (6) \\
&= \sum \phi((t^{-1} \rightarrow a)_1((t^{-1} \rightarrow a)_2{}^{-1} \rightarrow t^0))\bar{\psi}^2((t^{-1} \rightarrow a)_2{}^0) \text{ by } (22) \\
&= \sum \phi((S(n) \rightarrow a)_1((S(n) \rightarrow a)_2{}^{-1} \rightarrow t))\bar{\psi}^2((S(n) \rightarrow a)_2{}^0) \text{ by } (17) \\
&= \sum \phi((S(n) \rightarrow a_1)\beta(S(n)a_2{}^{-1}S^2(n))t)\bar{\psi}^2(S(n) \rightarrow a_2{}^0) \\
&= \sum \alpha(S(n) \rightarrow a_1)\bar{\psi}^2(S(n) \rightarrow a_2{}^0\beta(a_2{}^{-1})) \\
&= \sum \alpha(a_1)\bar{\psi}^2(\Omega(a_2)) \\
&= \sum (\bar{\psi}^2 \circ \Omega \circ (\leftharpoonup \alpha))(a).
\end{aligned}$$

Since ψ is a map in ${}^H_H\mathcal{YD}$, ψ commutes with Ω. So $N = \sum \bar{\psi}^2 \circ \Omega \circ (\leftharpoonup \alpha) = \Omega \circ \bar{\psi}^2 \circ (\leftharpoonup \alpha)$.

Proposition 8 If ${}^cN$ is the coNakayama automorphism, then ${}^cN = \bar{\psi}^2 \circ \Omega \circ (g \cdot)$.

Denote $\bar{\alpha}$ as the inverse of α, $\bar{g}$ as the inverse of g.

Corollary 1 $\leftharpoonup \alpha : A \rightarrow A$ is an algebra automorphism in ${}^H_H\mathcal{YD}$ and $\bar{\alpha} = \alpha \circ \psi = \alpha \circ \bar{\psi}$. $\bar{\alpha}$ is an algebra map in ${}^H_H\mathcal{YD}$.

Proof Since α is an algebra map, $\leftharpoonup \alpha$ is an algebra automorphism in ${}^H_H\mathcal{YD}$ by Proposition 7 and $\bar{\alpha}$ is an algebra map in ${}^H_H\mathcal{YD}$. So α is invertible in A^* under the $*$ − multiplication. Since α is a group-like element in A^*, $\bar{\alpha} = \psi^*(\alpha) = \alpha \circ \psi$ and $\bar{\alpha} = \bar{\psi}^*(\alpha) = \alpha \circ \bar{\psi}$.

Corollary 2 $g \cdot : A \rightarrow A$, $a \mapsto ga$ is a coalgebra automorphism in ${}^H_H\mathcal{YD}$.

By Corollary 1 and Corollary 2, we have the following two corollaries.

Corollary 3 The actions $\leftharpoonup \alpha$, $\leftharpoonup \bar{\alpha}$, $\alpha \rightharpoonup$, $\bar{\alpha} \rightharpoonup$ are algebra automorphisms in ${}^H_H\mathcal{YD}$. Dually, $g \cdot$, $\bar{g} \cdot$, $\cdot g$, $\cdot \bar{g}$ are coalgebra automorphisms in ${}^H_H\mathcal{YD}$.

Corollary 4 Actions $g \cdot \bar{g}$: $a \mapsto ga\bar{g}$, $\alpha \rightharpoonup (\ \ ) \leftharpoonup \bar{\alpha}$: $a \mapsto \alpha \rightharpoonup a \leftharpoonup \bar{\alpha}$ are both algebra and coalgebra automorphisms in ${}^H_H\mathcal{YD}$. $g \cdot \bar{g}$, $\alpha \rightharpoonup (\ \ ) \leftharpoonup \bar{\alpha}$ and ψ commute with each other.

4 The ψ^4-formula

Let $({}^H_H\mathcal{YD})^{op}$ be the opposite category of ${}^H_H\mathcal{YD}$, their objects are the same, and the braiding of $({}^H_H\mathcal{YD})^{op}$ is τ^{-1}. If (A, ϕ, t, ψ) is a bi-Frobenius algebra in ${}^H_H\mathcal{YD}$, it is easy to see that (A, ϕ, t, ψ) is also a bi-Frobenius algebra in $({}^H_H\mathcal{YD})^{op}$. Let N' be the Nakayama automorphism of bi-Frobenius algebra (A, ϕ, t, ψ) in $({}^H_H\mathcal{YD})^{op}$, N′ is defined by $\phi(aN'(b)) = \sum \phi(b^0(\bar{S}(b^{-1})$

$\rightarrow a))$ for $\forall a, b \in A$.

Proposition 9 Let N be the Nakayama automorphism of the bi-Frobenius algebra (A, ϕ, t, ψ) in ${}^H_H\mathcal{YD}$, N' be the Nakayama automorphism of the bi-Frobenius algebra (A, ϕ, t, ψ) in $({}^H_H\mathcal{YD})^{op}$, then $N = N' \circ \Omega = \Omega \circ N'$.

Proof for $\forall a, b \in A$, $\phi(aN(b)) = \sum \phi((a^{-1} \rightarrow b)a^0)$ by (23). We have

$$\begin{aligned}\phi(tN' \circ \Omega(a)) &= \sum \phi(t\,N'(S(n) \rightarrow a^0)\beta(a^{-1}))\\ &= \sum \phi((S(n) \rightarrow a^0)(\bar{S}(S(n)a^{-1}S^2(n)) \rightarrow t)\beta(a^{-2}))\\ &= \sum \phi((S(n) \rightarrow a^0)t\beta(\bar{S}(a^{-1}))\beta(a^{-2}))\\ &= \sum \phi((S(n) \rightarrow a)t)\\ &= \sum \phi((t^{-1} \rightarrow a)t^0)\\ &= \phi(tN(a)).\end{aligned}$$

Notice that N' is a left H-module and left H-comodule map, so $N = N' \circ \Omega = \Omega \circ N'$.

Let N be the Nakayama automorphism of bi-Frobenius algebra (A, ϕ, t, ψ) in ${}^H_H\mathcal{YD}$, we have $N = \Omega \circ \bar{\psi}^2 \circ (\leftharpoonup \alpha)$ by Proposition 7. By Proposition 9, we get $N' = \bar{\psi}^2 \circ (\leftharpoonup \alpha)$. Let N'' be the Nakayama automorphism of bi-Frobenius algebra $(A^{cop}, g \rightharpoonup \phi, t, \bar{\psi})$ in $({}^H_H\mathcal{YD})^{op}$. From

$$(g \rightharpoonup \phi)(agN'(b)\bar{g}) = \phi(agN'(b)) = \sum \phi(b^0(\bar{S}(b^{-1}) \rightarrow (ag)))$$

$$= \sum \phi(b^0(\bar{S}(b^{-1}) \rightarrow a)g) = \sum (g \rightharpoonup \phi)(b^0(\bar{S}(b^{-1}) \rightarrow a))$$

$$= (g \rightharpoonup \phi)(aN''(b))$$

for any $a, b \in A$, we get $N'' = gN'\bar{g} = g(\bar{\psi}^2 \circ (\leftharpoonup \alpha))\bar{g}$. On the other hand, the antipode of $(A^{cop}, g \rightharpoonup \phi, t, \bar{\psi})$ in $({}^H_H\mathcal{YD})^{op}$ is $\bar{\psi}$, modular function is $\alpha' = \alpha$, $\leftharpoonup \alpha' = \alpha \rightharpoonup$, and the natural automorphism Ω' is Ω^{-1}. Applying Proposition 7 to N'', we have $N'' = \Omega^{-1} \circ \psi^2 \circ (\alpha \rightharpoonup) = g(\bar{\psi}^2 \circ (\leftharpoonup \alpha))\bar{g}$. From the above argument, we get the following theorem as [9]:

Theorem 2 If (A, ϕ, t, ψ) is a bi-Frobenius algebra in ${}^H_H\mathcal{YD}$, α and g are the modular function and the modular element of A respectively, then

$$\psi^4 = \Omega \circ g(\bar{\alpha} \rightharpoonup (\ \ ) \leftharpoonup \alpha)\bar{g} = g(\bar{\alpha} \rightharpoonup (\ \ ) \leftharpoonup \alpha)\bar{g} \circ \Omega.$$

Remark By Corollary 4 and Ω commutes with ψ, we get that

$$\psi^{4m} = \Omega^m \circ g^m(\bar{\alpha}^m \rightharpoonup (\ \ ) \leftharpoonup \alpha^m)\bar{g}^m = g^m(\bar{\alpha}^m \rightharpoonup (\ \ ) \leftharpoonup \alpha^m)\bar{g}^m \circ \Omega^m$$

$\forall m \in \mathbf{N}$. Since g and α are group-like elements of A and A^* respectively and distinct group-like elements are k-independent, their orders are finite[6]. So the order of ψ is finite if and only if the order of Ω is finite.

Dually, using the coNakayama automorphism, we can also get the ψ^4-formula of bi-Frobenius algebra (A, ϕ, t, ψ) in ${}^H_H\mathcal{YD}$.

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左 Yetter-Drinfeld 模范畴中的双 Frobenius 代数

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摘要：考虑左 Yetter-Drinfeld 模范畴中的双 Frobenius 代数(A, ϕ, t, ψ). 证明了左 Yetter-Drinfeld 模范畴中的双 Frobenius 代数(A, ϕ, t, ψ)的对偶(A, t, ϕ, ψ^*)也是左 Yetter-Drinfeld 模范畴中的双 Frobenius 代数. 给出了右积分$\phi \in \int_{A^*}^r$，$t \in \int_A^r$，模函数α和模元g的模和余模结构，也给出了 Yetter-Drinfeld 模范畴中的双 Frobenius 代数的 Radford 的对极ψ^4公式.

关键词：Hopf 代数；Frobenius 代数；Yetter-Drinfeld 模范畴