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Applications of Operational Calculus: Rectangular Prisms and the Estimation of a Missing Datum

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Abstract

Operational equations can be used for the estimation of data missing from prismatic arrays. This paper continues the discussion of that subject. New interpolating equations for data in prismatic and rectangular arrays are illustrated.

Mathematics Subject Classifications: 65D07, 65D17

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1. Introduction

Operational equations are finite-difference equations that derive from the application of the shifting operator to the identities and approximations of trigonometry [1]. A representative application of operational methods is the development of polynomial, exponential, and trigonometric interpolating equations for eight- and nine-point rectangular prisms [2,3]. The estimation of data missing from these designs is another application of operational equations.

2. Discrete formulas for estimating a missing datum

Let a single letter such as G represent a datum at the corresponding vertex of the prism in Fig. 1. Let a two-letter combination such as CD represent a number at the midpoint of the corresponding edge of the prism, and let a four-letter combination like BDIG represent a number at the midpoint of the corresponding side of the prism. Many operational equations can be used to estimate a datum that is missing from the prismatic array [4,5]. The previously published equations are invariant under rotation of the data and they are often invariant under data translation, too.

There is another class of operational equations that are not invariant under data rotation. This deficiency limits their applications. Nevertheless, those equations are potentially useful to the experimentalist. They apply when the data can be satisfactorily approximated by an arbitrary function applied to monotonic numbers. The equations have the advantage of increased accuracy in this circumstance. They have that advantage because they are then numerical identities. Two examples help to clarify the preceding remarks. The example equations are Eqs. (1) and (2).

$$(BD–FH)(BG–CH) – (G–C)(BDIG–ACHF) = 0 \tag{1}$$

$$(BD–FH)(FGIH–ABDC)(BDIG–ACHF) + (CD–FG)(BD–FH)(BG–CH) – (G–C)(CD–FG)(BDIG–ACHF) – (FGIH–ABDC)(G–C)(BG–CH) = 0 \tag{2}$$

To illustrate the interesting property of Eqs. (1) and (2), let the data A .. I at the vertices of the prism in Fig. 1 be 1+t .. 9+t, respectively. The letter "t" represents a constant. Let Eq. (1) be rewritten as Eq. (3) in which u(x) represents an arbitrary function applied to its argument.

$$[u(3+t)–u(7+t)][u(9/2+t)–u(11/2+t)] – [u(7+t)–u(3+t)][BDIG–u(9/2+t)] = 0 \tag{3}$$

When Eq. (3) is solved for BDIG, the result is BDIG=u(11/2+t). The function u(11/2+t) is also the number at point BDIG when the data are generated by u(1+t) .. u(9+t) at A .. I in Fig. 1, respectively. In this circumstance, accurate measurements at points BD, FH, BG, CH, CD, FG, G, C, FGIH, ABDC and ACHF yield a reliable estimate of the missing datum at BDIG using either Eq. (1) or (2).

The class of numbers to which Eqs. (1) and (2) apply is wider than the illustration suggests. If "p" is a constant, the class of numbers can be represented by u(p+t), u(2p+t) .. u(9p+t) as A .. I in Fig. 1. The class of monotonic numbers is narrow. On the other hand, the class embracing "any function" is wide. The opportunities represented by operational equations like Eqs. (1) and (2) are potentially useful for replacing a missing or a corrupted datum. Measurement errors temper the accuracies of relationships like Eqs. (1) and (2) but modest errors do not necessarily destroy the economy they represent.

3. Equations for cubes and the problem of a missing datum

New methods for generating exponential equations for eight- or nine-point prisms have recently appeared [6]. The center point datum was estimated in the eight-point examples. That choice does not restrict the applications of the equations. The center point datum could be measured. The five basis equations are then used to estimate a datum missing from another vertex of the cube. To demonstrate this point, suppose corner-point G is the one where no measurement was made. Let the remaining data be generated by monotonic functions u(x) operating on the integers 1 .. 6, and 8, 9 at vertices A .. F, and H, I, respectively. Table 1 lists the true values of u(7) and the estimates of that number as obtained by the described procedure. The table illustrates that the method is satisfactory in cases where the data follow a simple exponential law. In other cases, the method is less satisfactory but still potentially useful. The five basis equations chosen for illustrating the example are the same ones applied in Section 3 of Ref. [6].

A remark in Section 1 of Ref. [6] indicates that the four basis equations used for the nine-point cube can have more than one solution containing positive, real numbers for J, K, and L that are needed to generate equations for the array. In the author's experience, these ambiguities seldom appear. When they appear, the question turns on the choice of one of two possibilities. A comparison test was suggested in order to choose between the alternatives. Another, bigger ambiguity is how to select a set of four or five basis equations that determine the values of J, K, and L.

A still bigger ambiguity is represented by the question of how to represent the eight-point cube. Should it be represented by the trilinear equation, by a polynomial equation containing second-order coefficients, by a polynomial equation containing second- and third-order coefficients, by a trigonometric equation, or by one of several possible exponential equations? These questions do not arise when there is tacit agreement that only one representation, the trilinear equation, is sanctioned. The same questions attend the problem of how best to represent a nine-point cube. In the case of the rectangle ACIG, as in Fig. 1 of Ref. [7], the situation is simpler. For example, if the data are monotonic increasing in alphabetical order, and if I ≥CG/A, an exponential interpolating equation is indicated. Otherwise, a polynomial or an exponential form may suffice. In either case, the user should beware of false curvature effects.

What is needed to alleviate these problems is a test that determines if eight data in prismatic array adhere to a polynomial, a trigonometric, or an exponential law. Ideally, the test will be short, easy to apply, definitive, and depend only on the numbers. An idea of the proposed test is afforded by the many relationships that have been proposed as methods for estimating one or two missing data. For example, let the data be substituted into Eqs. (9) and (10) in Ref. [4]. If the cited Eq. (10) yields the smaller residual, an exponential law is suggested. If a center point datum is available, the residual of Eq. (11) in Ref. [4] can be compared to the residuals of one of Eqs. (19)-(21) in Ref. [5]. The equation with the smaller residual suggests whether the data follow a polynomial or an exponential law. The examples illustrate the need for a standard. The proposed test is certain to be complicated by errors. That is another problem to be addressed.

4. Prisms and the method of least squares

Monotonic data A .. I in the eight- or nine-point prism, as in Fig. 1, can be represented by Eq. (4). For the eight-point cube, the letters J, K, and L are taken as Eqs. (15)-(17), respectively, in Ref. [2]. For the nine-point cube, J, K, and L are taken as Eqs. (3)-(5), respectively, in Ref. [4], respectively. R is an interpolated number.

$$R = (P)J^{(x+1)}K^{(y+1)}L^{(z+1)} + T \tag{4}$$

The unknowns P and T in Eq. (4) are estimated from the data, their coordinates, and Eq. (4) by the method of least squares. The resulting equations are exact on the data at vertices A-I if they are generated by a simple exponential function like 2^x+10. In other cases, interpolating equations based on Eq. (4) do not reproduce the original data. The new method is not limited by the suggested expressions for J, K, and L. Those parameters can also be estimated by another method such as is illustrated in Ref. [6]. Table 2 illustrates typical accuracies of the described methods as represented by sums of squares of deviations. If an exponential equation is desirable, and if the minimum of the sum of squares of deviations is the criterion of merit, the described approaches are potentially useful. They are easy to apply. Unfortunately, we have no standard prescribing the values of sums of squares of deviations by which to accept or reject an interpolating equation. This method is not recommended for non-monotonic data.

5. The representation of four-point rectangles

A previous paper pointed out a difficulty with the representation of a four-point rectangle by exponential means [7]. The data were A=70, B=180, C=90, D=75, as on face ABDC in Fig. 1. They rendered an exponential equation containing the square root of (–1). The SQRT(–1) is usually undesirable but this problem can sometimes be alleviated by an simple artifice. Let the same data be moved to the top of the prism in Fig. 1. They now appear as F=70, G=180, H=90, I=75. Let new data that mimic the numerical pattern of the original data be placed on the base of the prism, face ABDC. For example, let the new trial data be [1,16,9,4,70,180,90,75] as A .. I in the eight-point prism in Fig. 1.

An exponential equation for the new eight-point prism is easily found by the method described in Section 3 of Ref. [6]. A second exponential equation for the same prism is found by means of Eq. (14) in Ref. [2]. In these equations, set z=1. The two new equations, both lacking the square root of (–1), apply to the formerly troublesome rectangle. The new equations, with rounded coefficients, appear as Eqs. (5) and (6), respectively. Both of them reproduce the original data. Plotting them reveals that the rectangles they represent do not contain unwarranted curvature effects such as spurious extrema.

$$R = 225.3 - (102.0)(1.786)^{(x+1)} - (205.6)(0.7906)^{(y+1)} + (152.2)(1.786)^{(x+1)}(0.7906)^{(y+1)} \tag{5}$$

$$R = 1265 - (1109)(1.176)^{(x+1)} - (1482)(0.8754)^{(y+1)} + (1397)(1.176)^{(x+1)}(0.8754)^{(y+1)} \tag{6}$$

Several exponential representations of the four-point rectangle can be generated by the illustrated method. The original data can also appear at the bottom of the prism while the introduced numbers appear at the top of the prism. The user can change the magnitudes of the introduced numbers in order to change the properties of the interpolating equations. There are many possible choices. The many possibilities raise the question of how to choose among them. That is an opportunity for research.

6. The four-point cube

Methods for interpolating the four-point cube are desirable because of the importance of minimizing laboratory costs. The subject has been taken up in Refs. [8,9] but modern software makes other methods practical. For example, four-point sections of cubes can sometimes be represented by the cited methods or by Eq. (7). To illustrate, let measurements be taken at points HI, GI, DI, and E or IE in Fig. 1. Let the figure have 1 .. 9 placed at vertices A .. I, respectively.

$$R = (N)\exp(Jx + Ky + Lz) \quad (7)$$

The coordinates of the four points, and the measurements at those points, are often enough to estimate the unknowns N, J, K, and L. With those estimates, a missing datum at vertex I can be estimated. For example, let HI=f(17/2), GI=f(8), DI=f(13/2), E=f(5), where f(x) represents an arbitrary function of its argument. Following are simple functions f(x) and the estimates at vertex I as represented by [f(x), R]: [2^x, 512], [x^2, 88], [100/x, 10.6]. The true values of R are 512, 81, and 11.1, respectively. Another method for the same purpose turns on Eq. (8). Now let HI=u(17/2), GI=u(8), DI=u(13/2), IE=u(7). (IE is the midpoint of segment IE in Fig. 1.) Eq. (8) yields [x^2, 80], [ln(x+1), 2.3], [100/x, 11.2]. The true values of R are 81, 2.3, and 11.1, respectively. Four-point methods for cubes are interesting because of their economy.

$$R = (Jx + Ky + Lz)^N \quad (8)$$

The standard approaches illustrated in this section, the operational methods described above, and other methods that appear in the references, illustrate potentially useful approaches to the treatment of experimental data in rectangular and prismatic arrays.

Table 1. Estimation of the missing datum at vertex G in Fig. 1 by an eight-point exponential method plus the center point datum. The basis equations are described on p. 2148 of Ref. [6].

Function	Estimated G	G = M(7)
2^M	128	128
2^M+100	228	228
M^2	50.5	49.0
M^3	379	343
SQRT(M)	2.64	2.65
100/M	14.4	14.3
sinh(M/4)	2.76	2.79
cosh(M/4)	3.03	2.96
tan($9M^o$)	2.00	1.96
ln(M+1)	2.08	2.08

Table 2. Sums of squares of deviations of two least-squares equations from test surfaces. The equations are eight-point and nine-point versions of Eq. (4) as applied to Fig. 1.

Function	Eight-point	Nine-point
2^M	0	0
2^M+100	0	0
M^2	6.87	6.60
M^3	1625	1233
SQRT(M)	0.00198	0.00381
100/M	68.5	68.8
sinh(M/4)	0.00321	0.00309
cosh(M/4)	0.00572	0.00539
tan(9M^o)	0.434	0.435
ln(M+1)	0.00193	0.00231

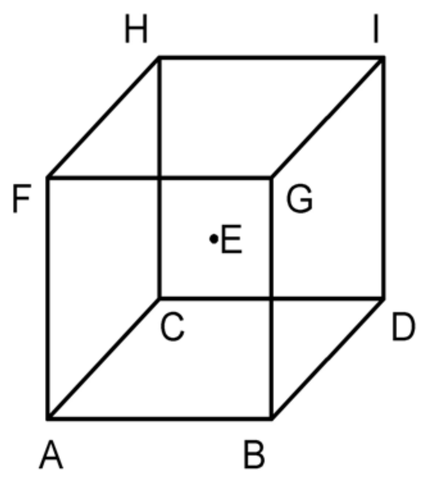

Fig. 1. The eight-point cube with center point E.

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