Applied Mathematical Sciences, Vol. 4, 2010, no. 15, 719 - 728

Compatibility of Type (α) and Weak Compatibility in Fuzzy Metric Spaces

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Abstract. The object of this paper is to establish a unique common fixed point theorem for six self-mappings satisfying a contractive condition of [8] through compatibility of type (α) and weak compatibility with different pairs of continuities in a fuzzy metric space. The established results generalize, extend, unify and fuzzify several existing fixed point results in metric spaces and fuzzy metric spaces.

Mathematics Subject Classification: 54H25, 47H10

Keywords: Fuzzy metric space, common fixed points, t-norm, compatible maps of type (α), weak compatible maps

1. Introduction

The theory of fuzzy sets was first introduced by Zadeh 15], after that a lot of research papers have been published on fuzzy sets. The motivation of introducing fuzzy metric space is the fact that in many situations the distance between two points is inexact due to fuzziness rather than randomness. Kramosil and Michalek [7] introduced the concept of fuzzy metric space by generalizing the concept of probabilistic metric space to fuzzy situation. George and Veeramani [3] modified this concept of fuzzy metric space and obtain a Hausdroff topology for this kind of fuzzy metric spaces. It appears that the study of Kramosil and Michalek [7] of fuzzy metric spaces paves the way for developing the smooth machinery in the field of fixed point theory for the study of contractive maps.

Sessa [10] initiated the tradition of improving commutativity condition in fixed point theorems by introducing the notion of weakly commuting maps in metric spaces. Jungck [7] soon enlarged this concept to compatible maps. The concepts of R-weakly commuting maps and compatible maps in fuzzy metric space have been introduced by Vasuki [14] and Mishra et al [9] respectively. Cho [1] introduced the concept of compatible maps of type (α) and compatible maps of type (β). In [6] Jungck and Rhoades termed a pair of self-map to be coincidentally commuting or equivalently weak compatible if they commute at their coincidence points. This concept is most general among all the commutativity concepts in this field as every pair of commuting maps or of compatible maps is weak compatible but the reverse is not true always.

In this paper we establish the existence of unique common fixed point of six self-maps through compatibility of type (α) and weak compatibility satisfying a contraction adopted in [8]. Our results generalize, extend, unify and fuzzify several existing fixed point results in metric spaces and fuzzy metric spaces.

2. Preliminaries

Definition 1: A binary operation $*:[0,1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a*b \le c*d$, whenever $a \le c$ and $b \le d$, for all a, b, c and d $\in [0, 1]$.

Examples of t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2: (Kramosil and Michalek [7]) **:** The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions for all x, y, z $\in$X and s, t > 0

(F.M-1) $M(x, y, 0) = 0$;

(F.M-2) $M(x, y, t) = 1$, for all $t > 0$ iff $x = y$;

(F.M-3) $M(x, y, t) = M(y, x, t)$;

(F.M-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$;

(F.M-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous.

Note that M(x, y, t) can be thought of as the degree of nearness between x and y with respect to t. We identify x = y with M(x, y, t) = 1, for all t > 0. The following example shows that every metric space induces a fuzzy metric space.

Example 1(George and Veeramani [3])**:** Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$.Let for all x, y$\in$ X, $M(x, y, t) = t/(t + d(x, y))$, for all t>0 & $M(x, y, 0)= 0$.

Then (X, M, $*$) is a fuzzy metric space. It is called the fuzzy metric space induced by the metric space (X, d).

Lemma 1 (Grabiec [4]) **:** For all x, y $\in$ X, M(x, y, .) is a non-decreasing function.

Definition 3 (Grabiec [4])**:**Let (X, M, $*$) be a fuzzy metric space**.** A sequence $\{x_n\}$ in X is said to convergent to a point $x\in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1,\ \forall\ t > 0$. Further, sequence $\{x_n\}$is said to be a Cauchy sequence if $\lim\limits_{n\to\infty} M(x_n, x_{n+p}, t)= 1$, for all t > 0 and for all p. The space is said to be complete if every Cauchy sequence in it converges to a point of it.

Remark 1: Since $*$ is continuous, it follows from F.M-4 that in a fuzzy metric space the limit of a sequence is unique, if it exists.
In this paper (X, M, $*$) will be considered to be the fuzzy metric space with condition
(F.M-6) $\quad \mathrm{Lim}_{t\to\infty} M(x, y, t) = 1$, for all $x, y \in X$ and $t > 0$.

Definition 4: A pair (A, S) of self mappings of a fuzzy metric space is said to be compatible maps of type (α) if (i) $\lim_{n\to\infty} M(ASx_n, S^2x_n, t) = 1$, for all t > 0 and

(i) $\quad \lim_{n\to\infty} M(SAx_n, A^2x_n, t) = 1$, for all t > 0,

when ever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$.

Definition 5: A pair (A, S) of self-mappings of a fuzzy metric space is said to be weak compatible or coincidentally commuting if A and S commute at their coincidence points i.e. for $x\in X$ if Ax = Sx then ASx = SAx.

Remark 2: If self-mappings A and S of a fuzzy metric space (X, M, $*$) are compatible maps of type (α), then they are weak compatible.

Lemma 2 [Cho 1]: Let $\{y_n\}$ be a sequence in a fuzzy metric space (X, M, $*$) with condition (F.M-6). If there exists a number $k \in (0, 1)$ such that,
$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$, for all t > 0. Then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 3 [Mishra et. al 9]: If for all $x, y\in X$ and $0 < k < 1$
$M(x, y, kt) \geq M(x, y, t)$, for all t > 0, then x = y.

3. Main Results

Theorem 3.1: Let (X, M, *) be a complete fuzzy metric space and let P, Q, S, T, A and B be self maps from X such that
(3.1.1) $\quad P(ST)(X) \cup Q(AB)(X) \subseteq AB(ST)(X)$;
(3.1.2) $\quad$ AB = BA, ST = TS, PB = BP, QT = TQ, AB(ST) = ST(AB);
(3.1.3) $\quad$ (P, AB) and (Q, ST) are compatible maps of type (α);
(3.1.4) $\quad$ one map from each of the above two pairs of (3.1.3) is continuous;
(3.1.5) $\quad \exists$ a constant $k \in (0, 1)$ such that

$$M^2(Px, Qy, kt) * [M(Px, ABx, kt)\, M(Qy, STy, kt)] * M^2(Qy, STy, kt)$$

$+ a\, M(Qy, STy, kt)\, M(ABx, Qy, 2kt)$

$\geq [p\, M(Px, ABx, t) + q\, M(ABx, STy, t)]\, M(ABx, Qy, 2kt), \forall\, x,y \in X, \forall\, t>0,$
where $0 < p, q < 1$, $-1< a < 1$ such that $p + q - a = 1$.
Then P, Q, S, T, A and B have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be a point in X. Construct sequences $\{x_n\}$ and $\{z_n\}$ in X such that

$PSTx_{2n} = ABSTx_{2n+1} = z_{2n+1}$ and $QABx_{2n+1} = ABSTx_{2n+2} = z_{2n+2}, \forall\, n.$ ... (1)

Step 1: Taking $x = STx_{2n}$, $y = ABx_{2n+1}$ & using $AB(ST) = ST(AB)$ in (3.1.5) we get, $M^2(PSTx_{2n},QABx_{2n+1},kt) * [M(PSTx_{2n}, ABSTx_{2n}, kt)\, M(QABx_{2n+1}, STABx_{2n+1},kt)] * M^2(QABx_{2n+1}, STABx_{2n+1},kt) + aM(QABx_{2n+1},STABx_{2n+1},kt)\, M(ABSTx_{2n},QABx_{2n+1}, 2kt)$
$\geq [pM(PSTx_{2n},ABSTx_{2n},t)+q\, M(ABSTx_{2n},STABx_{2n+1}, t)]\, M(ABSTx_{2n},QABx_{2n+1},2kt).$
Using (1) and as $AB(ST) = ST(AB)$, we have
$M^2(z_{2n+1}, z_{2n+2}, kt) * [M(z_{2n}, z_{2n+1}, kt)\, M(z_{2n+1}, z_{2n+2}, kt)] * M^2(z_{2n+1} , z_{2n+2}, kt)$
$+ a\, M(z_{2n+1}, z_{2n+2}, kt)\, M(z_{2n}, z_{2n+2}, 2kt) \geq [p + q]\, M(z_{2n}, z_{2n+1}, t)\, M(z_{2n}, z_{2n+2}, 2kt),$
then $M^2(z_{2n+1}, z_{2n+2}, kt) * [M(z_{2n}, z_{2n+1}, kt)\, M(z_{2n+1}, z_{2n+2}, kt)] + a\, M(z_{2n+1}, z_{2n+2},kt)\, M(z_{2n}, z_{2n+2}, 2kt) \geq [p + q]\, M(z_{2n}, z_{2n+1}, t)\, M(z_{2n}, z_{2n+2}, 2kt),$
so $M(z_{2n+1}, z_{2n+2}, kt)\, [M(z_{2n}, z_{2n+1}, kt) * M(z_{2n+1}, z_{2n+2}, kt)] + a\, M(z_{2n+1}, z_{2n+2}, kt)\, M(z_{2n}, z_{2n+2}, 2kt) \geq [p + q]\, M(z_{2n}, z_{2n+1}, t)\, M(z_{2n}, z_{2n+2}, 2kt).$
Implies $M(z_{2n+1}, z_{2n+2}, kt)\, M(z_{2n}, z_{2n+2}, 2kt) + a\, M(z_{2n+1}, z_{2n+2}, kt)\, M(z_{2n}, z_{2n+2},2kt) \geq [p + q]\, M(z_{2n}, z_{2n+1}, t) M(z_{2n}, z_{2n+2}, 2kt)$,
which gives $[1+ a]\, M(z_{2n+1}, z_{2n+2}, kt) M(z_{2n}, z_{2n+2}, 2kt) \geq [p + q]\, M(z_{2n}, z_{2n+1}, t)\, M(z_{2n}, z_{2n+2}, 2kt)$. Thus $M(z_{2n+1}, z_{2n+2}, kt) \geq \frac{p+q}{1+a} M(z_{2n}, z_{2n+1}, t)$, for all $t > 0$.
As $p + q - a = 1$, we get that $M(z_{2n+1} , z_{2n+2}, kt) \geq M(z_{2n} , z_{2n+1}, t)$, for all $t > 0$.
Similarly, if we take $x = STx_{2n+2}$, $y = ABx_{2n+1}$ in (3.1.5) we get

$M(z_{2n+2} , z_{2n+3}, kt) \geq M(z_{2n+1} , z_{2n+2}, t)$, for all $t > 0$.

Thus $M(z_{m+1} , z_{m+2}, kt) \geq M(z_m , z_{m+1}, t)$, for all $t > 0$ and for $m = 1, 2,...$
Therefore by Lemma 2, $\{z_n\}$ is a Cauchy sequence in X, which is compete. Hence $\{z_n\} \to u \in X$. Also its subsequences

$PSTx_{2n} \to u$ and $ABSTx_{2n} \to u$,
$QABx_{2n+1} \to u$ and $STABx_{2n+1} \to u$.
Let $STx_{2n} = v_n$ and $ABx_{2n+1} = w_{n+1}$, $\forall$ n, then
$Pv_n \to u$ and $ABv_n \to u$, ... (2)
$Qw_{n+1} \to u$ and $STw_{n+1} \to u$. ... (3)

Case 1: P, Q are continuous.

As P is continuous we have $P^2v_n \to Pu$. ... (4)
As (P, AB) is compatible of type (α), by (ii) we get that $ABPv_n \to Pu$. ... (5)

Step 2: Taking $x = Pv_n$, $y = w_{n+1}$ in (3.1.5) we get,
$M^2(P^2v_n, Qw_{n+1}, kt) * [M(P^2v_n, ABPv_n, kt)\, M(Qw_{n+1}, STw_{n+1}, kt)] *$
$M^2(Qw_{n+1}, STw_{n+1}, kt) + a\, M(Qw_{n+1}, STw_{n+1}, kt)\, M(ABPv_n, Qw_{n+1}, 2kt)$

$\geq [p\, M(P^2v_n, ABPv_n, t) + q\, M(ABPv_n, STw_{n+1}, t)]\, M(ABPv_n, Qw_{n+1}, 2kt)$.
Letting $n \to \infty$ and using (3), (4) and (5) we get that
$M^2(Pu, u, kt) * [M(Pu, Pu, kt)\, M(u, u, kt)] * M^2(u, u, kt) + aM(u, u, kt)\, M(Pu, u, 2kt)$
$\geq [p\, M(Pu, Pu, t) + q\, M(Pu, u, t)]\, M(Pu, u, 2kt)$, and
$M^2(Pu, u, kt) + a\, M(Pu, u, 2kt) \geq [p + q\, M(Pu, u, t)]\, M(Pu, u, 2kt)$.
Since M(x, y, .) is non-decreasing for all x, y in X we have,
$M(Pu, u, 2kt)\, M(Pu, u, t) + a\, M(Pu, u, 2kt) \geq [p + q\, M(Pu, u, t)]\, M(Pu, u, 2kt)$,
which gives $M(Pu, u, t) \geq \frac{p-a}{1-q}$.
As $p + q - a = 1$, we get that $M(Pu, u, t) \geq 1$, for all $t > 0$.
Hence we get $Pu = u$. ... (6)
The continuity of Q gives
$Q^2w_{n+1} \to Qu$. ... (7)
As (Q, ST) is compatible of type (α), by (ii) we get that
$STQw_{n+1} \to Qu$. ... (8)

Step 3: Taking $x = v_n$, $y = Qw_{n+1}$ in (3.1.5) we get,
$M^2(Pv_n, Q^2w_{n+1}, kt) * [M(Pv_n, ABv_n, kt)\, M(Q^2w_{n+1}, STQw_{n+1}, kt)] *$
$M^2(Q^2w_{n+1}, STQw_{n+1}, kt) + a\, M(Q^2w_{n+1}, STQw_{n+1}, kt)\, M(ABv_n, Q^2w_{n+1}, 2kt)$
$\geq [p\, M(Pv_n, ABv_n, t) + q\, M(ABv_n, STQw_{n+1}, t)]\, M(ABv_n, Q^2w_{n+1}, 2kt)$.
Letting $n \to \infty$ and using (2), (7) and (8) we get that
$M^2(u, Qu, kt) * [M(u, u, kt)\, M(Qu, Qu, kt)] * M^2(Qu, Qu, kt) + a\, M(Qu, Qu, kt)$
$M(u, Qu, 2kt) \geq [p\, M(u, u, t) + q\, M(u, Qu, t)]\, M(u, Qu, 2kt)$,
and $M^2(u, Qu, kt) + a\, M(u, Qu, 2kt) \geq [p + q\, M(u, Qu, t)]\, M(u, Qu, 2kt)$.
As in step 2, it follows that $Qu = u$.
Thus $Pu = Qu = u$. ... (9)
Step 4: Taking $x = v_n$, $y = u$ in (3.1.5) we get,
$M^2(Pv_n, Qu, kt) * [M(Pv_n, ABv_n, kt)\, M(Qu, STu, kt)] * M^2(Qu, STu, kt)$
$+ a\, M(Qu, STu, kt)\, M(ABv_n, Qu, 2kt)$
$\geq [p\, M(Pv_n, ABv_n, t) + q\, M(ABv_n, STu, t)]\, M(ABv_n, Qu, 2kt)$.
Letting $n \to \infty$ and using (2) and (9) we get that
$M^2(u, u, kt)*[M(u, u, kt)\, M(u, STu, kt)] * M^2(u, STu, kt)$
$+ a\, M(u, STu, kt)\, M(u, u, 2kt) \geq [p\, M(u, u, t) + q\, M(u, STu, t)]\, M(u, u, 2kt)$,
and $M(u, STu, kt) * M^2(u, STu, kt) + a\, M(u, STu, kt) \geq p + q\, M(u, STu, t)$,
so $M(u, STu, kt) + a\, M(u, STu, kt) \geq [p + q]\, M(u, STu, t)$,
implies $[1+a]\, M(u, STu, kt) \geq [p + q]\, M(u, STu, t)$,
i. e. $M(STu, u, kt) \geq \frac{p+q}{1+a} M(u, STu, kt)$.
As $p + q - a = 1$, we get that $M(STu, u, kt) \geq M(u, STu, t)$. Therefore by Lemma 3, we have $STu = u$. Thus $Pu = Qu = STu = u$. Now $P(ST)u = Pu = u$.

Step 5: As $P(ST)(X) \subseteq AB(ST)(X)$, there exists $z \in X$ such that
$u = P(ST)u = ABSTz$.
Writing $STz = v$. Therefore $u = ABv$. ... (10)

Taking x = v, y = u in (3.1.5) we get, $M^2(Pv, Qu, kt) * [M(Pv, ABv, kt) M(Qu, STu, kt)] * M^2(Qu, STu, kt) + a M(Qu, STu, kt) M(ABv, Qu, 2kt)$
$\geq [p M(Pv, ABv, t) + q M(ABv, STu, t)] M(ABv, Qu, 2kt)$,
So $M^2(Pv, u, kt) * [M(Pv, u, kt) M(u, u, kt)] * M^2(u, u, kt) + a M(u, u, kt) M(u, u, 2kt)$
$\geq [p M(Pv, u, t) + q M(u, u, t)] M(u, u, 2kt)$,
And $M(Pv, u, kt) * M(Pv, u, kt) + a \geq p M(Pv, u, t) + q$,
implies $M(Pv, u, kt) + a \geq p M(Pv, u, t) + q$,
i. e. $M(Pv, u, kt) + a \geq p M(Pv, u, kt) + q$,
which gives $M(Pv, u, kt) \geq \frac{q-a}{1-p}$,
As $p + q - a = 1$, we get that $M(Pv, u, kt) \geq 1$, for all $t > 0$, and therefore Pv = u.
Now Pv = ABv = u. As (P, AB) is compatible of type (α) and so is weak compatible. Hence Pu = ABu. Therefore Pu = ABu = Qu = STu = u.

Case 2: Maps AB and ST are continuous.
The continuity of AB implies
$(AB)^2 v_n \to ABu$. ... (11)
As (P, AB) is compatible maps of type (α), by (i) we get that
$PABv_n \to ABu$. ... (12)

Step 6: Taking $x = ABv_n$, $y = w_{n+1}$ in (3.1.5) we get,
$M^2(PABv_n, Qw_{n+1}, kt) * [M(PABv_n, (AB)^2v_n, kt) M(Qw_{n+1}, STw_{n+1}, kt)] * M^2(Qw_{n+1}, STw_{n+1}, kt) + a M(Qw_{n+1}, STx_{2n+1}, kt) M((AB)^2v_n, Qw_{n+1}, 2kt)$
$\geq [p M(PABv_n, (AB)^2v_n, t) + q M((AB)^2v_n, STw_{n+1}, t)] M((AB)^2v_n, Qw_{n+1}, 2kt)$.
Letting $n \to \infty$ and using (3), (11) and (12) we get that
$M^2(ABu, u, kt) * [M(ABu, ABu, kt) M(u, u, kt)] * M^2(u, u, kt)$
$+aM(u, u, kt) M(ABu, u,2kt) \geq [p M(ABu, ABu, t) + q M(ABu, u, t)] M(ABu, u, 2kt)$,
and $M^2(ABu, u, kt) + a M(ABu, u, 2kt) \geq [p + q M(ABu, u, t)] M(ABu, u, 2kt)$.
As in step 2, we get that ABu = u. ... (13)

Step 7: Taking x = u, $y = w_{n+1}$ in (3.1.5) we get,
$M^2(Pu, Qw_{n+1}, kt) * [M(Pu, ABu, kt) M(Qw_{n+1}, STw_{n+1}, kt)] * M^2(Qw_{n+1}, STw_{n+1},kt) + a M(Qw_{n+1}, STw_{n+1}, kt) M(ABu, Qw_{n+1}, 2kt)$
$\geq [p M(Pu, ABu, t) + q M(ABu, STw_{n+1}, t)] M(ABu, Qw_{n+1}, 2kt)$.
Letting $n \to \infty$ and using (3) and (13) we get
$M^2(Pu, u, kt) * [M(Pu, u, kt) M(u, u, kt)] * M^2(u, u, kt) + a M(u, u, kt) M(u, u, 2kt)$
$\geq [p M(Pu, u, t) + q M(u, u, t)] M(u, u, 2kt)$, and
$M^2(Pu, u, kt) * M(Pu, u, kt) + a \geq p M(Pu,u, t) + q$,
As in step 5, it follows that Pu = u. Hence Pu = ABu = u. ... (14)
The continuity of ST gives, $(ST)^2w_{n+1} \to STu$. ... (15)
As (Q, ST) is compatible of type (α), by (i) we get that
$QSTw_{n+1} \to STu$. ... (16)

Step 8: Taking $x = v_n$, $y = STw_{n+1}$ in (3.1.5) we get,
$M^2(Pv_n, QSTw_{n+1}, kt) * [M(Pv_n, ABv_n, kt) M(QSTw_{n+1}, (ST)^2w_{n+1}, kt)]$
$* M^2(QSTw_{n+1}, (ST)^2w_{n+1}, kt) + a M(QSTw_{n+1}, (ST)^2w_{n+1}, kt) M(ABv_n,QSTw_{n+1}, 2kt)$
$\geq [p M(Pv_n, ABv_n, t) + q M(ABv_n, (ST)^2w_{n+1}, t)] M(ABv_n, QSTw_{n+1}, 2kt)$.
Letting $n \to \infty$ using (2), (15) and (16) we get
$M^2(u, STu, kt) * [M(u, u, kt) M(STu, STu, kt)] * M^2(STu, STu, kt)$
$+ a M(STu, STu, kt) M(u, STu, 2kt) \geq [p M(u, u, t) + q M(u, STu, t)] M(u, STu, 2kt)$,
i. e. $M^2(u, STu, kt) + a M(u, STu, 2kt) \geq [p + q M(u, STu, t)] M(u, STu, 2kt)$,
As in step 2, it follows that $STu = u$. ... (17)

Step 9: Taking $x = u$, $y = u$ in (3.1.5), we get that
$M^2(Pu, Qu, kt) * [M(Pu, ABu, kt) M(Qu, STu, kt)] * M^2(Qu, STu, kt)$
$+ a M(Qu, STu, kt) M(ABu, Qu, 2kt)$
$\geq [p M(Pu, ABu, t) + q M(ABu, STu, t)] M(ABu, Qu, 2kt)$.
Using (14) and (17) we get
$M^2(u, Qu, kt) * [M(u, u, kt) M(Qu, u, kt)] * M^2(Qu, u, kt)$
$+ a M(Qu, u, kt) M(u, Qu, 2kt) \geq [p M(u, u, t) + q M(u, u, t)] M(u, Qu, 2kt)$,
then $M^2(u, Qu, kt) + a M(u, Qu, 2kt) \geq [p + q] M(u, Qu, 2kt)$,
Since $M(x, y, .)$ is non-decreasing for all x, y in X we have,
$M(u, Qu, 2kt) M(u, Qu, t) \geq [p + q - a] M(Qu, u, 2kt)$.
As $p +q - a = 1$, we get that $M(u, Qu, t) \geq 1$.
Thus $Qu = u$. Therefore $Qu = STu = u$.
Hence in both the cases we have $Pu = Qu = STu = ABu = u$.

Case 3: P and ST are continuous.
As P is continuous and (P, AB) is compatible of type (α), therefore (4) and (5) hold. Hence by step 2, it follows that $Pu = u$. Also as ST is continuous and (Q, ST) is compatible of type (α), therefore (15) and (16) hold. Hence by step 8, we have $STu = u$. Thus $Pu = STu = u$. Hence $PSTu = u$. Now as in step 5 (of case 1), it follows that
$Pu = ABu$. ... (18)
Therefore $Pu = ABu = STu = u$.
Taking $x = u$, $y = u$ in (3.1.5), as in step 9, we get $Qu = u$. Thus in this case also $Pu = Qu = ABu = STu = u$.

Case 4: Q and AB are continuous.
As Q is continuous and (Q, ST) is compatible of type (α), therefore (7) and (8) hold. Hence by step 3, it follows that $Qu = u$. Also from step 5, we have $STu = u$. Now as AB is continuous and (P, AB) is compatible of type (α), therefore (11) and (12) hold. Hence by step 6, we have $ABu = u$. Thus $Qu = STu = ABu = u$.
Taking $x = u$, $y = u$ in (3.1.5). As in step 5, we have $Pu = u$.
Thus in this case also $Pu = Qu = ABu = STu = u$. ... (19)
Hence in all the four cases we get that u is a common fixed point of P, Q, AB and ST.

Step 10: Taking $x = Bu$, $y = w_{n+1}$ in (3.1.5) we get,
$M^2(PBu, Qw_{n+1}, kt) * [M(PBu, ABBu, kt) M(Qw_{n+1}, STw_{n+1}, kt)] *$
$M^2(Qw_{n+1}, STw_{n+1}, kt) + a M(Qw_{n+1}, STw_{n+1}, kt) M(ABBu, Qw_{n+1}, 2kt)$
$\geq [p M(PBu, ABBu, t) + q M(ABBu, STw_{n+1}, t)] M(ABBu, Qw_{n+1}, 2kt)$.
Now PBu = BPu = Bu. Also ABBu = BABu = B(ABu) = Bu. Thus
$M^2(Bu, Qw_{n+1}, kt) * [M(Bu, Bu, kt) M(Qw_{n+1}, STw_{n+1}, kt)] * M^2(Qw_{n+1}, STw_{n+1}, kt)$
$+ a M(Qw_{n+1}, STw_{n+1}, kt) M(Bu, Qw_{n+1}, 2kt)$
$\geq [p M(Bu, Bu, t) + q M(Bu, STw_{n+1}, t)] M(Bu, Qw_{n+1}, 2kt)$.
Letting $n \to \infty$ and using (3) we get that
$M^2(Bu, u, kt) *[M(Bu, Bu, kt) M(u, u, kt)] * M^2(u, u, kt)+ a M(u, u, kt) M(Bu, u, 2kt)$
$\geq [p M(Bu, Bu, t) + q M(Bu, u, t)] M(Bu, u, 2kt)$.
As in step 2, it follows that Bu = u. Now ABu = u and Bu = u gives Au = u.
Therefore Pu = ABu = Bu = Au = u .

Step 11: Taking x = u, y = Tu in (3.1.5) we get,
$M^2(Pu, QTu, kt) * [M(Pu, ABu, kt) M(QTu, STTu, kt)] * M^2(QTu, STTu, kt)$
$+ a M(QTu, STTu, kt) M(ABu, QTu, 2kt)$
$\geq [p M(Pu, ABu, t) + q M(ABu, STTu, t)] M(ABu, QTu, 2kt)$.
As STTu = TSTu = T(STu) = Tu. Also QTu = TQu = Tu. Therefore
$M^2(Pu, Tu, kt) * [M(Pu, ABu, kt) M(Tu, Tu, kt)] * M^2(Tu, Tu, kt)$
$+ a M(Tu, Tu, kt) M(ABu, Tu, 2kt)$
$\geq [p M(Pu, ABu, t) + q M(ABu, Tu, t)] M(ABu, Tu, 2kt)$.
Using (19) we get that
$M^2(u, Tu, kt) * [M(u, u, kt) M(Tu, Tu, kt)] * M^2(Tu, Tu, kt)$
$+ a M(Tu, Tu, kt) M(u, Tu, 2kt) \geq [p M(u, u, t) + q M(u, Tu, t)] M(u, Tu, 2kt)$.
Then as in step 2, it follows tha Tu = u. Now STu = u and Tu = u gives Su = u.
Combining all the above results we get Pu = Qu = Su = Tu = Au = Bu = u.

Uniqueness: Let z be another common fixed point of P, Q, S, T, A and B. i. e. Pz = Qz = Sz = Tz = Az = Bz = z. Taking x = u, y = z in (3.1.5) we get,
$M^2(Pu, Qz, kt) * [M(Pu, ABu, kt) M(Qz, STz, kt)] * M^2(Qz, STz, kt)$
$+ a M(Qz, STz, kt) M(ABu, Qz, 2kt)$
$\geq [p M(Pu, ABu, t) + q M(ABu, STz, t)] M(ABu, Qz, 2kt)$,
i. e. $M^2(u, z, kt) * [M(u, u, kt) M(z, z, kt)] * M^2(z, z, kt) + a M(z, z, kt) M(u, z, 2kt)$
$\geq [p M(u, u, t) + q M(u, z, t)] M(u, z, 2kt)$.
As in step 2, it follows that u = z and thus u is a unique common fixed point of six self-maps P, Q, S, T, A and B.
Taking a = 0 we get;

Corollary 3.2: Let (X, M, *) be a complete fuzzy metric space and let P, Q, S, T, A and B be self-maps from X satisfying (3.11), (3.1.2) (3.1.3), (3.1.4) and

- there exists a constant $k \in (0, 1)$ such that

$M^2(Px, Qy, kt) * [M(Px, ABx, kt) M(Qy, STy, kt)] * M^2(Qy, STy, kt)$
$\geq [p M(Px, ABx, t) + q M(ABx, STy, t)] M(ABx,Qy, 2kt), \forall x, y \in X, \forall t > 0$,
for some $p, q \in (0, 1)$ with $p + q = 1$.

Then P, Q, S, T, A and B have a unique common fixed point in X.
Taking B = T = I in theorem 3.1, the quoted result of [8] follow:

Corollary 3.3: Let (X, M, *) be a complete fuzzy metric space with condition FM-6 and P, Q, S and A be self-maps from X satisfying

- $PS(X) \cup QA(X) \subseteq AS(X)$;
- SA =AS;
- the pairs (P, A) and (Q, S) are compatible maps of type (α);
- either A and S or else P and Q or else P and S or else Q and A are continuous
- $\exists$ a constant $k \in (0, 1)$ such that

$$M^2(Px, Qy, kt) * [M(Px, Ax, kt)\, M(Qy, Sy, kt)] * M^2(Qy, Sy, kt)$$
$$+ a\, M(Qy, Sy, kt)\, M(Ax, Qy, 2kt)$$
$$\geq [p\, M(Px, Ax, t) + q\, M(Ax, Sy, t)]\, M(Ax ,Qy, 2kt), \ \forall\, x, y \in X, \forall\, t > 0,$$

for some $p, q \in (0, 1)$ and $a \in (-1, 1)$ with $p + q - a = 1$.

Then P, Q, A and S have a unique common fixed point in X.

Taking A = S = I, in corollary 3.3 we have

Corollary 3. 4: Let (X, M, *) be a complete fuzzy metric space and let P and Q be self-maps from X such that

- $\exists$ a constant $k \in (0, 1)$ such that

$$M^2(Px, Qy, kt) * [M(Px, x, kt)\, M(Qy, y, kt)] * M^2(Qy, y, kt)$$
$$+ a\, M(Qy, y, kt)\, M(x, Qy, 2kt) \geq [p\, M(Px, x, t) + q\, M(x, y, t)]\, M(x, Qy, 2kt),$$

$\forall\, x, y \in X, \forall\, t > 0$, where $p, q \in (0, 1)$ and $a \in (-1, 1)$ with $p + q - a = 1$.

Then P and Q have a unique common fixed point in X.

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Received: June, 2009