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Generalized Heat Kernel Related to the Operator L_m^k and Spectrum

T. Panyatip[1] **and A. Kananthai**[2]

[1,2]Department of Mathematics
Chiangmai University
Chiangmai, 50200, Thailand
[1,2]E-mail: malamnka@science.cmu.ac.th

Abstract

In this paper, we study the equation

$$\frac{\partial}{\partial t}u(x,t) + c^2 L_m^k u(x,t) = 0$$

with the initial condition $u(x,0) = f(x)$ for $x \in \mathbb{R}^n$, where the operator L_m^k is defined by

$$L_m^k = (-1)^{mk}\left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^m - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^m\right]^k,$$

$p + q = n$ is the dimension of the space $\mathbb{R}^n$, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is a given generalized function, k and m are a positive integer and c is a positive constant. We obtain the solution of such equation which is related to the spectrum and the kernel. Moreover, such the kernel has interesting properties and also related to the kernel of an extension of the heat equation.

Mathematics Subject Classification: 35G20, 47F05

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1 Introduction

It is well known that for the heat equation

$$\frac{\partial}{\partial t}u(x,t) = c^2 \triangle u(x,t) \tag{1}$$

with the initial condition $u(x,0) = f(x)$, where $\triangle$ is the Laplace operator defined by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

and $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$, we obtain the solution in the convolution form $u(x,t) = E(x,t) * f(x)$ where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} e^{-\frac{|x|^2}{4c^2 t}}. \tag{2}$$

$E(x,t)$ is call the heat kernel, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$, see [1, p. 208-209]. We can extend (1) to the equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Box u(x,t) \tag{3}$$

where $\Box$ is the ultra-hyperbolic operator which is defined by

$$\Box = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2},$$

$p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$. we obtain

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega e^{c^2 t\left[\left(\sum_{j=p+1}^{p+q} \xi_j^2\right) - \left(\sum_{i=1}^{p} \xi_i^2\right)\right] + i(\xi, x)} d\xi \tag{4}$$

where $i = \sqrt{-1}$ and $\sum_{i=1}^{p} \xi_i^2 > \sum_{j=p+1}^{p+q} \xi_j^2$, see [5].

On the other hand, diamond heat equation

$$\frac{\partial}{\partial t} u(x,t) + c^2 \Diamond u(x,t) = 0 \tag{5}$$

where $\Diamond$ is the diamond operator which is defined by

$$\Diamond = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2,$$

$p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$, we obtain

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega e^{-c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right) + i(\xi, x)} d\xi$$

and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed $t > 0$, see [4].

Now the purpose of this work is to study the equation

$$\frac{\partial}{\partial t} u(x,t) + c^2 L_m^k u(x,t) = 0, \tag{6}$$

with initial condition $u(x,0) = f(x)$ for $x \in \mathbb{R}^n$ where the operator L_m^k is defined by

$$L_m^k = (-1)^{mk} \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^m - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^m \right]^k \tag{7}$$

$p + q = n$ is the dimension of the space $\mathbb{R}^n$, $u(x,t)$ is an unknown function, $f(x)$ is a given generalized function, k and m is a positive integer and c is a positive constant.

We obtain $u(x,t) = E(x,t) * f(x)$, as a solution of (6) which satisfies $u(x,0) = f(x)$, where

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega e^{\left[-c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^m \right)^k + i(\xi,x)\right]} d\xi \tag{8}$$

and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed $t > 0$. The function $E(x,t)$ is called the kernel or elementary solution of (6).

2 Preliminary Notes

Definition 2.1. *Let $f(x) \in L_1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ defined by*

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x) dx. \tag{9}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ and $dx = dx_1, dx_2, \ldots, dx_n$

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \hat{f}(\xi) d\xi. \tag{10}$$

Definition 2.2. *The spectrum of the kernel $E(x,t)$ defined by (13) is the bounded support of the Fourier transform $\widehat{E(x,t)}$, for any fixed $t > 0$.*

Definition 2.3. *Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$ and denote*

$$\Gamma_+ = \{\xi \in \mathbb{R}^n : \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \cdots - \xi_{p+q}^2 > 0 \text{ and } \xi_1 > 0\}$$

to be the set of an interior of the forward cone and $\overline{\Gamma}_+$ denotes the closure of Γ_+.

Let Ω be the spectrum of $E(x,t)$ defined by (2.2) and $\Omega \subset \overline{\Gamma}_+$. Let $\widehat{E(\xi,t)}$ be the Fourier transform of $E(x,t)$ which is defined by

$$\widehat{E(\xi,t)} = \begin{cases} \frac{1}{(2\pi)^{\frac{n}{2}}} e^{\left[-c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^m \right)^k\right]} & \text{for } \xi \in \Gamma^+; \\ 0 & \text{for } \xi \notin \Gamma^+. \end{cases} \tag{11}$$

Lemma 2.4. *Let The operator L defined by*

$$L = \frac{\partial}{\partial t} + c^2 L_m^k, \tag{12}$$

where L_m^k is the product operator iterated k-times and is defined by

$$L_m^k = (-1)^{mk}\left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^m - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^m\right]^k$$

$p+q=n$ is the dimension of the $\mathbb{R}^n$, $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, k and m are a positive integer and c is a positive constant. Then we obtain

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega e^{\left[-c^2t\left(\left(\sum_{i=1}^p \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k + i(\xi,x)\right]} d\xi \tag{13}$$

is the elementary solution of (12) *in the spectrum $\Omega\subset\mathbb{R}^n$ for $t>0$.*

Proof. Let $E(x,t)$ be the kernel or elementary solution of L_m^k operator and let δ be the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t}E(x,t) + c^2 L_m^k E(x,t) = \delta(x)\delta(t).$$

Applying the Fourier transform to the both sides of the above equation, we have

$$\frac{\partial}{\partial t}\widehat{E(\xi,t)} + c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right]^k \widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\delta(t).$$

Hence, we obtain

$$\widehat{E(\xi,t)} = \frac{H(t)}{(2\pi)^{n/2}} e^{-c^2t\left(\left(\sum_{i=1}^p \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k}$$

where $H(t)$ is the Heaviside function. Since $H(t)=1$ for $t>0$,

$$\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}} e^{-c^2t\left(\left(\sum_{i=1}^p \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k}$$

which has been already by (11). By inverse Fourier transform, we have

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\left[-c^2t\left(\left(\sum_{i=1}^p \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k + i(\xi,x)\right]} d\xi$$

Since Ω is the spectrum of $E(x,t)$, we obtain

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega e^{\left[-c^2t\left(\left(\sum_{i=1}^p \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k + i(\xi,x)\right]} d\xi$$

for $t>0$. □

3 Main Results

Theorem 3.1. *Given the equation*

$$\frac{\partial}{\partial t}u(x,t) + c^2 L_m^k u(x,t) = 0 \tag{14}$$

with initial condition

$$u(x,0) = f(x) \tag{15}$$

where L_m^k is the operator iterated k-times and defined by

$$L_m^k = (-1)^{mk}\left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^m - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^m\right]^k$$

$p+q=n$ is the dimension of the space $\mathbb{R}^n$, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is a given generalized function, k and m are a positive integer and c is a positive constant. Then

$$u(x,t) = E(x,t) * f(x) \tag{16}$$

is a solution of (14) *which satisfies* (15)*, where $E(x,t)$ is given by* (13)*.*

Proof. Taking the Fourier transform to the both sides of the (14), we obtain

$$\frac{\partial}{\partial t}\hat{u}(\xi,t) = -c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right]^k \hat{u}(\xi,t).$$

Thus

$$\hat{u}(\xi,t) = K(\xi)e^{-c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k}, \tag{17}$$

where $K(\xi)$ is constant and $\hat{u}(\xi,0) = K(\xi)$.

Now, by (15) we have

$$K(\xi) = \hat{u}(\xi,0) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)dx. \tag{18}$$

and by the inversion in (10), (17), (18) we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\hat{u}(\xi,t)d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\xi,x)}e^{-i(\xi,y)}f(y)e^{-c^2 t\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right]^k}d\xi dy. \end{aligned}$$

$$u(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\left[-c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^m\right)^k + i(\xi, x-y)\right]} f(y) d\xi dy. \quad (19)$$

Set

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\left[-c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^m\right)^k + i(\xi, x)\right]} d\xi \quad (20)$$

Since the integral in (20) is divergent, therefore we choose $\Omega \subset \mathbb{R}^n$ be the spectrum of $E(x,t)$ and by (12), we have

$$\begin{aligned} E(x,t) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\left[-c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^m\right)^k + i(\xi, x)\right]} d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\Omega} e^{\left[-c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^m\right)^k + i(\xi, x)\right]} d\xi \quad (21) \end{aligned}$$

Thus (19) can be written in the convolution form

$$u(x,t) = E(x,t) * f(x).$$

Moreover, since $E(x,t)$ exists, we see that

$$\begin{aligned} \lim_{t \to 0} E(x,t) &= \frac{1}{(2\pi)^n} \int_{\Omega} e^{i(\xi,x)} d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\xi,x)} d\xi \\ &= \delta(x), \quad \text{for} \quad x \in \mathbb{R}^n. \quad (22) \end{aligned}$$

holds (see [1,p.396, equation (10.2.19b)]).

Thus for the solution $u(x,t) = E(x,t) * f(x)$ of (14), then we have

$$u(x,0) = \lim_{t \to 0} u(x,t) = \lim_{t \to 0} E(x,t) * f(x) = \delta(x) * f(x) = f(x)$$

which satisfies (15). This complete the proof. □

Theorem 3.2. *The kernel $E(x,t)$ is defined by (21) has the following properties:*

(1) $E(x,t) \in C^\infty(\mathbb{R}^n \times (0,\infty))$ *the space of continuous with infinitely differentiable,*

(2) $(\frac{\partial}{\partial t} + c^2 L_m^k) E(x,t) = 0$, *for $t > 0$,*

(3) $|E(x,t| \leq \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}$, *for $t > 0$,*
where $M(t)$ is a function of $t > 0$ in the the spectrum Ω and Γ denote the Gamma function. Thus $E(x,t)$ is bounded for any fixed $t > 0$.

(4) $\lim_{t\to 0} E(x,t) = \delta(x)$.

Proof. (1) From (21), since

$$\frac{\partial^n}{\partial x^n}E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \frac{\partial^n}{\partial x^n} e^{\left[-c^2t\left(\left(\sum_{i=1}^p \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k + i(\xi,x)\right]}d\xi$$

Thus $E(x,t) \in C^\infty$ for $x \in \mathbb{R}^n$ and $t > 0$.
(2) By computing directly, we obtain $\left(\frac{\partial}{\partial t} + c^2 L_m^k\right) E(x,t) = 0$
(3) We have

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega e^{\left[-c^2t\left(\left(\sum_{i=1}^p \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k + i(\xi,x)\right]}d\xi$$

$$|E(x,t)| \le \frac{1}{(2\pi)^n}\int_\Omega e^{-c^2t\left[\left(\sum_{i=1}^p \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right]^k}d\xi$$

By changing to bipolar coordinates

$$\xi_1 = r\omega_1, \xi_2 = r\omega_2, \ldots, \xi_p = r\omega_p$$

and

$$\xi_{p+1} = s\omega_{p+1}, \xi_{p+2} = s\omega_{p+2}, \ldots, \xi_{p+q} = s\omega_{p+q},$$

where $\sum_{i=1}^p \omega_i^2 = 1$ and $\sum_{j=p+1}^{p+q}\omega_j^2 = 1$ Thus

$$|E(x,t)| \le \frac{1}{(2\pi)^n}\int_\Omega e^{-c^2t\left(r^{2m}-s^{2m}\right)^k} r^{p-1}s^{q-1}drdsd\omega_p d\omega_q$$

where $d\xi = drdsd\omega_p d\omega_q$ and $d\omega_p$, $d\omega_q$ are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Since $\omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ and we suppose $0 \le r \le R$ and $0 \le s \le T$ where R and T are constants. Thus we obtain

$$\begin{aligned}|E(x,t)| &\le \frac{\omega_p\omega_q}{(2\pi)^n}\int_0^R\int_0^T e^{-c^2t\left(r^{2m}-s^{2m}\right)^k} r^{p-1}s^{q-1}drds\\ &= \frac{\omega_p\omega_q}{(2\pi)^n}M(t) \quad \text{for any fixed } t>0 \text{ in the spectrum } \Omega\\ &= \frac{2^{2-n}}{\pi^{n/2}}\frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})},\end{aligned}$$

where

$$M(t) = \int_0^R\int_0^T e^{-c^2t\left(r^{2m}-s^{2m}\right)^k} r^{p-1}s^{q-1}drds$$

is a function of t, $\omega_p = \frac{2\pi^{p/2}}{\Gamma(\frac{p}{2})}$ and $\omega_q = \frac{2\pi^{p/2}}{\Gamma(\frac{q}{2})}$. Thus, for any fixed $t > 0$, $E(x,t)$ is bounded.
(4) obvious by (22). □

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