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Another Approach to Solution of Fuzzy Differential Equations

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Abstract

In this paper, we have introduced and studied a new technique for getting the solution of fuzzy initial value problem.

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1 Introduction

Fuzzy differential equations are a natural way to model dynamical systems under uncertainty. First order linear fuzzy differential equations are one of the simplest fuzzy differential equations, which appear in many applications.

In the recent years, the topic of FDEs has been investigated extensively. The concept of a fuzzy derivative was first introduced by S. L. Chang and L. A. Zadeh in [3]. In this paper, we have introduced and studied a new technique for getting the solution of fuzzy initial value problem. The organized paper is as follows: In the first three sections, we recall some concepts and introductory materials to deal with the fuzzy initial value problem. In sections four and five, we present Runge-Kutta method of order three and its iterative solution for solving Fuzzy differential equations. The proposed algorithm is illustrated by an example in the last section.

2 Preliminary

A trapezoidal fuzzy number u is defined by four real numbers $k < \ell < m < n,$ where the base of the trapezoidal is the interval $[k, n]$ and its vertices at $x = \ell$, $x = m$. Trapezoidal fuzzy number will be written as $u = (k, \ell, m, n)$. The membership function for the trapezoidal fuzzy number $u = (k, \ell, m, n)$ is defined as the following :

$$u(x) = \begin{cases} \dfrac{x-k}{\ell-k}, & k \le x \le \ell \\ 1, & \ell \le x \le m \\ \dfrac{x-n}{m-n}, & m \le x \le n \end{cases} \tag{1}$$

we will have :

(1) $u > 0$ if $k > 0$;
(2) $u > 0$ if $\ell > 0$;
(3) $u > 0$ if $m > 0$;
and (4) $u > 0$ if $n > 0$.

Let us denote R_F by the class of all fuzzy subsets of R (i.e. $u : R \to [0,1]$) satisfying the following properties:

(i) $\forall u \in R_F$, u is normal, i.e. $\exists x_0 \in R$ with $u(x_0) = 1$;
(ii) $\forall u \in R_F$, u is convex fuzzy set (i.e. $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, $\forall t \in [0,1]$, $x, y \in R$);
(iii) $\forall u \in R_F$, u is upper semi continuous on R;
(iv) $\overline{\{x \in R; u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Then R_F is called the space of fuzzy numbers (see e.g. [5]).
Obviously $R \subset R_F$. Here $R \subset R_F$ is understood as

$$R = \{\chi_{\{x\}}; x \text{ is usual real number}\}.$$

We define the r-level set, $x \in R$;

$$[u]_r = \{x \setminus u(x) \geq r\}, \qquad 0 \leq r \leq 1; \tag{2}$$

Clearly, $[u]_0 = \{x \setminus u(x) > 0\}$ is compact,

which is a closed bounded interval and we denote by $[u]_r = [\underline{u}(r), \overline{u}(r)]$. It is clear that the following statements are true,

1. $\underline{u}(r)$ is a bounded left continuous non decreasing function over [0,1],
2. $\overline{u}(r)$ is a bounded right continuous non increasing function over [0,1],
3. $\underline{u}(r) \leq \overline{u}(r)$ for all $r \in (0,1]$,

for more details see [1],[2].

Let $D: R_F \times R_F \rightarrow R_+ \cup \{0\}$,

$D(u,v) = \mathrm{Sup}_{r\in[0,1]} \max \{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$, be *Hausdorff distance* between fuzzy numbers, where $[u]_r = [\underline{u}(r), \overline{u}(r)]$, $[v]_r = [\underline{v}(r), \overline{v}(r)]$. The following properties are well-known (see e.g. [6]):

$$D(u + w, v + w) = D(u, v), \ \forall\ u,v,w \in R_F,$$
$$D(k.u, k.v) = |k| D(u, v), \ \forall\ k \in R,\ u,v \in R_F,$$
$$D(u + v, w + e) \leq D(u,w) + D(v,e), \ \forall\ u, v, w, e \in R_F$$

and (R_F, D) is a complete metric space.

Lemma 2.1

If the sequence of non-negative numbers $\{W_n\}_{n=0}^{N}$ satisfy

$$|W_{n+1}| \leq A|W_n| + B, \quad 0 \leq n \leq N - 1,$$

for the given positive constants A and B, then

$$|W_n| \leq A^n |W_0| + B\frac{A^n - 1}{A - 1}, \qquad 0 \leq n \leq N.$$

Lemma 2.2

If the sequence of numbers $\{W_n\}_{n=0}^{N}$, $\{V_n\}_{n=0}^{N}$ satisfy

$$|W_{n+1}| \leq |W_n| + A \max\{|W_n|, |V_n|\} + B,$$
$$|V_{n+1}| \leq |V_n| + A \max\{|W_n|, |V_n|\} + B,$$

for the given positive constants A and B, then denoting

$U_n = |W_n| + |V_n|$, $0 \leq n \leq N$,

we have, $$U_n \leq \overline{A}^n U_0 + \overline{B}\frac{\overline{A}^n - 1}{\overline{A} - 1}, \ 0 \leq n \leq N,$$

where $\overline{A} = 1 + 2A$ and $\overline{B} = 2B$.

Lemma 2.3

Let $F(t, u, v)$ and $G(t, u, v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F. Then for arbitrarily fixed r, $0 \leq r \leq 1$,

$$D\left(y(t_{n+1}), y^{(0)}(t_{n+1})\right) \leq h^2 L(1 + 2C),$$

where L is a bound of partial derivatives of F and G, and

$$C = \max\left\{\left| G\left[t_N, \underline{y}(t_N; r), \overline{y}(t_{N-1}; r)\right]\right|, r \in [0,1]\right\} < \infty.$$

Theorem 2.4

Let $F(t, u, v)$ and $G(t, u, v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F. Then for arbitrarily fixed r, $0 \le r \le 1$, the numerical solutions of $\underline{y}(t_{n+1};r)$ and $\overline{y}(t_{n+1};r)$ converge to the exact solutions $\underline{Y}(t;r)$ and $\overline{Y}(t;r)$ uniformly in t.

Theorem 2.5

Let $F(t, u, v)$ and $G(t, u, v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F and $2Lh < 1$. Then for arbitrarily fixed $0 \le r \le 1$, the iterative numerical solutions of $\underline{y}^{(j)}(t_n;r)$ and $\overline{y}^{(j)}(t_n;r)$ converge to the numerical solutions $\underline{y}(t_n;r)$ and $\overline{y}(t_n;r)$ in $t_0 \le t_n \le t_N$, when $j \to \infty$.

3 Fuzzy Initial Value Problem

Consider a first-order fuzzy initial value differential equation is given by

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \tag{3}$$

where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of the crisp variable t and the fuzzy variable y, y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a trapezoidal or a trapezoidal shaped fuzzy number.

We denote the fuzzy function y by $y = [\underline{y}, \overline{y}]$. It means that the r-level set of $y(t)$ for $t \in [t_0, T]$ is

$$[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)],$$

$$[y(t_0)]_r = [\underline{y}(t_0;r), \overline{y}(t_0;r)], \quad r \in (0,1]$$

we write $f(t, y) = [\underline{f}(t, y), \overline{f}(t, y)]$ and

$$\underline{f}(t, y) = F[t, \underline{y}, \overline{y}],$$

$$\overline{f}(t, y) = G[t, \underline{y}, \overline{y}].$$

Because of $y' = f(t, y)$ we have

$$\underline{f}(t, y(t); r) = F[t, \underline{y}(t;r), \overline{y}(t;r)] \tag{4}$$

$$\overline{f}(t, y(t); r) = G[t, \underline{y}(t;r), \overline{y}(t;r)] \tag{5}$$

By using the extension principle, we have the membership function

$$f(t, y(t))(s) = \sup\{y(t)(\tau) \backslash s = f(t,\tau)\}, \quad s \in R \tag{6}$$

so fuzzy number $f(t, y(t))$. From this it follows that

$$[f(t, y(t))]_r = [\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)], \quad r \in (0,1], \tag{7}$$

where

$$\underline{f}(t, y(t); r) = \min\{f(t, u) | u \in [y(t)]_r\} \tag{8}$$

$$\overline{f}(t, y(t); r) = \max\{f(t, u) | u \in [y(t)]_r\}. \tag{9}$$

Definition 3.1 A function f: $R \to R_F$ is said to be fuzzy continuous function, if for an arbitrary fixed $t_0 \in R$ and $\in > 0$, $\delta > 0$ such that

$$|t - t_o| < \delta \Rightarrow D\,[f(t), f(t_0)] < \in$$

exists.

Throughout this paper we also consider fuzzy functions which are continuous in metric D. Then the continuity of $f(t,y(t);r)$ guarantees the existence of the definition of $f(t, y(t); r)$ for $t \in [t_0, T]$ and $r \in [0,1]$ [4]. Therefore, the functions G and F can be definite too.

4 Runge -Kutta method of order three

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \tag{10}$$

Assuming the following Runge-Kutta method with three slopes

$$y(t_{n+1}) = y(t_n) + W_1K_1 + W_2K_2 + W_3K_3 \tag{11}$$

where

$$K_1 = hf(t_n, y(t_n))$$

$$K_2 = hf(t_n + c_2h, y(t_n) + a_{21}K_1)$$

$$K_3 = hf(t_n + c_3h, y(t_n) + a_{31}K_1 + a_{32}K_2)$$

and the parameters $W_1, W_2, W_3, c_2, c_3, a_{21}, a_{31}$ & a_{32} are chosen to make y_{n+1} closer to $y(t_{n+1})$.There are eight parameters to be determined .Now, Taylor's series expansion about t_n gives

$$y(t_{n+1}) = y(t_n) + \frac{h}{1!}y'(t_n) + \frac{h^2}{2!}y''(t_n) + \frac{h^3}{3!}y'''(t_n) + \ldots$$

$$= y(t_n) + \frac{h}{1!}f(t_n, y(t_n)) + \frac{h^2}{2!}\left[f_t + ff_y\right]_{t_n} + \frac{h^3}{3!}\left[f_{tt} + 2ff_{ty} + f^2f_{yy} + f_y(f_t + ff_y)\right]_{t_n} + \ldots \tag{12}$$

If we set

$$K_1 = hf_n$$

$$K_2 = hf\left(t_n + c_2h, y(t_n) + a_{21}K_1\right)$$
$$= h\left\{ f_n + \frac{h}{1!}\left[c_2 f_t + a_{21} ff_y\right]_{t_n} + \frac{h^2}{2!}\left[c_2^{\ 2} f_{tt} + 2c_2 a_{21} ff_{ty} + a_{21}^{\ 2} f^2 f_{yy}\right]_{t_n} + \dots \right\}$$

$$K_3 = hf\left(t_n + c_3h, y(t_n) + a_{31}K_1 + a_{32}K_2\right)$$
$$= h\left\{ f_n + c_3hf_t + \left[a_{31}K_1 + a_{32}K_2\right]f_y + \frac{1}{2!}\begin{bmatrix} c_3^{\ 2}h^2 f_{tt} + 2c_3h(a_{31}K_1 + a_{32}K_2)f_{ty} \\ +(a_{31}K_1 + a_{32}K_2)^2 f_{yy} \end{bmatrix} + \dots \right\}$$

$$= h\left\{ \begin{array}{l} f_n + \frac{h}{1!}\left[c_3 f_t + (a_{31} + a_{32}) f_n f_y\right]_{t_n} + \frac{h^2}{2!}\begin{bmatrix} 2\left(c_2 f_t + a_{21} ff_y\right)a_{32} f_y + c_3^{\ 2} f_{tt} \\ + 2c_3 a_{31} f_n + 2c_3 a_{32} f_{ty} f_n \\ + \left(a_{31}^{\ 2} + a_{32}^{\ 2} + 2a_{31}a_{32}\right) f_n^{\ 2} f_{yy} \end{bmatrix}_{t_n} \\ + \frac{h^3}{3!}\begin{bmatrix} 3\left(c_2^{\ 2} f_{tt} + 2c_2 a_{21} ff_{ty} + a_{21}^{\ 2} f^2 f_{yy}\right)a_{32} f_y \\ + \left(6c_3 a_{32} f_{ty} + 6a_{31} f_n a_{32} f_{yy}\right)\left(c_2 f_t + a_{21} ff_y\right) \end{bmatrix}_{t_n} \\ + \dots \end{array} \right\}$$

Substituting the values of K_1, K_2 & K_3 in (11), we get

$$y(t_{n+1}) = y(t_n) + \left[W_1 + W_2 + W_3\right]hf_n + h^2\left[W_2\left(c_2 f_t + a_{21} ff_y\right) + W_3\left(c_3 f_t + (a_{31} + a_{32}) f_n f_y\right)\right]_{t_n} +$$
$$+ \frac{h^3}{2}\left[W_2\left(c_2^{\ 2} f_{tt} + 2c_2 a_{21} ff_{ty} + a_{21}^{\ 2} f^2 f_{yy}\right) + W_3\left(\begin{array}{l} 2\left(c_2 f_t + a_{21} ff_y\right)a_{32} f_y + c_3^{\ 2} f_{tt} + 2c_3 a_{31} f_n + \\ 2c_3 a_{32} f_n f_{ty} + \left(a_{31}^{\ 2} + a_{32}^{\ 2} + 2a_{31}a_{32}\right) f_n^{\ 2} f_{yy} \end{array} \right) \right]_{t_n}$$
$$+ \dots \tag{13}$$

Comparing the coefficients of h, h^2 & h^3 in (12) & (13), we obtain

$$a_{21} = c_2\,, \qquad a_{31} + a_{32} = c_3\,, \qquad W_1 + W_2 + W_3 = 1,$$
$$c_2 W_2 + c_3 W_3 = \frac{1}{2} \qquad c_2^{\,2} W_2 + c_3^{\,2} W_3 = \frac{1}{3} \qquad c_2 a_{32} W_3 = \frac{1}{6}. \tag{14}$$

Multiplying the fourth and fifth equations by $c_2\, a_{32}$ and using the sixth equation of (14), we get

$$c_2^{\,2} a_{32} W_2 + c_3\left(\frac{1}{6}\right) = \frac{1}{2} c_2 a_{32}\,, \qquad c_2^{\,3} a_{32} W_2 + c_3^{\,2}\left(\frac{1}{6}\right) = \frac{1}{3} c_2 a_{32}\,.$$

Eliminating W_2 from these two equations, we find that no solution exists unless

$$\frac{3c_2 a_{32} - c_3}{6c_2^{\,2} a_{32}} = \frac{2c_2 a_{32} - c_3^{\,2}}{6c_2^{\,3}\ a_{32}} \quad or \quad a_{32} = \frac{c_3(c_3 - c_2)}{c_2(2 - 3c_2)}. \tag{15}$$

Usually, c_2, c_3 are arbitrarily chosen and a_{32} is determined from (15). However, if $c_2 = c_3$, then we immediately obtain from the fourth and fifth equations of (14), that $c_2 = \frac{2}{3}$. The values of the remaining parameters are obtained from (14).

When $c_2 = c_3$, we get $c_2 = \frac{2}{3}$ and $a_{21} = \frac{2}{3}$. We get the values of the other parameters as $a_{31} = 0,\ a_{32} = \frac{2}{3},\ W_1 = \frac{2}{8},\ W_2 = \frac{3}{8}\ \&\ W_3 = \frac{3}{8}.$

Runge-Kutta method is obtained as

$$y(t_{n+1}) = y(t_n) + \frac{1}{8}\left[2K_1 + 3K_2 + 3K_3\right] \tag{16}$$

Where

$$K_1 = hf\,(t_n, y(t_n))$$
$$K_2 = hf\,(t_n + \frac{2h}{3}, y(t_n) + \frac{2}{3}K_1)$$
$$and \qquad K_3 = hf\,(t_n + \frac{2h}{3}, y(t_n) + \frac{2}{3}K_2).$$

5 Runge-Kutta method of order three for solving Fuzzy Differential Equations

Let $Y = [\underline{Y}, \overline{Y}]$ be the exact solution and $y = [\underline{y},\ \overline{y}]$ be the approximated solution of the fuzzy initial value problem (3).

Let $[Y(t)]_r = [\underline{Y}\ (t;r), \overline{Y}\ (t;r)]$, $[y(t)]_r = [\underline{y}\ (t;r),\ \overline{y}\ (t;r)]$.

Throughout this argument, the value of r is fixed. Then the exact and approximated solution at t_n are respectively denoted by

$$[Y(t_n)]_r = [\underline{Y}(t_n; r), \overline{Y}(t_n; r)],$$

$$[y(t_n)]_r = [\underline{y}(t_n; r), \overline{y}(t_n; r)] \ (0 \le n \le N).$$

The grid points at which the solution is calculated are

$$h = \frac{T - t_0}{N}, t_i = t_0 + ih, \ 0 \le i \le N.$$

Then we obtain, $\underline{Y}(t_{n+1}; r) = \underline{Y}(t_n; r) + \frac{1}{8}[2K_1+3K_2+3K_3]$,

where

$$K_1 = h\,F[t_n, \underline{Y}(t_n;r), \overline{Y}(t_n;r)]$$

$$K_2 = h\,F[t_n+\frac{2h}{3}, \underline{Y}(t_n;r) + \frac{2}{3}K_1, \overline{Y}(t_n;r)+\frac{2}{3}K_1] \quad (17)$$

$$K_3 = h\,F[t_n+\frac{2h}{3}, \underline{Y}(t_n;r) + \frac{2}{3}K_2, \overline{Y}(t_n;r)+\frac{2}{3}K_2]$$

and

$$\overline{Y}(t_{n+1}; r) = \overline{Y}(t_n; r) + \frac{1}{8}[2K_1+3K_2+3K_3],$$

where

$$K_1 = h\,G[t_n, \underline{Y}(t_n;r), \overline{Y}(t_n;r)]$$

$$K_2 = h\,G[t_n+\frac{2h}{3}, \underline{Y}(t_n;r) + \frac{2}{3}K_1, \overline{Y}(t_n;r)+\frac{2}{3}K_1] \quad (18)$$

$$K_3 = h\,G[t_n+\frac{2h}{3}, \underline{Y}(t_n;r) + \frac{2}{3}K_2, \overline{Y}(t_n;r)+\frac{2}{3}K_2]$$

Also we have

$$\underline{y}(t_{n+1}; r) = \underline{y}(t_n; r) + \frac{1}{8}[2K_1+3K_2+3K_3],$$

where

$$K_1 = h\,F[t_n, \underline{y}(t_n;r), \overline{y}(t_n;r)]$$

$$K_2 = h\,F[t_n+\frac{2h}{3}, \underline{y}(t_n;r) + \frac{2}{3}K_1, \overline{y}(t_n;r)+\frac{2}{3}K_1] \quad (19)$$

$$K_3 = h\,F[t_n+\frac{2h}{3}, \underline{y}(t_n;r) + \frac{2}{3}K_2, \overline{y}(t_n;r)+\frac{2}{3}K_2]$$

and

$$\overline{y}(t_{n+1}; r) = \overline{y}(t_n; r) + \frac{1}{8}[2K_1+3K_2+3K_3],$$

where

$$K_1 = h\,G[t_n, \underline{y}(t_n;r), \overline{y}(t_n;r)]$$

$$K_2 = h\,G[t_n+\frac{2h}{3}, \underline{y}(t_n;r) + \frac{2}{3}K_1, \overline{y}(t_n;r)+\frac{2}{3}K_1] \quad (20)$$

$$K_3 = h\,G[t_n+\frac{2h}{3}, \underline{y}(t_n;r) + \frac{2}{3}K_2, \overline{y}(t_n;r)+\frac{2}{3}K_2]$$

Clearly, $\underline{y}\,(t\,;\,r)$ and $\overline{y}\,(t\,;\,r)$ converge to $\underline{Y}\,(t\,;\,r)$ and $\overline{Y}\,(t\,;\,r)$, respectively whenever $h \to 0$.

6 Numerical Results

In this section, the exact solutions and approximated solutions obtained by Euler's method and Runge-Kutta method of order three are plotted in figure 1 and figure 2.

Example 6.1

Consider the initial value problem

$$\begin{cases} y'(t) = f(t), & t \in [0,1] \\ y(0) = (0.8 + 0.125r \ , 1.1 - 0.1r). \end{cases}$$

The exact solution at $t = 1$ is given by

$$Y(1\,;r) = [(0.8 + 0.125r)e \ , (\ 1.1 - 0.1r)e] \ , \quad 0 \le r \le 1.$$

Using iterative solution of Runge-Kutta method of order three, we have

$$\underline{y}\,(0;\,r) = 0.8 + 0.125r\,,$$
$$\overline{y}\,(0;\,r) = 1.1 - 0.1r$$

and by

$$\underline{y}^{(0)}(t_{i+1};r) = \underline{y}\,(t_i;r) + h\,\underline{y}\,(t_i;r)$$
$$\overline{y}^{(0)}(t_{i+1};r) = \overline{y}\,(t_i;r) + h\,\overline{y}\,(t_i;r),$$

where i = 0, 1, . . .,$N - 1$ and $h = \frac{1}{N}$. Now, using these equations as an initial guess for following iterative solutions respectively,

$$\underline{y}^{\mathrm{j}}(t_{\mathrm{i}+1};\,r) = \underline{y}\,(t_{\mathrm{i}};\,r) + \frac{1}{8}[2K_1+3K_2+3K_3],$$

where

$$K_1 = h\ \underline{y}\ (t_{\mathrm{i}};\,r)$$
$$K_2 = h\,(\underline{y}\ (t_{\mathrm{i}};\,r) + \frac{2}{3}K_1)$$
$$K_3 = h\,(\underline{y}\ (t_{\mathrm{i}};\,r) + \frac{2}{3}K_2).$$

and

$$\overline{y}^{\mathrm{j}}(t_{\mathrm{i}+1};\,r) = \overline{y}\,(t_{\mathrm{i}};\,r) + \frac{1}{8}[2K_1+3K_2+3K_3],$$

where

$$K_1 = h\ \overline{y}\ (t_{\mathrm{i}};\,r)$$
$$K_2 = h\,(\overline{y}\ (t_{\mathrm{i}};\,r) + \frac{2}{3}K_1)$$
$$K_3 = h\,(\overline{y}\ (t_{\mathrm{i}};\,r) + \frac{2}{3}K_2).$$

and j = 1, 2, 3. Thus, we have $\underline{y}\left(t_i;r\right) = \underline{y}^{(3)}\left(t_i;r\right)$ and $\overline{y}\left(t_i;r\right) = \overline{y}^{(3)}\left(t_i;r\right)$, for $i = 1 \ldots N$.

Therefore, $\underline{Y}(1;r) \approx \underline{y}^{(3)}(1;r)$ and $\overline{Y}(1;r) \approx \overline{y}^{(3)}(1;r)$ are obtained.

Table 3, shows estimation of error for different values of $r \in [0,1]$ and h.

By minimizing the step size h, the solution by exact method and RK method almost coincides.

r	Exact solution
0	2.174625,2.990110
0.2	2.242583,2.935744
0.4	2.310540,2.881379
0.6	2.378497,2.827013
0.8	2.446454,2.772647
1	2.514411,2.718282

TABLE 1: Exact solution

h / r	0.1	0.01
0	1.958468,2.692893	2.174515,2.989958
0.2	2.019670,2.643931	2.242468,2.935595
0.4	2.080872,2.594970	2.310422,2.881232
0.6	2.142074,2.546008	2.378375,2.826869
0.8	2.203276,2.497046	2.446329,2.772506
1	2.264478,2.448085	2.514283,2.718143

TABLE 2: Approximated solution

h / r	0.1	0.01
0	0.513374	0.000262
0.2	0.514726	0.000264
0.4	0.516077	0.000265
0.6	0.517428	0.000266
0.8	0.518779	0.000266
1	0.520130	0.000267

TABLE 3: Error for different values of r and h.

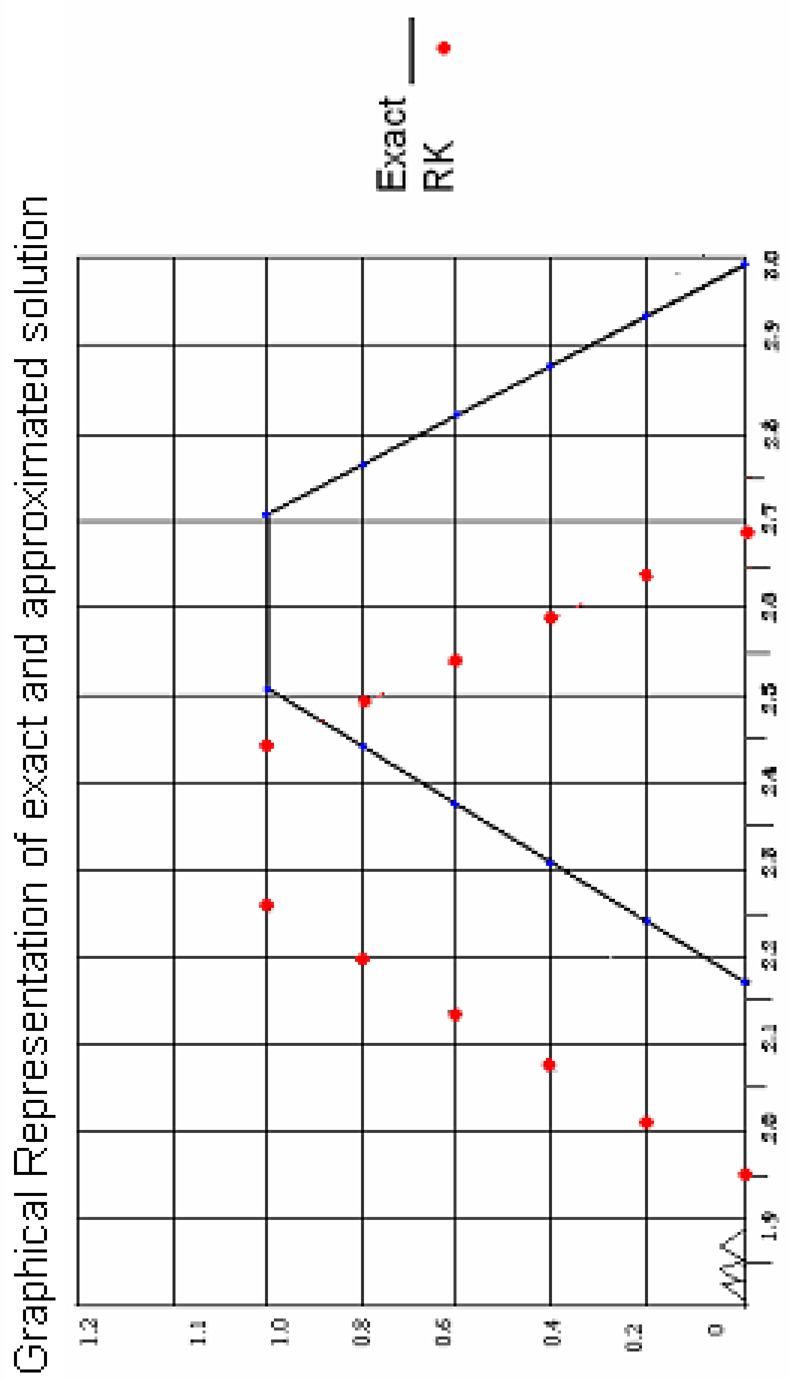
Graphical Representation of exact and approximated solution
Exact
RK
1.2
1.1
1.0
0.8
0.6
0.4
0.2
0

Figure 1 : h =0.1

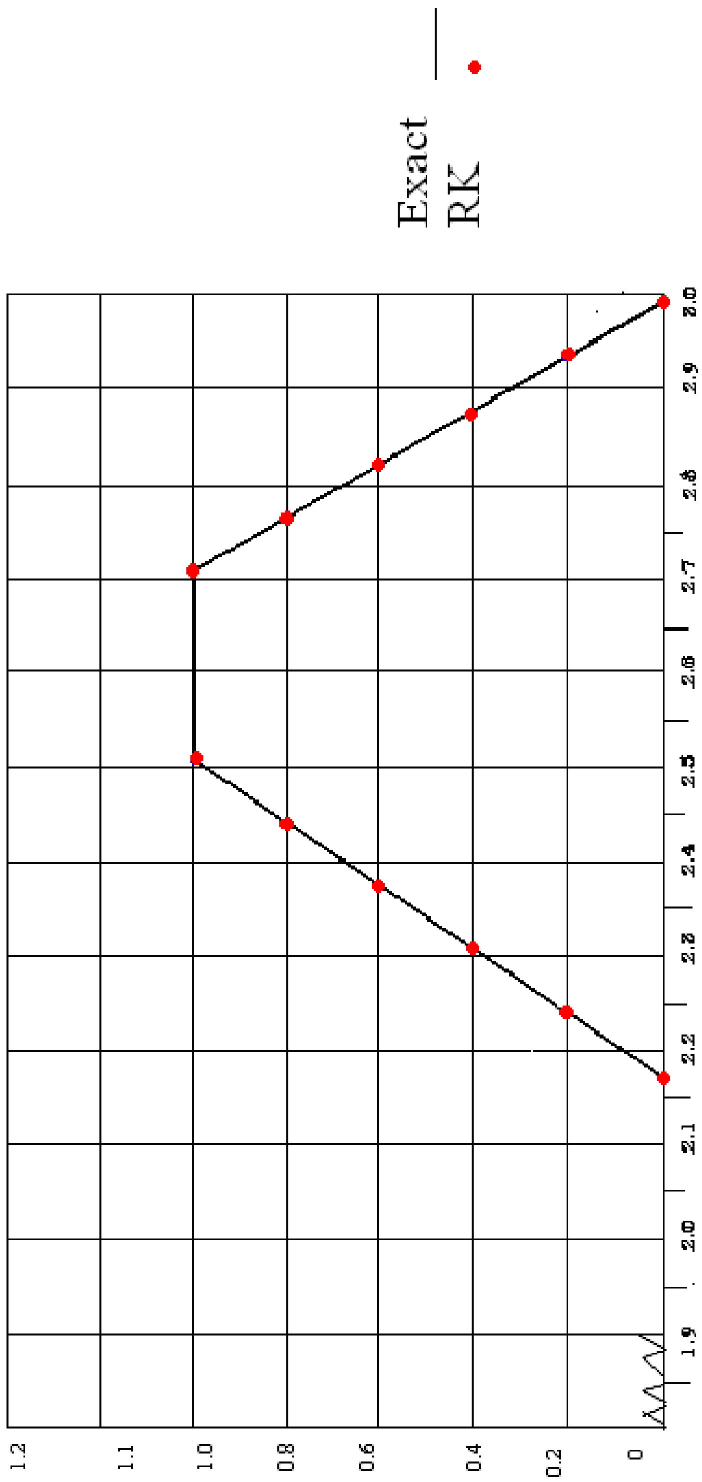

Figure 2 : h = 0.01

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