

Influence of Irregularity and Rigidity on the Propagation of Torsional Wave

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Abstract

The paper is devoted to investigate the effect of presence of rigid boundary on the propagation of torsional surface wave in a homogeneous layer over a semi infinite heterogeneous half space with an irregularity at the interface. The heterogeneity has been considered both in rigidity and density. It has been assumed that rigidity and density are varying exponentially with depth. In this paper the irregularity has been taken in the half-space in the form of a triangle. The study reveals that under assumed conditions, a torsional surface wave propagates in the medium. The velocities of torsional surface waves have been obtained. It is also observed that for a layer over a homogeneous half-space, the velocity of torsional surface waves does not coincides with that of Love waves.

Keywords: Torsional waves; heterogeneous half-space; irregular boundary

Introduction

Study the propagation of torsional waves in the assumed medium having a non planar boundary is an interesting matter for its similarity to most of real situations. Because of its closeness to the natural situation, the study of the effect of irregular boundaries on the propagation of waves in an elastic medium has gained much in importance. As the analytical treatment of the irregularities of the surface in entails formidable mathematical difficulties, most of the research workers particularly in this branch, concentrated their effort with considerable success in considering the cases of slightly curved surfaces of different types. Sato [15] studied the propagation of Love waves in

a layer with sharp change in thickness while De Noyer [8] considered the same in a layer over a half space with sinusoidal interface. Mal [3] studied the problem when the thickness of the layer abruptly increases throughout a certain length of the path. Bhattacharya [7] discussed dispersion curves for Love waves due to irregularity in the thickness of the transversely isotropic crystal layer. Chattopadhyay [1] discussed the effects of irregularities and non-homogeneities in the crystal layer on the propagation of Love waves. Wolf [4] discussed the propagation of Love waves in an isotropic layer with irregular boundary. Sezawa [10] discussed Love waves generated from a source of a certain depth. Chattopadhyay and Pal [2] discussed the propagation of SH waves in an anisotropic layer with irregular boundary, and the displacement fields are obtained. They determined the reflected field in the anisotropic layer when an SH wave is incident on an irregular boundary in the shape of triangular notch.

Although much information is available on the propagation of surface waves, such as Rayleigh waves, Love waves and Stoneley waves, etc., torsional waves have not drawn much attention and only scanty literature is available on the propagation of such waves. Lord Rayleigh [11] in his remarkable paper showed that an isotropic homogeneous elastic half-space does not allow torsional surface waves to propagate. Later on, Meissner [5] pointed out that in an inhomogeneous elastic half-space with quadratic variation of shear modulus and density varying linearly with depth, torsional surface waves do exist. Recently, Vardoulakis [9] has shown that torsional surface wave also propagate in a Gibson half-space, that is a half-space in which the shear modulus varies linearly with depth but the density remains unchanged. Torsional waves in an initially stressed cylinder have been studied by Dey and Dutta [13] and the existence and propagation of torsional surface waves in an elastic half-space with void pores has been discussed by Dey et al [14]. Selim [12] studied the propagation of torsional surface waves in heterogeneous half- space with irregular free surface.

In this paper an attempt has been made to assess the possible propagation of torsional surface waves in a homogeneous layer over a semi infinite heterogeneous half space with linearly varying rigidity and density in presence of an irregular boundary. It is observed that such a medium allows torsional waves to propagate.

Formulation

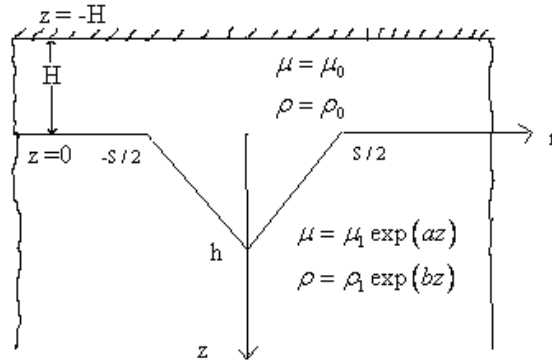


Fig.1: Geometry of the problem

Consider a homogeneous layer of thickness H , over a vertically heterogeneous half-space. The heterogeneity has been considered both in density and rigidity. Considering the origin of the cylindrical co-ordinate system at the interface of the layer and the z -axis downward positive, the following variation in rigidity and density is taken:

$$\left. \begin{aligned} \text{For the layer, } \mu &= \mu_0 \text{ and } \rho = \rho_0. \\ \text{For the half-space, } \mu &= \mu_1 \exp(az) \text{ and } \rho = \rho_1 \exp(bz). \end{aligned} \right\}$$

where μ and ρ are rigidity and density of the media, respectively, and a, b are constants having dimensions that are inverse of length.

We assume that the irregularity is of the form of a rectangle with the length $2d$ and depth h .

The equation of irregularity has been taken as

$$z = \varepsilon f(r) = \begin{cases} = h \left(b - \frac{2r}{s} \right), & r < \frac{s}{2} \\ = h \left(b + \frac{2r}{s} \right), & r > -\frac{s}{2} \\ = 0, & |r| \geq \frac{s}{2} \end{cases} \quad (1)$$

and $\varepsilon = \frac{h}{2d} \ll 1$, h is the maximum height of irregularity and s is the maximum width of the depth

Equations of motion

The dynamical equations of motion are

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= \rho \frac{\partial^2 u}{\partial t^2}, \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} &= \rho \frac{\partial^2 v}{\partial t^2}, \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} &= \rho \frac{\partial^2 w}{\partial t^2}, \end{aligned} \quad (2)$$

where σ_{rr} , $\sigma_{\theta\theta}$, σ_{zz} , σ_{rz} , $\sigma_{r\theta}$ and $\sigma_{\theta z}$ are the respective stress components and u , v and w are the respective displacement components.

The stress-strain relations are

$$\left. \begin{aligned} \sigma_{rr} &= \lambda\Omega + 2\mu e_{rr}, & \sigma_{\theta\theta} &= \lambda\Omega + 2\mu e_{\theta\theta} \\ \sigma_{zz} &= \lambda\Omega + 2\mu e_{zz}, & \sigma_{r\theta} &= 2\mu e_{r\theta} \\ \sigma_{rz} &= 2\mu e_{rz}, & \sigma_{\theta z} &= 2\mu e_{\theta z} \end{aligned} \right\} \quad (3)$$

where λ and μ are Lamé's constants and $\Omega = \left(\frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} + \frac{\partial w}{\partial z} \right)$ denotes the dilatation.

The strain-displacement relations are

$$\left. \begin{aligned} e_{rr} &= \frac{\partial u}{\partial r}, & e_{\theta\theta} &= \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r}, & e_{zz} &= \frac{\partial w}{\partial z} \\ e_{r\theta} &= \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, & e_{\theta z} &= \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \\ e_{rz} &= \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \end{aligned} \right\} \quad (4)$$

The torsional wave is characterized by the displacements

$$u = 0, \quad w = 0, \quad v = v(r, z, t) \quad (5)$$

In view of eqs. (3), (4) and (5), the dynamical equations of motion governing torsional waves reduce to

$$\frac{\partial}{\partial r} \sigma_{r\theta} + \frac{\partial}{\partial z} \sigma_{z\theta} + \frac{2}{r} \sigma_{r\theta} = \rho(z) \frac{\partial^2 v}{\partial t^2}, \quad (6)$$

with $v(r, z, t)$ being the displacement along the θ (azimuthal) direction. The respective stresses are then related to the displacement component v through the following relations

$$\begin{aligned} \sigma_{r\theta} &= \mu(z) \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right), \\ \sigma_{z\theta} &= \mu(z) \left(\frac{\partial v}{\partial z} \right). \end{aligned} \tag{7}$$

Using eq.(7), eq.(6) takes the form

$$\mu(z) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) v + \frac{\partial}{\partial z} \left(\mu(z) \frac{\partial v}{\partial z} \right) = \rho(z) \frac{\partial^2 v}{\partial t^2} \tag{8}$$

We assume a solution of eq. (8) of the form

$$v = V(z) J_1(Kr) \exp(i\omega t) \tag{9}$$

where $V(z)$ is the solution of the following equation

$$V''(z) + \frac{\mu'(z)}{\mu(z)} V'(z) - K^2 \left(1 - \frac{c^2}{c_s^2} \right) V(z) = 0 \tag{10}$$

In the above, $c = \frac{\omega}{K}$, K is the wave number, $c_s = \sqrt{\frac{\mu}{\rho}}$, $J_1(Kr)$ is the Bessel function of the first kind and dash denotes the differentiation with respect to z .

Solution for the layer

For the layer, $\mu = \mu_0$ and $\rho = \rho_0$. (11)

Using eq. (11), eq.(10) for the layer takes the form

$$\frac{d^2 V}{dz^2} - K^2 \left(1 - \frac{c^2}{c_0^2} \right) V(z) = 0 \tag{12}$$

where $c_0 = \sqrt{\frac{\mu_0}{\rho_0}}$.

The solution of eq. (12) can be written as

$$V(z) = A_1 \exp \left\{ \sqrt{\left(1 - \frac{c^2}{c_0^2} \right)} Kz \right\} + A_2 \exp \left\{ -\sqrt{\left(1 - \frac{c^2}{c_0^2} \right)} Kz \right\}, \tag{13}$$

and hence the displacement for the torsional wave in the homogeneous layer is

$$v_0 = \left[A_1 \exp \left\{ \sqrt{\left(1 - \frac{c^2}{c_0^2}\right)} Kz \right\} + A_2 \exp \left\{ -\sqrt{\left(1 - \frac{c^2}{c_0^2}\right)} Kz \right\} \right] J_1(Kr) \exp(i\omega t) \quad (14)$$

Solution for the half-space

For the half-space, $\mu = \mu_1 \exp(az)$ and $\rho = \rho_1 \exp(bz)$. (15)

Using eq.(15), eq. (10) for the half-space takes the form

$$\frac{d^2V}{dz^2} + a \frac{dV}{dz} - K^2 \left(1 - \frac{c^2 \exp(bz)}{c_1^2 \exp(az)} \right) V(z) = 0, \quad (16)$$

where $c_1 = \sqrt{\frac{\mu_1}{\rho_1}}$.

Substituting $V(z) = \phi(z) \exp\left(-\frac{az}{2}\right)$ in eq. (16) to eliminate the term $\frac{dV}{dz}$, we obtain

$$\frac{d^2\phi}{dz^2} + \left[\frac{K^2 c^2 (1+bz)}{c_1^2 (1+az)} - K^2 \left(1 + \frac{a^2}{4K^2} \right) \right] \phi(z) = 0 \quad (17)$$

Introducing the dimensionless quantities $\gamma = \sqrt{\left(1 + \frac{a^2}{4K^2} - \frac{bc^2}{ac_1^2}\right)}$ and

$\eta = \frac{2\gamma K}{a}(1+az)$ eq. (17) reduces to

$$\frac{d^2\phi}{d\eta^2} + \left[-\frac{1}{4} + \frac{R}{\eta} \right] \phi(\eta) = 0, \quad (18)$$

where $R = \frac{c^2(a-b)K}{c_1^2 a^2 2\gamma}$.

We are interested in the solution of eq. (18) which is bounded and vanishes as $z \rightarrow \infty$, therefore we search for the solution which gives $V(z) \rightarrow 0$ as $z \rightarrow \infty$. This condition is equivalent to $\lim_{\eta \rightarrow \infty} \phi(\eta) \rightarrow 0$. Therefore the solution of eq. (18) satisfying the above condition may be written as

$$\phi(\eta) = DW_{R, \frac{1}{2}}(\eta),$$

where $W_{R, \frac{1}{2}}(\eta)$ is the Whittaker function.

Hence, the displacement for the torsional wave in the heterogeneous half-space is

$$v_1 = DW_{R, \frac{1}{2}} \left[\frac{2\gamma K}{a} (1 + az) \right] \exp\left(-\frac{az}{2}\right) J_1(Kr) \exp(i\omega t) \quad (19)$$

Boundary conditions

The following boundary conditions must be satisfied:

- (i) At the rigid boundary $z = -H$
displacement component $v_0 = 0$
- (ii) continuity of the displacement
component $v_0 = v_1$
- (iii) continuity of stress component
 $(\sigma_{r\theta})_0 = (\sigma_{r\theta})_1$ at $z = \varepsilon f(r)$

where $(\sigma_{r\theta})_0 = l\mu_0 \frac{\partial v_1}{\partial \theta} + n\mu_0 \frac{\partial v_1}{\partial z}$

$$(\sigma_{r\theta})_1 = l\mu_1 \frac{\partial v_2}{\partial \theta} + n\mu_1 \frac{\partial v_2}{\partial z}$$

where $(l, 0, n)$ are components of unit normal (to the interface at $z = 0$)

$$l = 0, n = 1 \text{ at } z = -H$$

$$l = \frac{-\varepsilon f'}{\sqrt{1 + \varepsilon^2 f'^2}}, n = \frac{1}{\sqrt{1 + \varepsilon^2 f'^2}} \text{ at } z = \varepsilon f(r)$$

where $f' = \frac{df}{dr}$

The boundary condition (iii) may be written as

$$\begin{aligned} &\mu_0 \left(\frac{\partial v_1}{\partial r} - \frac{v_1}{r} \right) \left(\frac{-\varepsilon f'}{\sqrt{1 + \varepsilon^2 f'^2}} \right) + \frac{1}{\sqrt{1 + \varepsilon^2 f'^2}} \mu_0 \frac{\partial v_1}{\partial z} \\ &= \mu_1 \left(\frac{\partial v_1}{\partial r} - \frac{v_1}{r} \right) \left(\frac{-\varepsilon f'}{\sqrt{1 + \varepsilon^2 f'^2}} \right) + \frac{1}{\sqrt{1 + \varepsilon^2 f'^2}} \mu_1 \frac{\partial v_1}{\partial z} \end{aligned} \quad (20) \text{ at } z = \varepsilon f(r)$$

where $f' = \frac{df(r)}{dr}$

Expanding Whittaker's function up to linear terms and substituting into the set of boundary conditions and eliminating A_1, A_2 and D , we obtained

$$\left[(2\mu_1 - \mu_0) \left\{ K \left(\frac{Kr}{4} + \frac{K^3 r^3}{96} + \frac{K^5 r^5}{1536} \right) \right\} \varepsilon f' + 2\mu_1 \left\{ \frac{a}{1+a\varepsilon f(r)} - \left(\gamma K + \frac{a}{2} \right) + \frac{(1-R)\gamma K}{1+(1-R)\frac{\gamma K}{a}(1+a\varepsilon f(r))} \right\} \right]$$

$$= \cot \left\{ \left(\frac{c^2}{c_0^2} - 1 \right)^{\frac{1}{2}} (KH + K\varepsilon f(r)) \right\} 2\mu_0 K \left(\frac{c^2}{c_0^2} - 1 \right)^{\frac{1}{2}} \quad (21)$$

This gives the velocity of torsional surface waves in a homogeneous layer over a vertically heterogeneous half-space with triangular irregularity at the interface.

Particular cases

Case I : Considering $b=0$, i.e. when the half-space is with constant density, equation (21) takes the form

$$\left[(2\mu_1 - \mu_0) \left\{ K \left(\frac{Kr}{4} + \frac{K^3 r^3}{96} + \frac{K^5 r^5}{1536} \right) \right\} \varepsilon f' + 2\mu_1 \left\{ \frac{a}{1+a\varepsilon f(r)} - \left(\gamma_1 K + \frac{a}{2} \right) + \frac{(1-R_1)\gamma K}{1+(1-R_1)\frac{\gamma K}{a}(1+a\varepsilon f(r))} \right\} \right]$$

$$= \cot \left\{ \left(\frac{c^2}{c_0^2} - 1 \right)^{\frac{1}{2}} (KH + K\varepsilon f(r)) \right\} 2\mu_0 K \left(\frac{c^2}{c_0^2} - 1 \right)^{\frac{1}{2}} \quad (22)$$

$$\text{where } \gamma_1 = \sqrt{1 + \frac{a^2}{4K^2}}, \quad R_1 = \frac{c^2 K}{c_1^2 a 2\gamma}$$

Case II : When $a \rightarrow 0, b \rightarrow 0$ and $\varepsilon \rightarrow 0$, i.e. the layer and the half-space have both constant density and rigidity, the equation (22), takes the form

$$\frac{\mu_0 \left(\frac{1 - \frac{c^2}{c_0^2}}{c_0^2} \right)^{\frac{1}{2}}}{\mu_1 \left(\frac{c^2}{c_1^2} - 1 \right)^{\frac{1}{2}}} = \tan \left\{ KH \left(\frac{c^2}{c_0^2} - 1 \right)^{\frac{1}{2}} \right\} \quad (23)$$

The equation (23) gives the velocity of Love waves in a homogeneous layer over a homogeneous half-space bounded by a rigid boundary.

It may be mentioned here that in the absence of rigid boundary the equation giving the velocity of Love wave in the as given in Ewing, Jardetzky, and Press[6] is

$$\frac{\mu_1 \left(1 - \frac{c^2}{c_1^2}\right)^{\frac{1}{2}}}{\mu_0 \left(\frac{c^2}{c_0^2} - 1\right)^{\frac{1}{2}}} = \tan \left\{ KH \left(\frac{c^2}{c_0^2} - 1\right)^{\frac{1}{2}} \right\} \quad (24)$$

So, from equation (23) and (24) we can say that the velocity of Love wave in homogeneous layer over a homogeneous medium bounded by a rigid boundary is not same with that of a free boundary.

Example of applications and discussions of the results

The values of c/c_0 have been computed for $a/K = 2.50$ and $b/K = 0.250$ for different values of KH from equation (23) and (24) and are presented in Fig. 2 and Fig. 3 by taking $(c_0/c_1)^2 = 0.2$ respectively.

Therefore, from this study we can say that the presence of rigid layer much effects the propagation of Love waves.

Table 1

Curve no	$\frac{\mu_0}{\mu_1}$
1	0.5
2	1.5
3	2.5

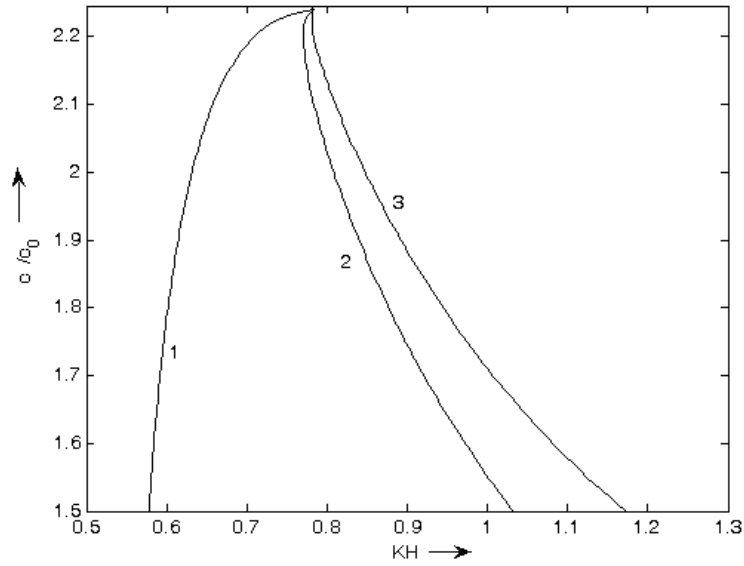


Fig. 2: Curve of c/c_0 versus KH from equation (23) for $(c_0/c_1)^2=0.2$

Table 2

Curve no	$\frac{\mu_0}{\mu_1}$
1	0.5
2	1.5
3	2.5

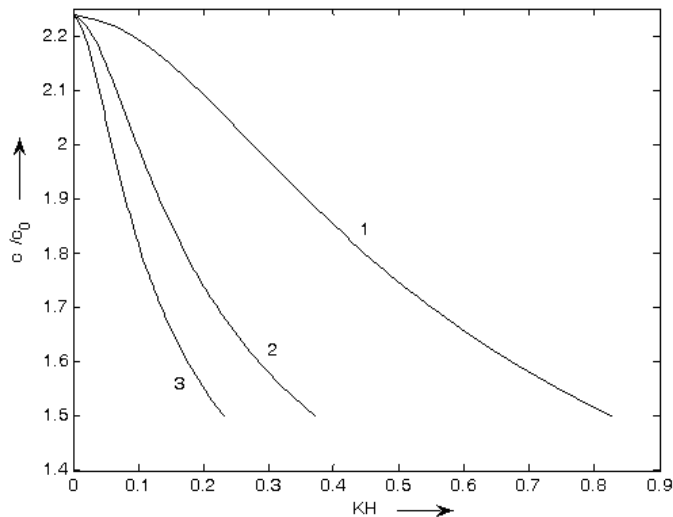


Fig. 3: Curve of c/c_0 versus KH from equation (24) for $(c_0/c_1)^2=0.2$

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