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On the Positive Solutions of the Difference Equation System $x_{n+1} = \frac{1}{y_{n-k}}$, $y_{n+1} = \frac{x_{n-k}}{y_{n-k}}$

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Abstract

In this work, we study the positive solutions of the difference equation system $x_{n+1} = \frac{1}{y_{n-k}}$, $y_{n+1} = \frac{x_{n-k}}{y_{n-k}}$, $n = 0, 1, 2, \ldots$ where x_{-k}, $x_{-k+1}, \ldots, x_0$ and $y_{-k}, y_{-k+1}, \ldots, y_0$ are positive real numbers.

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1 Introduction

Our aim in this paper is to investigate the solutions of the difference equation system

$$x_{n+1} = \frac{1}{y_{n-k}} \ , \ y_{n+1} = \frac{x_{n-k}}{y_{n-k}} \ , \ n = 0, 1, 2, \ldots \tag{1}$$

where

$$x_{-k}, x_{-k+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0 \in R^+ \tag{2}$$

Some papers related to this subject are the following:
Cinar[1] has investigated the periodicity of the difference equation system $x_{n+1} = \frac{1}{y_n}$, $y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}$. Yang, Liu and Bai [2] study the behavior of

positive solutions of the equation system $x_n = \frac{a}{y_{n-p}}$, $y_n = \frac{by_{n-p}}{x_{n-q}y_{n-q}}$. Also Ozban [3] has investigated the periodic nature of solutions of the system of rational difference equations $x_{n+1} = \frac{1}{y_{n-k}}$, $y_{n+1} = \frac{y_n}{x_{n-m}y_{n-m-k}}$.

Similar to the references above, in this paper, we define (1) with conditions (2) and investigate the solutions of this difference equation system.

2 Main Results

Theorem 2.1 *Assume that (2) holds and $k \geq 1$ be a integer. Let $\{x_n, y_n\}$ be a solution of equation system (1). Then all the solutions of the equation system (1) are periodic with $3(k+1)$.*

From our assumption (2), we have all the solutions of the equation system (1) which have positive. So, we have:

$$
\begin{array}{ll}
x_{n+1} = \frac{1}{y_{n-k}} & y_{n+1} = \frac{x_{n-k}}{y_{n-k}} \\
x_{n+2} = \frac{1}{y_{n-k+1}} & y_{n+2} = \frac{x_{n-k+1}}{y_{n-k+1}} \\
\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots \\
x_{n+k+1} = \frac{1}{y_n} & y_{n+k+1} = \frac{x_n}{y_n} \\
x_{n+k+2} = \frac{1}{y_{n+1}} = \frac{y_{n-k}}{x_{n-k}} & y_{n+k+2} = \frac{x_{n+1}}{y_{n+1}} = \frac{1}{x_{n-k}} \\
x_{n+k+3} = \frac{y_{n-k+1}}{x_{n-k+1}} & y_{n+k+3} = \frac{1}{x_{n-k+1}} \\
x_{n+2k+2} = \frac{y_n}{x_n} & y_{n+2k+2} = \frac{1}{x_n} \\
x_{n+2k+3} = \frac{y_{n+1}}{x_{n+1}} = x_{n-k} & y_{n+2k+3} = \frac{1}{x_{n+1}} = y_{n-k} \\
x_{n+2k+4} = x_{n-k+1} & y_{n+2k+4} = y_{n-k+1} \\
\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots \\
x_{n+3k+2} = x_{n-1} & y_{n+3k+2} = y_{n-1} \\
x_{n+3k+3} = x_n & y_{n+3k+3} = y_n
\end{array}
\quad (3)
$$

Therefore, the proof is completed.

Theorem 2.2 *Suppose that (2) holds and $k \geq 1$ be a integer. Let $\{x_n, y_n\}$ be a solution of equation system (1) with $x_{-k} = a_0,\ x_{-k+1} = a_1,\ \dots,\ x_0 = a_k$*

and $y_{-k} = b_0,\ y_{-k+1} = b_1,\ \ldots,\ y_0 = b_k$. *Then for* $n = 0, 1, 2, \ldots$ *all the solutions of the equation system (1) are*

$$
\begin{array}{ll}
x_{3(k+1)n+1} = \frac{1}{b_0} \ , & y_{3(k+1)n+1} = \frac{a_0}{b_0} \\
x_{3(k+1)n+2} = \frac{1}{b_1} \ , & y_{3(k+1)n+2} = \frac{a_1}{b_1} \\
\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\
x_{3(k+1)n+k} = \frac{1}{b_{k-1}} \ , & y_{3(k+1)n+k} = \frac{a_{k-1}}{b_{k-1}} \\
x_{3(k+1)n+k+1} = \frac{1}{b_k} \ , & y_{3(k+1)n+k+1} = \frac{a_k}{b_k} \\
x_{3(k+1)n+k+2} = \frac{b_0}{a_0} \ , & y_{3(k+1)n+k+2} = \frac{1}{a_0} \\
x_{3(k+1)n+k+3} = \frac{b_1}{a_1} \ , & y_{3(k+1)n+k+3} = \frac{1}{a_1} \\
\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\
x_{3(k+1)n+2k+2} = \frac{b_k}{a_k} \ , & y_{3(k+1)n+2k+2} = \frac{1}{a_k} \\
x_{3(k+1)n+2k+3} = a_0 \ , & y_{3(k+1)n+2k+3} = b_0 \\
x_{3(k+1)n+2k+4} = a_1 \ , & y_{3(k+1)n+2k+4} = b_1 \\
\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\
x_{3(k+1)n+3k+2} = a_{k-1} \ , & y_{3(k+1)n+3k+2} = b_{k-1} \\
x_{3(k+1)n+3k+3} = a_k \ , & y_{3(k+1)n+3k+3} = b_k
\end{array}
\qquad (4)
$$

From our assumption (2), all the solutions of the equation system (1) are positive. Now, assume that $n > 0$ and that our assumption holds for $n - 1$. We try to show that the result holds for n. From our assumption for $n - 1$, we have the following:

$$
\begin{array}{ll}
x_{3(k+1)n-3k-2} = \frac{1}{b_0} & y_{3(k+1)n-3k-2} = \frac{a_0}{b_0} \\
x_{3(k+1)n-3k-1} = \frac{1}{b_1} & y_{3(k+1)n-3k-1} = \frac{a_1}{b_1} \\
\dots\dots\dots\dots & \dots\dots\dots\dots \\
x_{3(k+1)n-2k-2} = \frac{1}{b_k} & y_{3(k+1)n-2k-2} = \frac{a_k}{b_k} \\
x_{3(k+1)n-2k-1} = \frac{b_0}{a_0} & y_{3(k+1)n-2k-1} = \frac{1}{a_0} \\
\dots\dots\dots\dots & \dots\dots\dots\dots \\
x_{3(k+1)n-k-1} = \frac{b_k}{a_k} & y_{3(k+1)n-k-1} = \frac{1}{a_k} \\
x_{3(k+1)n-k} = a_0 & y_{3(k+1)n-k} = b_0 \\
\dots\dots\dots\dots & \dots\dots\dots\dots \\
x_{3(k+1)n-1} = a_{k-1} & y_{3(k+1)n-1} = b_{k-1} \\
x_{3(k+1)n} = a_k & y_{3(k+1)n} = b_k
\end{array}
\tag{5}
$$

Also for n, we have the following:

$$
\begin{array}{ll}
x_{3(k+1)n+1} = \frac{1}{y_{3(k+1)n-k}} = \frac{1}{b_0} & y_{3(k+1)n+1} = \frac{x_{3(k+1)n-k}}{y_{3(k+1)n-k}} = \frac{a_0}{b_0} \\
x_{3(k+1)n+2} = \frac{1}{y_{3(k+1)n-k+1}} = \frac{1}{b_1} & y_{3(k+1)n+2} = \frac{x_{3(k+1)n-k+1}}{y_{3(k+1)n-k+1}} = \frac{a_1}{b_1} \\
\dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots \\
x_{3(k+1)n+k+1} = \frac{1}{y_{3(k+1)n}} = \frac{1}{b_k} & y_{3(k+1)n+k+1} = \frac{x_{3(k+1)n}}{y_{3(k+1)n}} = \frac{a_k}{b_k} \\
x_{3(k+1)n+k+2} = \frac{1}{y_{3(k+1)n+1}} = \frac{b_0}{a_0} & y_{3(k+1)n+k+2} = \frac{x_{3(k+1)n+1}}{y_{3(k+1)n+1}} = \frac{1}{a_0} \\
\dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots \\
x_{3(k+1)n+2k+2} = \frac{b_k}{a_k} & y_{3(k+1)n+2k+2} = \frac{1}{a_k}
\end{array}
\tag{6}
$$

$$
\begin{array}{ll}
x_{3(k+1)n+2k+3} = \frac{1}{y_{3(k+1)n+k+2}} = a_0 & y_{3(k+1)n+2k+3} = \frac{x_{3(k+1)n+k+2}}{y_{3(k+1)n+k+2}} = b_0 \\
x_{3(k+1)n+2k+4} = a_1 & y_{3(k+1)n+2k+4} = b_1 \\
\dots\dots\dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots \\
x_{3(k+1)n+3k+2} = a_{k-1} & y_{3(k+1)n+2k+3} = b_{k-1} \\
x_{3(k+1)n+3k+3} = a_k & y_{3(k+1)n+3k+3} = b_k
\end{array}
\qquad (7)
$$

Therefore, the proof is completed by induction.

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