

Variational Iteration Method for Solving Coupled Schrödinger-KdV Equation

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Abstract

In this letter, we will consider variational iteration method (VIM) for finding analytical solutions of the coupled Schrödinger–KdV equation. The available analytical solutions of the coupled Schrödinger–KdV equation obtained by multiple traveling wave method are compared with VIM to examine the accuracy of the method. The results illustrate that VIM is an attractive method in solving the systems of nonlinear equations.

Keywords: Variational iteration method (VIM); the coupled Schrödinger–KdV equation.

1. Introduction

The solutions of the nonlinear evolution equations play an important role in the field of nonlinear wave phenomena. The exact solutions facilitate the verification of numerical methods when they exist. In the last few decades, substantial progress has been made on researches in this area and it continues with

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this direction. Multiple traveling wave solutions of nonlinear evolution equations such as the coupled Schrödinger–KdV equation [1,2] have been successfully applied to get exact solutions by Fan [3]. The model equation for the coupled Schrödinger–KdV equation can be presented in the following form

$$\begin{aligned} iU_t &= U_{xx} + Uw, \\ w_t + 6ww_x + w_{xxx} &= \left(|U|^2\right)_x, \end{aligned} \quad (1)$$

where $i = \sqrt{-1}$.

The VIM was first proposed by He [4,5] and systematically illustrated in 1999 [6] and used to give approximate solutions of the problem of seepage flow in porous media with fractional derivatives. The VIM is useful to obtain exact and approximate solutions of linear and nonlinear differential equations. In this method, general Lagrange multipliers are introduced to construct correction functionals for the problems. The multipliers can be identified optimally via the variational theory. There is no need of linearization or discretization, and large computational work and round-off errors is avoided. It has been used to solve effectively, easily and accurately a large class of nonlinear problems with approximation [7, 8]. The main goal of the present study is to find the analytic solutions of the coupled Schrödinger–KdV equation by the variational iteration method.

2. Basic concepts of VIM

To illustrate the basic concepts of VIM, we consider the following differential equation

$$Lu + Nu = g(t), \quad (2)$$

where L, N and $g(t)$ are the linear operator, the nonlinear operator and a heterogeneous term, respectively. The variational iteration method was proposed by He where a correction functional for Eq. (1) can be written as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda [Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)] d\tau, \quad n \geq 0. \quad (3)$$

It is obvious that the successive approximations u_j , $j \geq 0$ can be established by determining λ , a general Lagrangian multiplier, which can be identified optimally via the variational theory. The function \tilde{u}_n is a restricted variation

which means $\delta \tilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(t), n \geq 0$ of the solution $u(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0 . When λ determined, then several approximations $u_j(t), j \geq 0$, follow immediately. Consequently, the exact solution may be obtained by using

$$u = \lim_{n \rightarrow \infty} u_n. \tag{4}$$

3. Analytical solution

In this section, two different solutions of the coupled Schrödinger–KdV equation [3] will be examined by using the VIM. By using $U = u + iv$, one can separate Eq. (1) into real and imaginary parts. Therefore, one can get a (1 + 1)-dimensional tripled system in the following form

$$\begin{aligned} u_t - v_{xx} - vw &= 0 \\ v_t + u_{xx} + uw &= 0 \\ w_t + 6ww_x + w_{xxx} - 2uu_x - 2vv_x &= 0 \end{aligned} \tag{5}$$

In order to obtain VIM solution of the Eq. (5), we construct a correction functional which reads

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda_1 \left(\frac{\partial u_n(x, \tau)}{\partial \tau} - \frac{\partial^2 \tilde{v}_n(x, \tau)}{\partial x^2} - \tilde{v}_n(x, \tau) \tilde{w}_n(x, \tau) \right) d\tau, \tag{6}$$

$$v_{n+1}(x, t) = v_n(x, t) + \int_0^t \lambda_2 \left(\frac{\partial v_n(x, \tau)}{\partial \tau} + \frac{\partial^2 \tilde{u}_n(x, \tau)}{\partial x^2} + \tilde{u}_n(x, \tau) \tilde{w}_n(x, \tau) \right) d\tau, \tag{7}$$

$$\begin{aligned} w_{n+1}(x, t) = w_n(x, t) + \int_0^t \lambda_3 \left(\frac{\partial w_n(x, \tau)}{\partial \tau} + 6w_n(x, \tau) \frac{\partial^2 \tilde{w}_n(x, \tau)}{\partial x^2} - \right. \\ \left. \frac{\partial^3 \tilde{w}_n(x, \tau)}{\partial x^3} - 2u_n(x, \tau) \frac{\partial \tilde{u}_n(x, \tau)}{\partial x} - 2v_n(x, \tau) \frac{\partial \tilde{v}_n(x, \tau)}{\partial x} \right) d\tau, \end{aligned} \tag{8}$$

where λ_1, λ_2 and λ_3 are the general Lagrangian multipliers and $\tilde{u}_n(x, t), \tilde{v}_n(x, t)$ and $\tilde{w}_n(x, t)$ denote restricted variations, i.e. $\delta \tilde{u}_n = \delta \tilde{v}_n = \delta \tilde{w}_n = 0$.

Its stationary conditions can be obtained as follows

$$1 + \lambda_1(\tau)|_{\tau=t} = 0, \quad \lambda_1'(\tau) = 0, \quad (9)$$

$$1 + \lambda_2(\tau)|_{\tau=t} = 0, \quad \lambda_2'(\tau) = 0, \quad (10)$$

$$1 + \lambda_3(\tau)|_{\tau=t} = 0, \quad \lambda_3'(\tau) = 0. \quad (11)$$

The Lagrange multipliers can therefore, be identified as $\lambda_1 = \lambda_2 = \lambda_3 = -1$ and the variational iteration formula is obtained in the form of:

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left(\frac{\partial u_n(x, \tau)}{\partial \tau} - \frac{\partial^2 v_n(x, t)}{\partial x^2} - v_n(x, t) w_n(x, t) \right) d\tau, \quad (12)$$

$$v_{n+1}(x, t) = v_n(x, t) - \int_0^t \left(\frac{\partial v_n(x, \tau)}{\partial \tau} + \frac{\partial^2 u_n(x, t)}{\partial x^2} + u_n(x, t) w_n(x, t) \right) d\tau, \quad (13)$$

$$w_{n+1}(x, t) = w_n(x, t) - \int_0^t \left(\frac{\partial w_n(x, \tau)}{\partial \tau} + 6w_n(x, t) \frac{\partial^2 w_n(x, t)}{\partial x^2} - \frac{\partial^3 w_n(x, t)}{\partial x^3} - 2u_n(x, t) \frac{\partial u_n(x, t)}{\partial x} - 2v_n(x, t) \frac{\partial v_n(x, t)}{\partial x} \right) d\tau. \quad (14)$$

To examine the VIM for the coupled Schrödinger–KdV equation, one soliton solution and one Jacobi doubly periodic wave solution [3] are studied in the following sections.

3.1. Jacobi periodic solution to coupled Schrödinger–KdV equation

The Jacobi exact solutions are given [3] by

$$u = a_1 \sqrt{\frac{2c_2}{b_2(2-m^2)}} dn \left(\sqrt{\frac{c_2}{2-m^2}} \xi \right) \cos(\theta), \quad (15)$$

$$v = a_1 \sqrt{\frac{2c_2}{b_2(2-m^2)}} dn \left(\sqrt{\frac{c_2}{2-m^2}} \xi \right) \sin(\theta), \quad (16)$$

$$w = \frac{1}{4} \left(2p + 4c_2 + \frac{a_1^2}{b_2} \right) - \frac{2c_2}{(2-m^2)} dn^2 \left(\sqrt{\frac{c_2}{2-m^2}} \xi \right), \quad (17)$$

where a_1, b_2, p, c_2 are arbitrary constants and $\xi = x + 2pt$ and $\theta = px + \frac{1}{4}(2p + 2p^2 - 2c_2 - \frac{a_1^2}{b_2})t$. For simplicity, $a_1 = b_2 = p = c_2 = 1$ is used in the analyses. Then, Eqs. (15)–(17) take the following form for $m = 0$

$$u = \cos(x + t/4), \tag{18}$$

$$v = \sin(x + t/4), \tag{19}$$

$$w = 3/4. \tag{20}$$

By using an initial condition to Eqs. (18)–(20), one can write $u_0 = u(x, 0) = \cos(x)$, $v_0 = v(x, 0) = \sin(x)$ and $w_0 = w(x, 0) = 3/4$. Now we start with initial approximations and by the above iteration formula (12)–(14), we have

$$u_1(x, t) = \cos(x) - \frac{1}{4}t \sin(x), \tag{21}$$

$$v_1(x, t) = \sin(x) + \frac{1}{4}t \cos(x), \tag{22}$$

$$w_1(x, t) = \frac{3}{4}, \tag{23}$$

$$u_2(x, t) = \cos(x) - \frac{1}{4}t \sin(x) - \frac{1}{32}t^2 \cos(x), \tag{24}$$

$$v_2(x, t) = \sin(x) + \frac{1}{4}t \cos(x) - \frac{1}{32}t^2 \sin(x), \tag{25}$$

$$w_2(x, t) = \frac{3}{4}, \tag{26}$$

⋮

In Fig. 1, the exact distribution of $u(x, t)$ for the intervals $-3 \leq x \leq 3$ and $0 \leq t \leq 1$ is plotted. The approximated solution of $u_3(x, t)$ is plotted in Fig. 2 for the same interval. In Fig. 3, the exact distribution of $v(x, t)$ for the intervals $-3 \leq x \leq 3$ and $0 \leq t \leq 1$ is plotted. The approximated solution of $v_3(x, t)$ is plotted in Fig. 4 for the same interval. Since the obtained w is exact, it is not plotted. Fig. 5 shows the exact distributions of $u(x, t)$ and $v(x, t)$ for the interval $-9 \leq x \leq 9$ at $t = 1$. VIM solutions of $u_3(x, t)$ and $v_3(x, t)$ for the interval $-9 \leq x \leq 9$ at $t = 1$ are represented in Fig. 6. Fig. 7 shows the comparison between exact and VIM solutions of $u(x, t)$ and $v(x, t)$ for the same interval of x at $t = 1$.

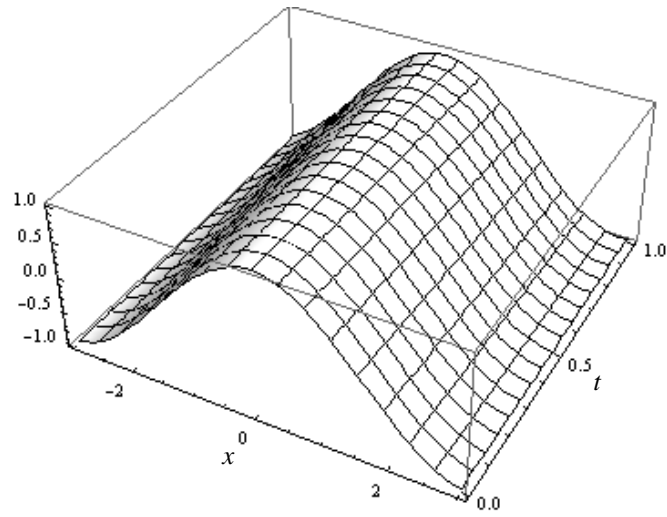


Fig. 1. Exact solution of $u(x, t)$ for the intervals $-3 \leq x \leq 3$ and $0 \leq t \leq 1$.

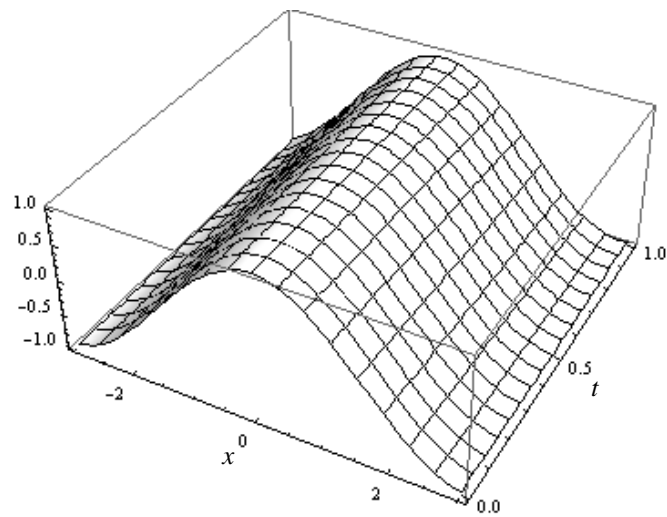


Fig.2. VIM solution of $u_3(x, t)$ for the intervals $-3 \leq x \leq 3$ and $0 \leq t \leq 1$.

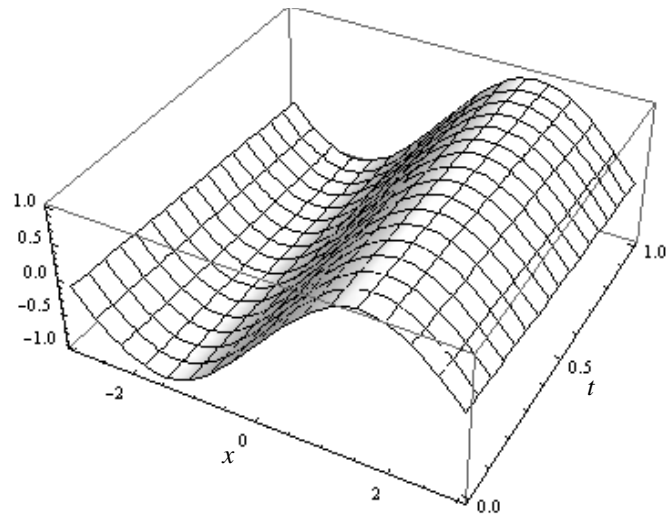


Fig. 3. Exact solution of $v(x, t)$ for the intervals $-3 \leq x \leq 3$ and $0 \leq t \leq 1$.

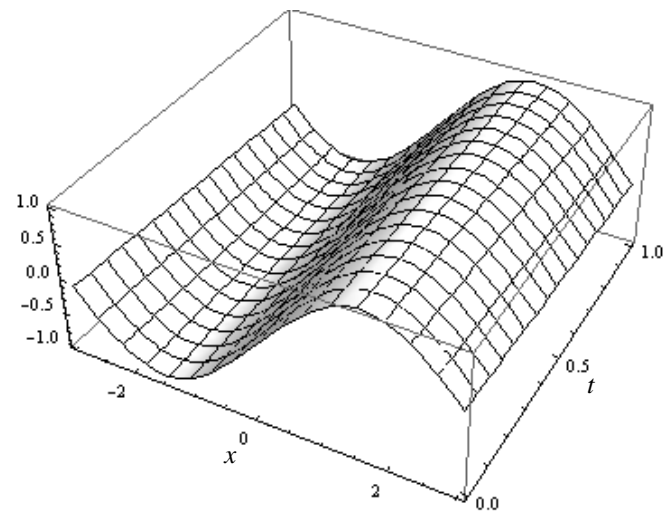


Fig.4. VIM solution of $v_3(x, t)$ for the intervals $-3 \leq x \leq 3$ and $0 \leq t \leq 1$.

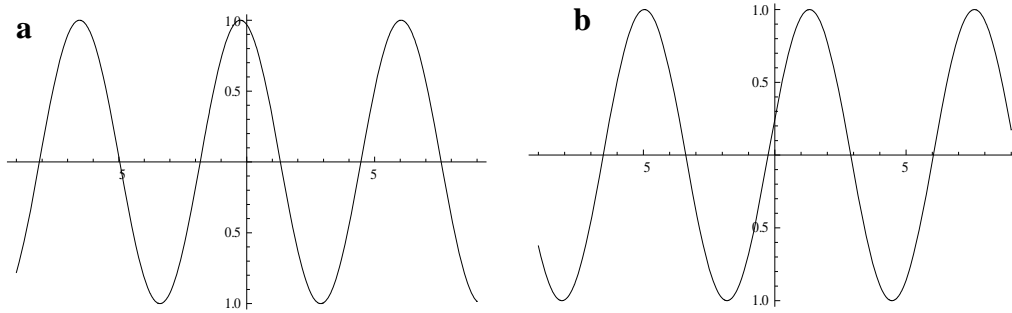


Fig. 5. Exact solutions of $u(x, t)$ (a) and $v(x, t)$ (b) for the interval $-9 \leq x \leq 9$ at $t = 1$.

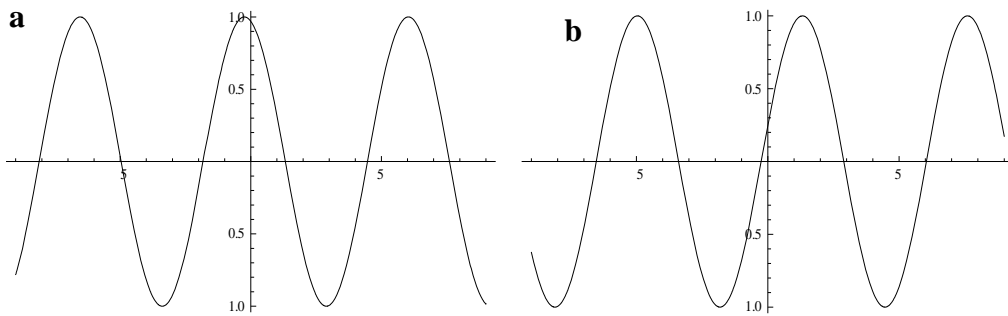


Fig. 6. VIM solutions of $u_3(x, t)$ (a) and $v_3(x, t)$ (b) for the interval $-9 \leq x \leq 9$ at $t = 1$.

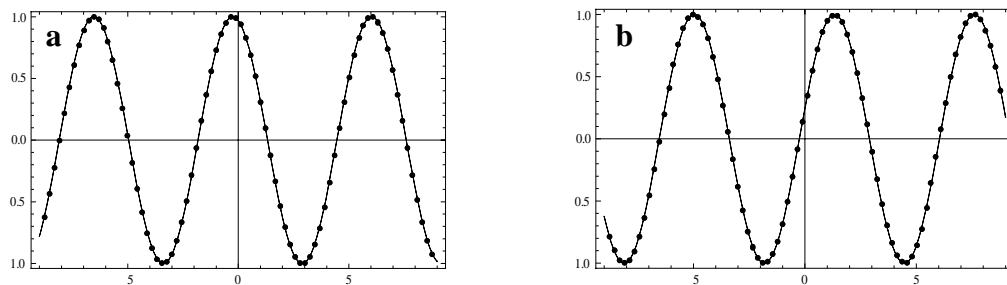


Fig. 7. The comparison between exact and VIM solutions of $u(x, t)$ (a) and $v(x, t)$ (b) for the interval $-9 \leq x \leq 9$ at $t = 1$. Line stands for the figures of the VIM solution and point for the exact solution.

3.2. Soliton solution to coupled Schrödinger–KdV equation

The soliton exact solutions are given [3] by

$$u = a_1 \sqrt{\frac{2c_2}{b_2}} \tanh\left(\sqrt{\frac{-c_2}{2}} \xi\right) \cos(\theta), \tag{27}$$

$$v = a_1 \sqrt{\frac{2c_2}{b_2}} \tanh\left(\sqrt{\frac{-c_2}{2}} \xi\right) \sin(\theta), \tag{28}$$

$$w = -\frac{1}{2} \left(2p + 2c_2 + \frac{a_1^2}{b_2}\right) + c_2 \tanh^2\left(\sqrt{\frac{-c_2}{2}} \xi\right), \tag{29}$$

where $\xi = x + 2pt$ and $\theta = px + \frac{1}{4}(2p + 2p^2 - 2c_2 - \frac{a_1^2}{b_2})t$ and $a_1, b_2, p, c_2 < 0$ are arbitrary constants. For simplicity, $a_1 = 1, b_2 = -4, p = 1, c_2 = -2$ are used in the analyses. Then, Eqs. (27)–(29) takes the following form

$$u = \tanh(x + 2t) \cos(x + 17t/8), \tag{30}$$

$$v = \tanh(x + 2t) \sin(x + 17t/8), \tag{31}$$

$$w = 7/8 - 2 \tanh^2(x + 2t). \tag{32}$$

By using an initial condition to Eqs. (30)–(32), one can write $u_0(x, t) = u(x, 0) = \tanh(x) \cos(x)$ and $v_0(x, t) = v(x, 0) = \tanh(x) \sin(x)$ and $w_0(x, t) = w(x, 0) = 7/8 - 2 \tanh^2(x)$. Now we start with initial approximations and by the above iteration formula (12)–(14), we have

$$u_1(x, t) = \tanh(x) \cos(x) - t \left(-2 \cos(x) \operatorname{sech}^2(x) + \tanh(x) \sin(x) + 2 \tanh(x) \sin(x) \operatorname{sech}^2(x) - \tanh(x) \sin(x) (7/8 - 2 \tanh^2(x)) \right), \tag{33}$$

$$v_1(x, t) = \tanh(x) \sin(x) - t \left(-2 \sin(x) \operatorname{sech}^2(x) - \tanh(x) \cos(x) + 2 \tanh(x) \cos(x) \operatorname{sech}^2(x) \tanh(x) \cos(x) (7/8 - 2 \tanh^2(x)) \right), \tag{34}$$

$$w_1(x, t) = 7/8 - 2 \tanh^2(x) - t \left(32 \tanh(x) \operatorname{sech}^4(x) - 16 \tanh^3(x) \operatorname{sech}^2(x) - 2 \tanh(x) \sin(x) \left(\sin(x) \operatorname{sech}^2(x) + \tanh(x) \cos(x) \right) - 2 \tanh(x) \cos(x) \left(\cos(x) \operatorname{sech}^2(x) - \tanh(x) \sin(x) \right) - 24 \tanh(x) \operatorname{sech}^2(x) (7/8 - 2 \tanh^2(x)) \right), \tag{35}$$

⋮

In Fig. 8, the exact distribution of $u(x, t)$ for the intervals $-3 \leq x \leq 3$ and $0 \leq t \leq 1$ is plotted. The approximated solution of $u_4(x, t)$ is plotted in Fig. 9 for the same interval. In Fig. 10, the exact distribution of $v(x, t)$ for the intervals $-3 \leq x \leq 3$ and $0 \leq t \leq 1$ is plotted. The approximated solution of $v_4(x, t)$ is plotted in Fig. 11 for the same interval. Similarly, exact and approximated distributions of $w(x, t)$ for the same intervals are plotted in Figs. 12 and 13, respectively. Fig. 14 shows the exact distributions of $u(x, t)$, $v(x, t)$ and $w(x, t)$ for the interval $-9 \leq x \leq 9$ at $t=1$. VIM solutions of $u_4(x, t)$, $v_4(x, t)$ and $w_4(x, t)$ for the interval $-9 \leq x \leq 9$ at $t=1$ are represented in Fig. 15. Fig. 16 shows the comparison between exact and VIM solutions of $u(x, t)$, $v(x, t)$ and $w(x, t)$ for the same interval of x at $t=1$.

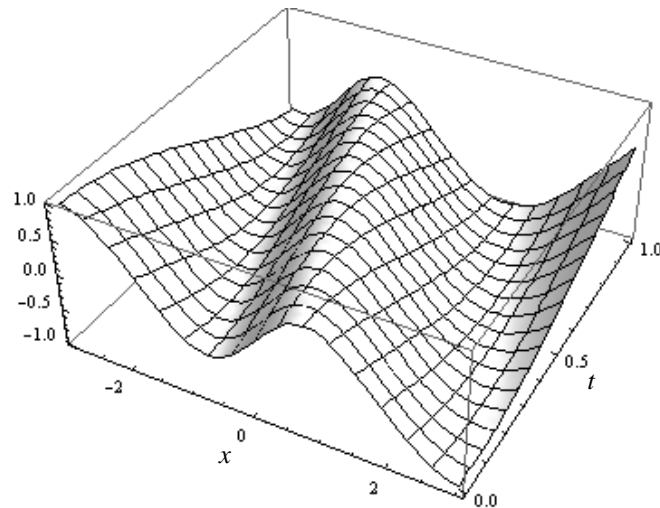


Fig. 8. Exact solution of $u(x, t)$ for the intervals $-3 \leq x \leq 3$ and $0 \leq t \leq 1$.

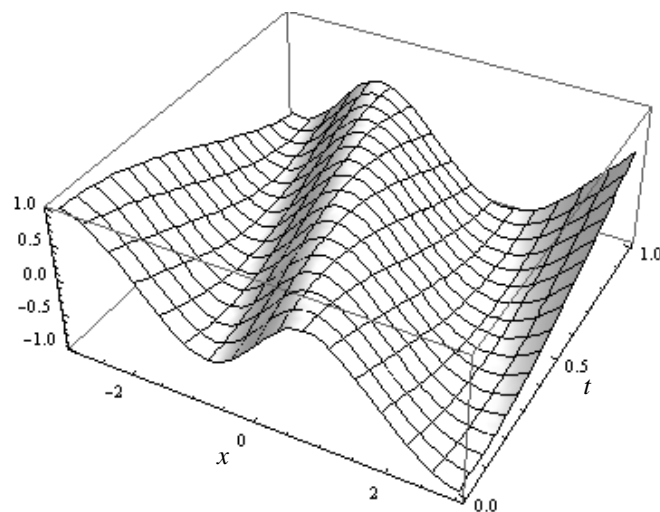


Fig. 9. VIM solution of $u_4(x, t)$ for the intervals $-3 \leq x \leq 3$ and $0 \leq t \leq 1$.

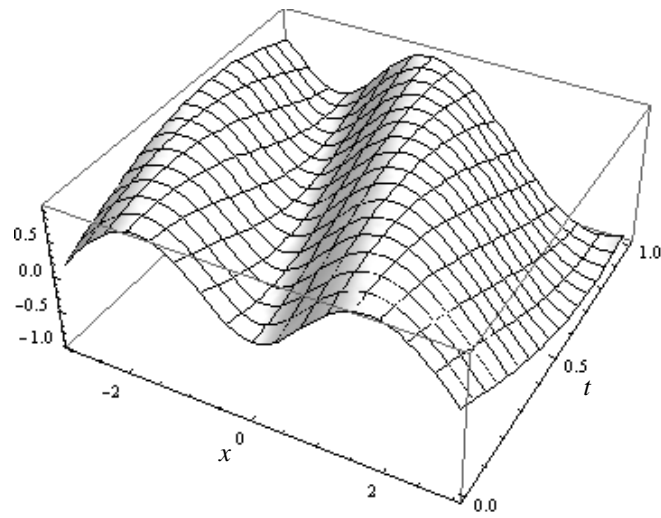


Fig. 10. Exact solution of $v(x, t)$ for the intervals $-3 \leq x \leq 3$ and $0 \leq t \leq 1$.

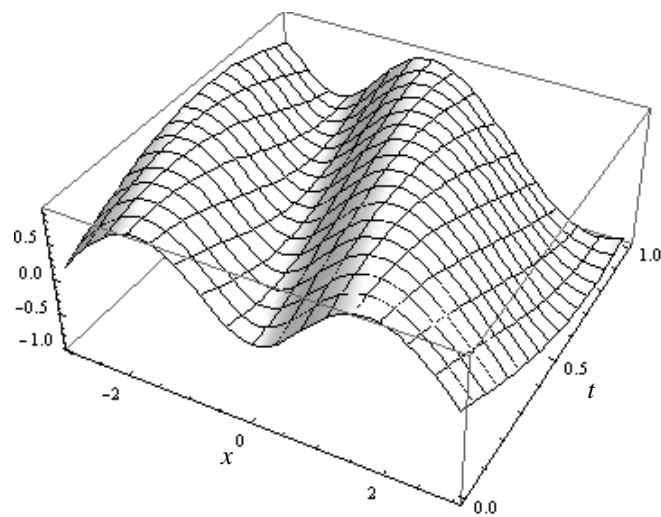


Fig. 11. VIM solution of $v_4(x, t)$ for the intervals $-3 \leq x \leq 3$ and $0 \leq t \leq 1$.

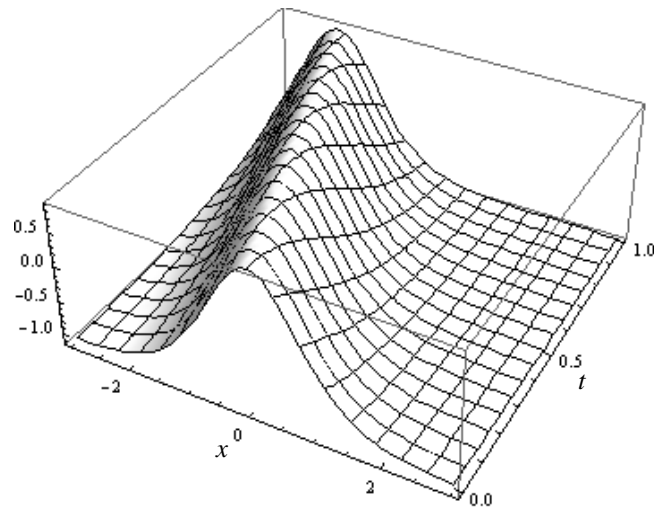


Fig. 12. Exact solution of $w(x, t)$ for the intervals $-3 \leq x \leq 3$ and $0 \leq t \leq 1$.

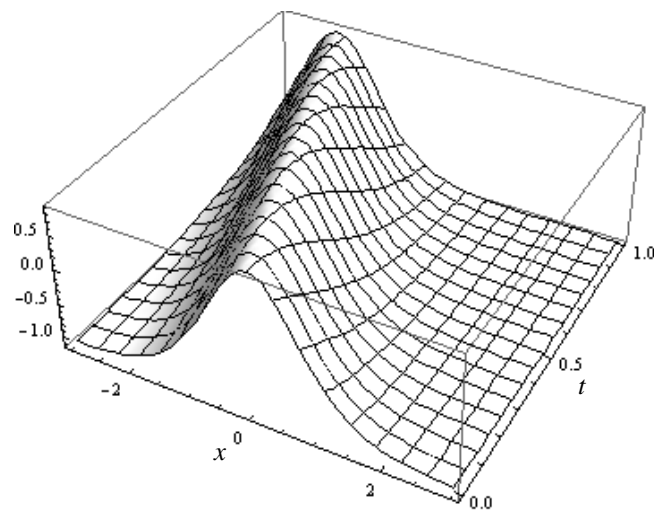


Fig. 13. VIM solution of $w_4(x, t)$ for the intervals $-3 \leq x \leq 3$ and $0 \leq t \leq 1$.

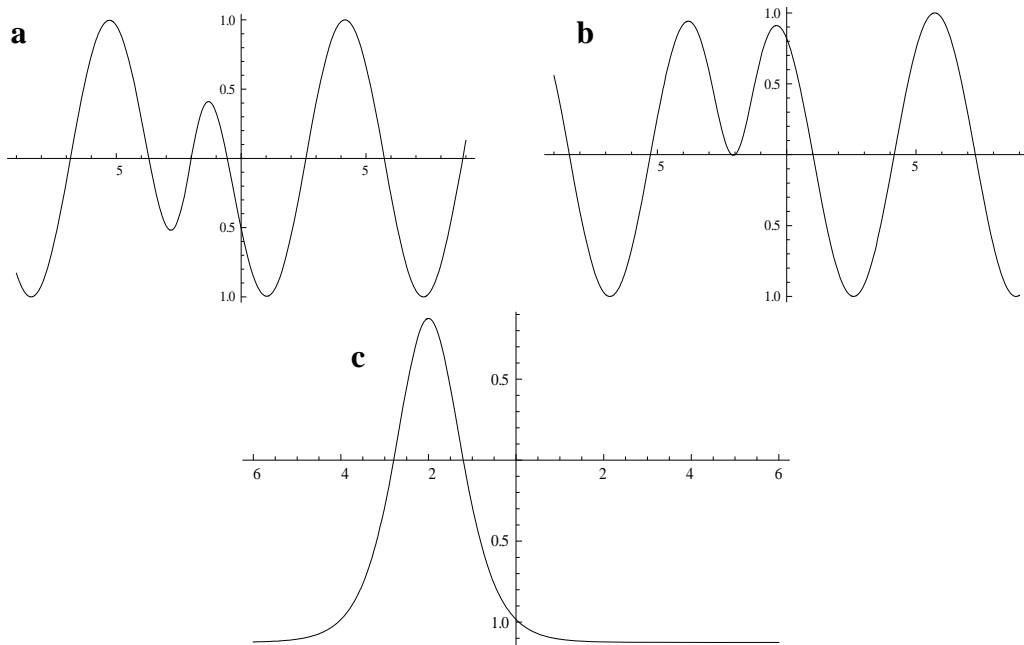


Fig. 14. Exact solutions of $u(x, t)$ (a) and $v(x, t)$ (b) and $w(x, t)$ (c) for the interval $-9 \leq x \leq 9$ at $t = 1$.

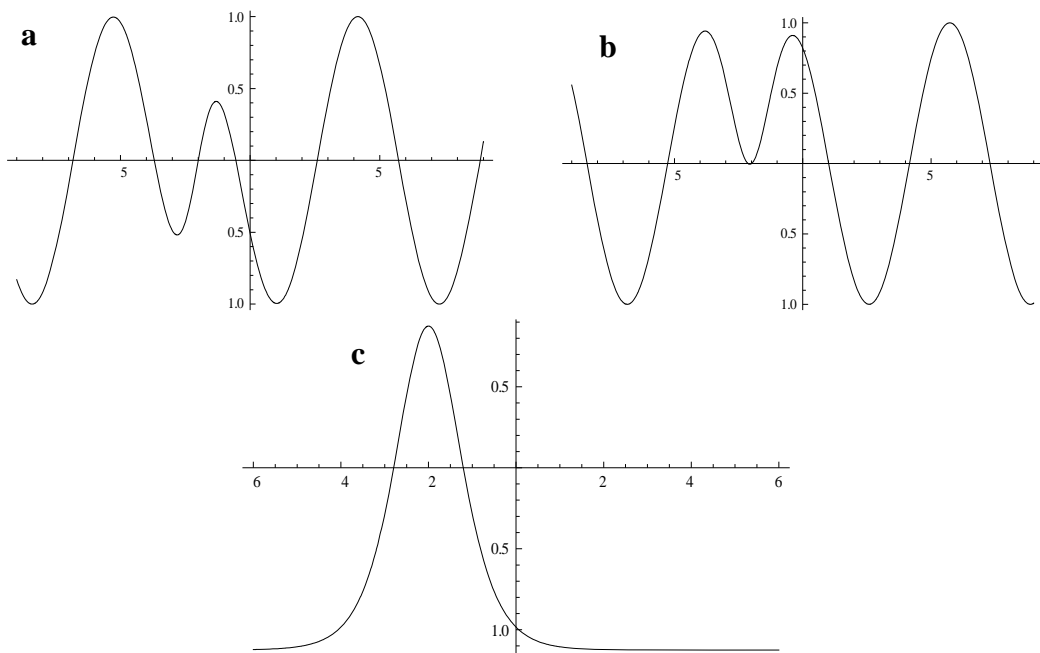


Fig. 15. VIM solutions of $u_4(x, t)$ (a) and $v_4(x, t)$ (b) $w_4(x, t)$ (c) for the interval $-9 \leq x \leq 9$ at $t = 1$.

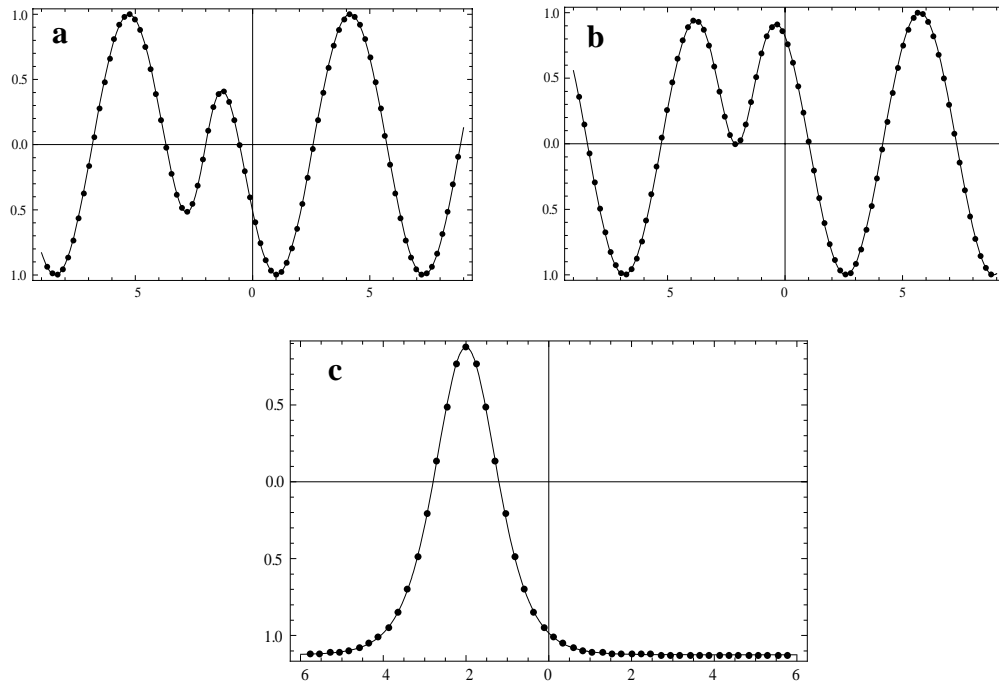


Fig. 16. the comparison between exact and VIM solutions of $u(x, t)$ (a) and $v(x, t)$ (b) and $w(x, t)$ (c) for the interval $-9 \leq x \leq 9$ at $t=1$. Line stands for the figures of the VIM solution and point for the exact solution.

4. Discussion

In this work, we proposed the variational iteration method (VIM) for solving the coupled Schrödinger–KdV equation and compared our results with the exact solution. A clear conclusion can be drawn from the results that the VIM provides highly accurate solutions for non-linear differential equations. This method can give very good approximations by means of a few terms for most cases. The results revealed that the VIM is a powerful mathematical tool for solving non-linear partial differential equations having wide applications in engineering.

References

- [1] N.N. Rao, Nonlinear wave modulations in plasmas, *Pramana J. Phys.* 49 (1997) 109–127.
- [2] S.V. Singh, N.N. Rao, P.K. Shukla, Nonlinearly coupled Langmuir and dust-acoustic waves in a dusty plasma, *J. Plasma Phys.* 60 (1998) 551–567.

- [3] E. Fan, Multiple traveling wave solutions of nonlinear evolution equations using a unified algebraic method, *J. Phys. A* 35 (2002) 6853–6872.
- [4] J.H. He, Approximate solution of nonlinear differential equations with convolution product nonlinearities, *Comput. Methods Appl. Mech. Eng.* 167 (12) (1998) 69–73.
- [5] J.H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, *Comput. Methods Appl. Mech. Eng.* 167 (12) (1998) 57–68.
- [6] J.H. He, Variational iteration method –a kind of non-linear analytical technique: some examples, *Int. J. Non-Linear Mech.* 34 (4) (1999) 699–708.
- [7] H. Tari, D.D. Ganji, M. Rostamian, Approximate solutions of K(2,2), KdV and modified KdV equations by variational iteration method, homotopy perturbation method and homotopy analysis method, *Int. J. Nonlinear Sci. Numer. Simul.* 8 (2) (2007) 203–
- [8] E. Yusufoglu, Variational iteration method for construction of some compact and noncompact structures of Klein–Gorden equations, *Int. J. Nonlinear Sci. Numer. Simul.* 8 (2) (2007) 153–158.

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