Applied Mathematical Sciences, Vol. 4, 2010, no. 17, 849 - 856

A Lotka-Volterra Model with Impulsive Effects on the Prey and Stage Structure on the Predator[1]

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Abstract

In this paper, we consider a Lotka-Volterra model with impulsive effects on the prey and stage structure on the predator. We prove that all solutions of the system are uniformly ultimately bounded, sufficient conditions of the global attractivity of predator-extinction periodic solution and the permanence of the system are obtained. These results show that the behavior of impulsive effects on the prey play an important role for the permanence of the system. Our results provide reliable tactical basis for the biological resource management.

Mathematics Subject Classification: 34C05, 92D25

Keywords: Stage structure; Impulsive; Global attractivity; Permanence

1. Introduction

The predator-prey models with stage structure for the predator were introduced or investigated by Jiao et al.[1].Since the immature predator takes τ units of time to mature, the death toll during the juvenile period should be considered, so time delays have important biological meanings in age-structured models. Hence many stage structured models with time delay were extensively studied by Wang and Chen et al.[2]. In recently years, impulsive systems are found in many domains of applied sciences[3]. The investigation of impulsive delay differential equations is beginning, and impulsive delay differential equations are almost analyzed in theory by Liu and Ballinger [4]. Time delay and

[1]This work is supported by the Nature Science Foundation of Hubei Province of China(2008CDB075), the Youth Science Foundation of Educational Department of Hubei Province in China, the Science Foundation(Q20082906), and the Doctoral Foundation of Hubei Institute for Nationalities and the Research Program for Outstanding Groups Science Foundation of Educational Department of Hubei Province in China(T200804).

impulse are introduced into predator-prey models with stage structure, which greatly enriches biologic background, but the system become nonautonomous, which causes us greatly difficult in studying the model.

2. Model formulation

The model we consider is based on the following predator-prey system

$$\begin{cases} x'(t) = x(t)(r - ax(t) - by(t)), \\ y'(t) = cx(t)y(t) - dy(t). \end{cases} \tag{2.1}$$

where $x(t)$ and $y(t)$ are densities of the prey and the predator, respectively, $r > 0$ is the intrinsic growth rate of prey, $a > 0$ is the coefficient of intraspecific competition, $b > 0$ is the per-capita rate of predation of the predator. d is the death rate of predator, c denotes the product of the per-capita rate of predation and the rate of conversing pest into predator. According to the nature of biological resource management, developing (2.1) by introducing the stocking on prey at fixed moments and harvesting mature predator population throughout the whole year or continuously. we consider the following impulsive delay differential equation:

$$\begin{cases} \left.\begin{array}{l} x'(t) = rx(t)(1 - \frac{x(t)}{k}) - \frac{\beta x(t)y_2(t)}{1+\alpha x(t)}, \\ y_1'(t) = \frac{k\beta x(t)y_2(t)}{1+\alpha x(t)} - e^{-\omega\tau}\frac{k\beta x(t-\tau)y_2(t-\tau)}{1+\alpha x(t-\tau)} - \omega y_1(t), \\ y_2'(t) = e^{-\omega\tau}\frac{\lambda\beta x(t-\tau)y_2(t-\tau)}{1+\alpha x(t-\tau)} - \omega y_2(t) - \mu y_2^2(t), \end{array}\right\} \quad t \neq nT, \\ \left.\begin{array}{l} \Delta x(t) = -px(t), \\ \Delta y_1(t) = 0, \Delta y_2(t) = 0, \end{array}\right\} \quad t = nT, n = 1, 2 \ldots \\ (\varphi_1(s), \varphi_2(s), \varphi_3(s)) \in C_+ = C([-\tau, 0], R_+^3), \varphi_i(0) > 0, i = 1, 2, 3. \end{cases} \tag{2.2}$$

Where $y_1(t), y_2(t)$ represent the immature and mature predator densities respectively. β is the predation rate of predator, ω is the death rate of predator, we assume that the death rate of mature populations are of a logistic nature, that is, proportional to the square of the population with proportionality constant μ. α is the saturation which represents that a certain amount of predators can prey on a limited amount of preys, although the preys are numerous. λ represents the conversion rate at which ingested prey in excess of what is needed for maintenance is translated into predator population increase. $p(0 \leq p < 1)$ represents partial impulsive harvest to preys by catching or pesticides, τ is the mean length of the juvenile period, the capacity rate k is concerned with the resources which maintain the evolution of the population, T is the period of the impulsive of the prey. In this paper, we always assume the immature predator population can not prey the prey population. Because the first and third equations of (2.2) do not contain $y_1(t)$,we can simplify model and restrict our attention to the following model:

$$\begin{cases} \left.\begin{array}{l} x'(t) = rx(t)(1 - \frac{x(t)}{k}) - \frac{\beta x(t) y_2(t)}{1+\alpha x(t)}, \\ y_2'(t) = e^{-\omega\tau} \frac{\lambda\beta x(t-\tau) y_2(t-\tau)}{1+\alpha x(t-\tau)} - \omega y_2(t) - \mu y_2^2(t), \end{array}\right\} \quad t \neq nT, \\ \Delta x(t) = -px(t), \Delta y_2(t) = 0, \quad t = nT, n = 1, 2 \ldots \\ (\phi_1(s), \phi_3(s)) \in C_+ = C([-\tau, 0], R_+^2), \phi_i(0) > 0, i = 1, 3. \end{cases} \tag{2.3}$$

From the biological point of view, we only consider system (2.2) in the biological meaning region: $D = \{(x, y_1, y_2) | x, y_1, y_2 \geq 0\}$,

3. Some important lemmas

Before we have the main results we need to give some lemmas which will be used in the next.

Definition 3.1 System (2.3) is said to be uniformly persistent if there is an $\eta > 0$ (independent of the initial data) such that every solution $(S(t), I(t))$ of system (2.3) satisfies $\lim_{t\to\infty} \inf S(t) \geq \eta, \lim_{t\to\infty} \inf I(t) \geq \eta$.

Definition 3.2 System (2.3) is said to be permanent if there exists a compact region $D \in \Omega$ such that every solution of system (2.3) will eventually enter and remain in region D.

Lemma 3.1[5]. Considering the following delay equation:

$$x'(t) = ax(t - \tau) - bx(t) - cx^2(t),$$

where a, b, c, τ are all positive constants and $x(t) > 0$ for $t \in [-\tau, 0]$.

(1)If $a < b$ then $\lim_{t\to\infty} x(t) = 0$, (2)If $a > b$, then $\lim_{t\to\infty} x(t) = \frac{a-b}{c}$.

Lemma 3.2[6]. Let $v^* = \frac{b}{a} \frac{1-e^{-aT}}{1-p-e^{-aT}}$, considering the following impulsive system:

$$\begin{cases} v'(t) = v(t)(a - bv(t)), & t \neq nT, \\ v(t^+) = (1 - p)v(t), & t = nT, n = 1, 2, \ldots, \end{cases} \tag{3.1}$$

where $a, b > 0, 0 \leq p < 1$,then there exists a unique positive periodic solution of system (2.2) $\widetilde{v}(t) = \frac{ae^{a(t-nT)}}{av^* - b + be^{a(t-nT)}}, t \in (nT, (n + 1)T]$, which is globally asymptotically stable.

Lemma 3.3 There exists a constant $L = \frac{k^2(\omega+r)^2}{4r\omega} > 0$, such that $x(t) \leq \frac{L}{k}, y_1(t) \leq L, y_2(t) \leq \frac{\lambda L}{k}$ for each positive solution $(x(t), y_1(t), y_2(t))$ of (2.2) with all t large enough.

4. Predator-extinction periodic solution and global attractivity of the periodic solution

We begin the analysis of (2.3) by first demonstrating the existence of a 'predator-extinction' solution, in which predator individuals are entirely absent from the population permanently, i.e.,

$$y_2(t) = 0, t \geq 0 \ . \tag{4.1}$$

This is motivated by the fact that $y^* = 0$ is an equilibrium solution for the variable $y_2(t)$, as it leaves $\dot{y}_2(t) = 0$. Assuming (4.1), we know that the growth of the prey in the time-interval $nT < t \leq (n+1)T$ and give some basic properties of the following subsystem of (2.3)

$$\begin{cases} x'(t) = x(t)(r - \frac{r}{k}x(t)), t \neq nT, \\ x(t^+) = (1-p)x(t), t = nT, n = 1, 2 \ldots, \end{cases} \tag{4.2}$$

By Lemma 3.2, system (4.2) has a globally asymptotically stable positive periodic

$$\widetilde{x}(t) = \frac{kx^*}{x^* + (k - x^*)e^{-r(t-nT)}}, t \in (nT, (n+1)T], n \in N,$$

therefore, system (2.3) has a predator-extinction periodic solution

$$(\widetilde{x}(t), 0) = (\frac{kx^*}{x^* + (k - x^*)e^{-r(t-nT)}}, 0), t \in (nT, (n+1)T], n \in N,$$

which is globally asymptotically stable, where $x^* = \frac{k[(1-p)-e^{-rT}]}{1-e^{-rT}}$

Theorem 4.1 Let $(x(t), y_1(t), y_2(t))$ be any solution of system (2.3), if

$$R_1 = \frac{\lambda\beta e^{-\omega\tau}}{\omega}\frac{R_0}{1+\alpha R_0} < 1, \tag{4.3}$$

where $R_0 = \frac{kx^*}{x^*+(k-x^*)e^{-rT}}$, $x^* = \frac{k[(1-p)-e^{-rT}]}{1-e^{-rT}}$, then the 'predator-extinction' periodic solution $(\widetilde{x}(t), 0, 0)$ is globally attractive.

Proof: It is clear that the global attraction of predator-extinction periodic solution $(\widetilde{x}(t), 0, 0)$ of system (2.2) is equivalent to the global attraction of predator-extinction $(\widetilde{x}(t), 0)$ of system (2.3), so we only dedicate to prove system (2.3).

Since $\lambda\beta e^{-\omega\tau}\frac{R_0+\varepsilon_0}{1+\alpha(R_0+\varepsilon_0)} < \omega$, we can choose ε_0 to be sufficiently small such that

$$\lambda\beta e^{-\omega\tau}\frac{\frac{kx^*}{x^*+(k-x^*)e^{-rT}}+\varepsilon_0}{1+\alpha[\frac{kx^*}{x^*+(k-x^*)e^{-rT}}+\varepsilon_0]} < \omega \ . \tag{4.4}$$

where $x^* = \frac{k[(1-p)-e^{rT}]}{1-e^{-rT}}$, It following from the first equation of system (2.3) that $x'(t) \leq rx(t)(1 - \frac{x(t)}{k})$, so we consider the following comparison impulsive differential system

$$\begin{cases} x_1'(t) = rx_1(t)(1 - \frac{x_1(t)}{k}), t \neq nT, \\ x_1(t^+) = (1-p)x_1(t), t = nT, \end{cases} \tag{4.5}$$

by Lemma 3.2, system (4.5) has a globally asymptotically stable positive periodic solution $\widetilde{x_1}(t) = \frac{kx_1^*}{x_1^*+(k-x_1^*)e^{-r(t-nT)}}, t \in (nT, (n+1)T], n \in N$, from comparison theorem of impulsive equation, we have $x(t) \le x_1(t)$ and $x_1(t) \to \widetilde{x}(t)$ as $t \to \infty$. Then there exists an integer $k_2 > k_1, t > k_2$ such that

$$x(t) \le x_1(t) < \widetilde{x}(t) + \varepsilon_0 < \frac{kx^*}{x^*+(k-x^*)e^{-rT}} + \varepsilon_0 =: \rho, t \in (nT, (n+1)T], n > k_2, \tag{4.6}$$

From (2.3) and (4.6), we have that

$$y_2'(t) \le e^{-\omega\tau}\frac{\lambda\beta\rho y_2(t-\tau)}{1+\alpha\rho} - \omega y_2(t) - \mu y_2^2(t), t > nT + \tau, n > k_2.$$

we consider the following impulsive equation

$$z'(t) = e^{-\omega\tau}\frac{\lambda\beta\rho z(t-\tau)}{1+\alpha\rho} - \omega z(t) - \mu z^2(t), t > nT + \tau, n > k_2.$$

From (4.4), we have $\lambda\beta e^{-\omega\tau}\frac{\rho}{1+\alpha\rho} < \omega$. According to Lemma 3.1, we have $\lim_{t\to\infty} y(t) = 0$, by using comparison theorem, we have $\lim_{t\to\infty} y_2(t) < \lim_{t\to\infty} y(t) = 0$. Incorporating into the positivity of $y_2(t)$, we know that $\lim_{t\to\infty} y_2(t) = 0$. Therefore, for any $\varepsilon_1 > 0$(sufficiently small), there exists an integer $k_3 > k_2$ such that $y_2(t) < \varepsilon_1$ for all $t > k_3T$, by system (2.3), we obtain that $x(t)(r - \frac{r}{k}x(t) - \beta\varepsilon_1) \le x'(t) \le rx(t)(1 - \frac{x(t)}{k})$, then we have $z_1(t) \le x(t) \le z_2(t)$, and $z_1(t) \to \widetilde{z}_1(t), z_2(t) \to \widetilde{z}_2(t)$ as $t \to \infty$, where $\widetilde{z}_1(t) = \frac{(r-\beta\varepsilon_1)e^{(r-\beta\varepsilon_1)(t-nT)}}{(r-\beta\varepsilon_1)v^*-r/k+(r/k)e^{(r-\beta\varepsilon_1)(t-nT)}}$, $v^* = \frac{r/k}{(r-\beta\varepsilon_1)}\frac{1-e^{-(r-\beta\varepsilon_1)T}}{1-p-e^{-(r-\beta\varepsilon_1)T}}$ and $\widetilde{z}_2(t) = \frac{kx^*}{x^*+(k-x^*)e^{-r(t-nT)}}$ for $t \in (nT, (n+1)T]$. By using comparison theorem of impulsive equation, for any $\varepsilon_2 > 0$ there exists an integer $k_4 > k_3$, such that $\widetilde{z_1}(t) - \varepsilon_2 < x(t) < \widetilde{x}(t) + \varepsilon_2$ for $t > k_4T$, let $\varepsilon_1 \to 0$, then it follows that $\widetilde{x}(t) - \varepsilon_2 < x(t) < \widetilde{x}(t) + \varepsilon_2, t \to \infty$. Because ε_2 arbitrary small, it follows that $x(t) \to \widetilde{x}(t)$ as $t \to \infty$. Therefore, predator-extinction periodic solution $(\widetilde{x}(t), 0)$ is globally attractive. This completes the proof.

5. Boundness and Permanence

The next work is to investigate the permanence of the system (2.3). Denote

$$R_2 = \frac{\lambda\beta e^{-\omega\tau}}{\omega}\frac{\eta_1}{1+\alpha\eta_1}, \eta_1 = \frac{(k-\lambda\beta L)(1-p-e^{-r(1-\frac{\lambda\beta L}{k})T})}{1-e^{-r(1-\frac{\lambda\beta L}{k})T}} \tag{5.1}$$

Theorem 5.1 Suppose $R_2 > 1$, then there is a positive constant q such each positive solution $(x(t), y_2(t))$ of system (2.3) satisfies $y_2(t) \ge q$ if t is large enough.

Proof: Suppose $(x(t), y_2(t))$ is any positive solution of system (2.3) with initial conditions (2.4). The second equation of system (2.3) may be rewritten as follows

$$y_2'(t) = (\lambda\beta e^{-\omega\tau}\tfrac{x(t)}{1+\alpha x(t)} - \omega - \mu y_2(t))y_2(t) - \lambda\beta e^{-\omega\tau}\tfrac{d}{dt}\int_{t-\tau}^{t}\tfrac{x(\theta)y_2(\theta)}{1+\alpha x(\theta)}d\theta \ . \quad (5.2)$$

Define $V(t) = y_2(t) + \lambda\beta e^{-\omega\tau}\int_{t-\tau}^{t}\frac{x(\theta)y_2(\theta)}{1+\alpha x(\theta)}d\theta$. Calculating the derivative of $V(t)$ along the solution of system (2.3), it follows from (5.2) that

$$V'(t) = (\lambda\beta e^{-\omega\tau}\frac{x(t)}{1+\alpha x(t)} - \omega - \mu y_2(t))y_2(t). \quad (5.3)$$

Due to Lemma 3.3, (5.3) can be written

$$V'(t) = (\lambda\beta e^{-\omega\tau}\frac{x(t)}{1+\alpha x(t)} - \omega - \mu\frac{\lambda L}{k})y_2(t)$$

for t large enough. Since $R_2 > 1$, then there exists sufficiently small $\varepsilon_3 > 0$ such that

$$\frac{\lambda\beta k e^{-\omega\tau}}{k\omega + \lambda\mu L}\frac{\eta_1 + \varepsilon_3}{1 + \alpha(\eta_1 + \varepsilon_3)} > 1,$$

We claim that for any $t_0 > 0$, it is impossible that $y_2(t) < y_2^*$ for all $t \geq t_0$. Suppose that the claim is not valid, then there is a $t_0 > 0$ such that $y_2(t) < y_2^*$ for all $t \geq t_0$. It follows from the first equation of (2.3) that for all $t \geq t_0$, $x'(t) > rx(t)(1 - \frac{\lambda\beta L}{k} - \frac{x(t)}{k})$. By comparison theorem of impulsive differential equation, we know that there exists a $t_1 (t_1 > t_0 + \omega)$ such that the following inequality holds for $t > t_1$

$$x(t) \geq \frac{k - \lambda\beta L}{[(k-\lambda\beta L)v_1^* - 1]e^{-r(1-\frac{\lambda\beta L}{k})(t-nT)} + 1} - \varepsilon_3, \quad (5.4)$$

Where $v_1^* = \dfrac{1 - e^{-r(1-\frac{\lambda\beta L}{k})T}}{(k-\lambda\beta L)(1 - p - e^{-r(1-\frac{\lambda\beta L}{k})T})}$. Thus

$$x(t) \geq \frac{(k-\lambda\beta L)(1 - p - e^{-r(1-\frac{\lambda\beta L}{k})T})}{1 - e^{-r(1-\frac{\lambda\beta L}{k})T}} - \varepsilon_3 =: \eta_1 - \varepsilon_3$$

for $t > t_1$. By (5.3) and (5.4), we have

$$V'(t) = (\lambda\beta e^{-\omega\tau}\frac{\eta_1 - \varepsilon_3}{1 + \alpha(\eta_1 - \varepsilon_3)} - \omega - \mu\frac{\lambda L}{k})y_2(t), t > t_1. \quad (5.5)$$

Set $y_2^m = \min_{t\in[t_1, t_1+\tau]} y_2(t)$, we will show that $y_2(t) \geq y_2^m$ for all $t \geq t_1$. Suppose the contrary, then there is a nonnegative constant $T_0 > 0$ such that

$y_2(t) \geq y_2^m$ for $t \in [t_1, t_1+\tau+T_0]$, $y_2(t_1+\tau+T_0) = y_2^m$ and $y_2'(t_1+\tau+T_0) < 0$. However, the second equation of system (2.3) imply that

$$y_2'(t_1+\tau+T_0) \geq (\lambda\beta e^{-\omega\tau}\frac{\eta_1}{1+\alpha\eta_1} - \omega - \mu\frac{\lambda L}{k})y_2^m$$

for $t > t_1$, which imply that $t \to \infty, V(t) \to \infty$. This a contradiction to $V(t) \leq L$. Hence, for any $t_0 > 0$, it is impossible that $y_2(t) < y_2^m$ for all $t \geq t_0$. Following, we are left to consider two cases:
(i) $y_2(t) \geq y_2^m$ for all t large enough; (ii) $y_2(t)$ oscillates about y_2^m for all t large enough. Let

$$q = \min\{\frac{y_2^m}{2}, y_2^m e^{-(\omega+\mu\frac{\lambda L}{k})T}\}.$$

In the following, we shall show that $y_2(t) \geq y_2^m$. There exist two positive constant t_1, t_2 such that $y_2(t_1) = y_2(t_1+t_2) = y_2^m$ and $y_2(t) < y_2^m$, for all $t_1 < t < t_1+t_2$. When t_1 is large enough, the inequality $x(t) > \eta_1$ holds true for $t_1 < t < t_1+t_2$. Since $y_2(t)$ is continuous and bounded and is not effected by impulses, we conclude that $y_2(t)$ is uniformly continuous. Hence there exists a constant T_1 (with $0 < T_1 < \tau$ and T_1 is independent of the choice of t_1) such that $y_2(t) > \frac{y_2^m}{2}$ for all $t_1 \leq t \leq t_1+T_1$. If $t_2 \leq T_1$, our aim is obtained. If $T_1 < t_2 \leq \tau$, from the second equation of (2.3) we have that $y_2(t) \geq -(\omega+\mu\frac{\lambda L}{k})y_2(t)$ for $t_1 < t \leq t_1+t_2$. Then we have $y_2(t) \geq y_2^m e^{-(\omega+\mu\frac{\lambda L}{k})T}$ for $t_1 < t \leq t_1+t_2 \leq t_1+\tau$ since $y_2(t_1) = y_2^m$. It is clear that $y_2(t) \geq q$ for $t_1 < t \leq t_1+t_2$. If $t_2 \geq \tau$, by the second equation of (2.3), then we have that $y_2(t) \geq q$ for $t_1+\tau \leq t \leq t_1+t_2$. Since the interval $[t_1, t_2]$ is arbitrarily chose (we only need t_1 to be large), we get that $y_2(t) \geq q$ for all t large enough. In view of our arguments above, the choice of q is independent of the positive solution of (2.3) which satisfies that $y_2(t) \geq q$ for sufficiently large t. The completes the proof.

Theorem 5.2 System (2.3) is permanent provided $R_2 > 1$.

Proof: Denote $(x(t), y_2(t))$ be the solution of system (2.3). From the first equation of (2.3), we have that $x'(t) > rx(t)(1 - \frac{\lambda\beta L}{k} - \frac{x(t)}{k})$. Similar to (5.4), we obtain that $x(t) \geq \tilde{z}_3(t) - \varepsilon_3 \geq \eta_1 - \varepsilon_3 ==: p$. By theorem 5.1, there exist positive constants p, q and T_2 such that $x(t) \geq p, y_2(t) \geq q$ for $t \geq T_2$. Set $\Omega = \{(x, y_2) \in R_+^2 | p \leq x(t) \leq \frac{L}{k}, q \leq y_2(t) \leq \frac{\lambda L}{k}\}$. Then Ω is a bounded compact region which has positive distance from coordinate axes. By theorem 5.1, one obtains that every solution of system (2.3) with initial condition (2.4) eventually enters and remains in the region Ω. This completes the proof.

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Received: August, 2009