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The Linear Stability of Traveling Waves to the Compound Kdv-Burgers Equation[1]

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Abstract

Herein the traveling waves of compound Kdv-Burgers are considered,.with the help of an analytic function $D(\lambda)$, the traveling wave is stable.

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1 Introduction

The compound Kdv-Burgers equation with diffusion terms

$$u_t + auu_x + bu^2u_x + ru_{xx} + \beta u_{xxx} = 0 \tag{1.1}$$

where nonlinearity coefficients a, b and dispersion coefficient β are positive constants,diffusion coefficient r isn't positive constant

when $r = 0$ equation (1) can be written the compound Kdv equation

$$u_t + auu_x + bu^2u_x + \beta u_{xxx} = 0 \tag{1.2}$$

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when $b = 0$ equation (1) can be written the Kdv-Burgers equation

$$u_t + auu_x + ru_{xx} + \beta u_{xxx} = 0 \tag{1.3}$$

Many authors[1−3] have investigated the existence and stability of equation (1.2) and (1.3),such as Zhang Weiguo[4]solve the four solitary solutions and six periodic waves by hypothesis undetermined method.C. Jose[5]discussed numerically the stability of the shock solution of equation(1.3) for $a = 2$.L.Pego[6]studied the stability of traveling wave solutions of a generalization of the Kdv-Burgers $u_t + u^p u_x + u_{xxx} = \alpha u_{xx}$, as the parameters p and α are varied, by Evan's function and numerical experiment,and obtained that linear instability took place when: a) for fixed positive wave velocity c and $p > 4, \alpha$ is made sufficiently small. b) for fixed positive α and $p > 4$,wave velocity c is made sufficiently small. c) for fixed positive α and wave velocity c,p is made sufficiently large.

2 Linear stability

In this section, we investigate the linear stability of oscillatory wave solution[7] $\phi(\xi)$ which satisfies:

1) $\phi(\xi) \to \begin{cases} 0, & \xi \to +\infty \\ u_0, & \xi \to -\infty \end{cases}$,where $u_0 = \frac{-3+\sqrt{9+48c}}{4}$

2) when $r^2 < \frac{8\beta c\sqrt{3+16c}}{\sqrt{3}+\sqrt{3+16c}}$ holds, $\phi(\xi)$ is monotonically decreasing and decay to zero as $\xi \to +\infty$, while $\phi(\xi) \to u_0$ in an oscillatory fashion as $\xi \to -\infty$

A natural one for our purposes is the space

$$BC(R, R) = \{u : R \to R | u \text{ is bounded and uniformly continuous}\}$$

supplied with supremum norm,see Henry[8]. There exists a solution of equation (1.1) with the initial value $u(x, 0) = u_0(x) \in BC(R, R)$,see [9].If (1.1) is recast in a moving coordinate frame,that is,in terms of variables $\xi = x - ct$ and t,assuming that $a = b = 1, r < 0$,it becomes

$$u_t + uu_\xi + u^2 u_\xi + ru_{\xi\xi} + \beta u_{\xi\xi\xi} = cu_\xi \tag{2.1}$$

To consider the stability,we use the linearized criterion. By substituting $u(x, t) = \phi(x - ct) + v(x - ct, t)$ into equation (1.1),neglecting the terms which are $O(v^2)$,the linear equation is

$$v_t = Lv \tag{2.2}$$

where $L = -\beta\partial_\xi^3 - r\partial_\xi^2 + c\partial_\xi - \partial_\xi f'(\phi), f(\phi) = \frac{\phi^2}{2} + \frac{\phi^3}{3}$

The linearized criterion for stability of the traveling wave $\phi(\xi)$ is that the spectrum of L(expect for 0) lies in the left half plane.

If λ is an eigenvalue of L with corresponding L^2 function Y,then Y is a solution of the differential equation

$$\lambda Y = LY \tag{2.3}$$

i.e.

$$\lambda Y = \partial_\xi(-\beta\partial_\xi^2 - r\partial_\xi + c - f'(\phi))Y = \partial_\xi L_c Y \tag{2.4}$$

equals to equation

$$\frac{dy}{d\xi} = A(\xi,\lambda)y \tag{2.5}$$

where $\quad y = (Y, Y', Y'')^T, \qquad A(\xi,\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\beta}(\lambda + \partial_\xi f'(\phi)) & -\frac{1}{\beta}(c - f'(\phi)) & -\frac{r}{\beta} \end{pmatrix}$

and we note asymptotical behavior matric $A^+ = \lim_{\xi\to+\infty} A(\xi,\lambda)$, $A^- = \lim_{\xi\to-\infty} A(\xi,\lambda)$.So

$$A^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{\lambda}{\beta} & -\frac{c}{\beta} & -\frac{r}{\beta} \end{pmatrix}, \qquad A^- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{\lambda}{\beta} & \frac{\sqrt{9+48c}-3-16c}{8\beta} & -\frac{r}{\beta} \end{pmatrix}$$

If μ is the eigenvalues of matric $A^\pm$,then

$$\hbar_+(\mu) = \mu^3 + \frac{r}{\beta}\mu^2 - \frac{c}{\beta}\mu + \frac{\lambda}{\beta} = 0 \tag{2.6}$$

and

$$\hbar_-(\mu) = \mu^3 + \frac{r}{\beta}\mu^2 - \frac{\sqrt{9+48c}-3-16c}{8\beta}\mu + \frac{\lambda}{\beta} = 0 \tag{2.7}$$

Obviously, if $\mu = i\tau(\tau \in R)$, then we find easily

$$S_e^+ = \{\lambda : \lambda = r\tau^2 + i(\beta\tau^3 + c\tau)\}$$

$$S_e^- = \{\lambda : \lambda = r\tau^2 + i(\beta\tau^3 + \tau(\frac{\sqrt{9+48c}-3-16c}{8}))\}$$

Because $r < 0$ and $S_e^+ \cup S_e^-$ determines the information of the essential spectral(see Henry[8]), so we know the essential spectrum $\sigma_e(L)$ of L lies entirely in the left half plane and cause no problem for stability. We discuss mainly the isolate spectrum $\sigma_n(L)$ by defining a analytic function $D(\lambda)$,which is Evan's function[10] clarifying the isolate spectrum.The complement of $S_e^+ \cup S_e^-$ consists of a number of disjoint connected components.Let Ω denote the component which contains the right half plane.Then the essential spectrum can be characterized by the following results which is proved in [6]

Proposition 3 The essential spectrum of L contains $S_e^+ \cup S_e^-$, but contains no point of the component Ω,in particular,points in the spectrum of lying in Ω,must be isolated eigenvalues.

Ω does not consist entirely of eigenvalues,however,this does not influence the discuss about stability because of the following proposition 6 and 7.

Assuming $\lambda = 0$,by (2.6) and (2.7) we compute the eigenvalues of A^+ are

$$\mu_1^+ = \frac{-r-\sqrt{r^2+4\beta c}}{2\beta}, \quad \mu_2^+ = 0, \quad \mu_3^+ = \frac{-r+\sqrt{r^2+4\beta c}}{2\beta} \tag{2.8}$$

and the eigenvalues of A^- are

$$\begin{aligned} \mu_1^- = 0, \quad \mu_2^- &= \tfrac{-2r-\sqrt{4r^2+2\beta(\sqrt{9+48c}-3-16c)}}{4\beta}, \\ \mu_3^- &= \tfrac{-2r+\sqrt{4r^2+2\beta(\sqrt{9+48c}-3-16c)}}{4\beta} \end{aligned} \tag{2.9}$$

Differentiating (2.6)in λ,when $\mu = 0$,we observe $\frac{d\mu}{d\lambda} = \frac{1}{c} > 0$ and find $\mu > 0$ holds if $\lambda > 0$,

Differentiating (2.7)in λ,when $\mu = 0$,we observe $\frac{d\mu}{d\lambda} = \frac{8}{\sqrt{9+48c}-3-16c} < 0$ and find $\mu < 0$ holds if $\lambda > 0$.

when $r^2 < \frac{8\beta c\sqrt{3+16c}}{\sqrt{3}+\sqrt{3+16c}}$, so $4r^2 + 2\beta(\sqrt{9+48c} - 3 - 16c) < 0$,therefore real part of the eigenvalues μ_2^- and μ_3^- are positive.

Then $A^\pm$ exists unique negative real part eigenvalue,therefore

Proposition 4 there exists $a > 0$, such that $\Omega_+ = \{\lambda : Re\lambda \geq -a\} \subset \Omega$.

Now let us define Evan's function $D(\lambda)$. Firstly we need the following contents.

$$\frac{dz}{d\xi} = -zA(\xi, \lambda) \tag{2.10}$$

is the adjoint system of (2.5),using the equation (1.31) of [11],it may be related to the adjoint equation

$$\lambda z = (-\beta\partial_\xi^2 + r\partial_\xi + c - f'(\phi))\partial_\xi z = L_c\partial_\xi z$$

$\mu_j^\pm (j = 1, 2, 3)$ denotes the eigenvalues $A^\pm$ and satisfies $Re\mu_1^\pm \leq Re\mu_j^\pm$, by the proposition (1.2)of [11], there exists unique solution ζ^+ of system (2.5)and unique solution η^- of system (2.10) which are analytic for $\lambda \in \Omega_+$, satisfy

$$\zeta^+ \sim exp(\mu_1^+\xi)v^+ \quad (as \quad \xi \to +\infty), \tag{2.11}$$

$$\eta^- \sim exp(-\mu_1^-\xi)w^- \quad (as \quad \xi \to -\infty) \tag{2.12}$$

where the notations

$$v^+ = (1, \mu_1^+, (\mu_1^+)^2)^T, \qquad v^- = (1, \mu_1^-, (\mu_1^-)^2)^T$$

and

$$w^+ = \frac{(\mu_1^+(\mu_1^+ + \frac{r}{\beta}) - \frac{c}{\beta}, \mu_1^+ + \frac{r}{\beta}, 1)}{\hbar'_+(\mu_1^+)}, \qquad w^- = \frac{(\mu_1^-(\mu_1^- + \frac{r}{\beta}) - \frac{\sqrt{9+48c}-3-16c}{8\beta}, \mu_1^- + \frac{r}{\beta}, 1)}{\hbar'_-(\mu_1^-)}$$

satisfy

$$(A^\pm - \mu_1^\pm I)v^\pm = 0, \qquad w^\pm(A^\pm - \mu_1^\pm I) = 0, \qquad w^+ \cdot v^+ = w^- \cdot v^- = 1$$

by the proposition (1.4)of [11], we find

$$\zeta^+ \sim (\eta^- \cdot \zeta^+)(\lambda) exp(\mu_1^- \xi) v^-(\lambda) \quad (as \quad \xi \to -\infty)$$

Secondly we may define Evan's function $D(\lambda) = (\eta^- \cdot \zeta^+)(\xi, \lambda)$. If $D(\lambda)$ satisfies the below propositions,we can investigate the stability of oscillatory wave solution $\phi(\xi)$.

Proposition 5 Evan's function $D(\lambda)$ is independent of ξ and analytic for $\lambda \in \Omega_+$

Proof: because the solution ζ^+, η^- are analytic,so is $D(\lambda)$; computing

$$\frac{\partial D(\lambda)}{\partial \xi} = \frac{\partial \eta^-}{\partial \xi} \cdot \zeta^+ + \eta^- \cdot \frac{\partial \zeta^+}{\partial \xi} = -\eta^- A(\xi, \lambda) \cdot \zeta^+ + \eta^- \cdot A(\xi, \lambda)\zeta^+ = 0$$

so $D(\lambda)$ is independent of ξ. Using theorem 1.9 by [11],we have

Proposition 6 If $\lambda \in \Omega_+, \lambda$ is a eigenvalue of equation (2.3) if and only if $D(\lambda) = 0$.

To investigate stability , we must press for

Proposition 7 For $\lambda \in \Omega_+, \quad D(\lambda)$ satisfy i)$D(0) = 0$, ii)$D(\lambda) \neq 0$, for Re$\lambda > 0$, and iii) $\frac{dD(\lambda)}{d\lambda}\Big|_{\lambda=0} > 0$.

proof: Differentiating (2.1) in ξ, we observe

$$\beta\phi''' + r\phi'' - c\phi' + \phi\phi' + \phi'\phi^2 = 0 \tag{2.13}$$

comparing (2.13) with (2.4), and using the (1.34)of [11], we find $D(0) = 0$,i)holds.

Assuming that $D(\lambda) = 0$ when $Re\lambda > 0$, for $\lambda \in \Omega_+$, by the definition of $D(\lambda)$, we know there exists solution Y which decays to zero as $|\xi| \to -\infty$ together with its derivatives. Integrating (2.4) yields

$$\lambda \int_{-\infty}^{+\infty} Y = 0$$

so we can define

$$T(x) = \int_{-\infty}^{x} Y(w)dw$$

obviously $T \to 0$ as $|\xi| \to \infty$ and the derivatives of $T \to 0$ as $|\xi| \to \infty$, substituting T into (2.4), we observe T satisfies

$$\lambda T = (-\beta\partial_\xi^2 - r\partial_\xi + c - f'(\phi))\partial_\xi T \tag{2.14}$$

With the integration by parts and the asymptotic propositions of T,computing

$$0 < Re\lambda \int_{-\infty}^{+\infty} T\bar{T}d\xi = Re\int_{-\infty}^{+\infty}(-\beta\partial_\xi^2 - r\partial_\xi + c - f'(\phi))\partial_\xi T\bar{T}d\xi$$

$$= r\int_{-\infty}^{+\infty} |T^{'}|^2 d\xi + \frac{1}{2}\int_{-\infty}^{+\infty} |T|^2 f''(\phi)\phi' d\xi$$

because of the proposition of ϕ and $f''(\phi) = 1 + 2\phi > 0, \phi' \leq 0$,we know the right hand of the above inequality is negative,however the left hand is positive,This contradiction establishes the results.Then ii) holds.

we compute $\frac{dD(\lambda)}{d\lambda}\Big|_{\lambda=0}$ by using the (1.34) and (1.35)of [11], for $\lambda = 0, Y^+ = \zeta_1^+$ is unique solution of equation (2.5), that is

$$0 = \partial_\xi(-\beta\partial_\xi^2 - r\partial_\xi + c - f'(\phi))Y^+ \tag{2.15}$$

and $Z^- = \eta_m^-$ is unique solution of equation (2.10), that is

$$0 = (-\beta\partial_\xi^2 + r\partial_\xi + c - f'(\phi))\partial_\xi Z^- \tag{2.16}$$

Y^+ satisfy $Y^+e^{-\mu_1^+\xi} \to v_1$ (as $\xi \to +\infty$) and Z^- satisfy $Z^-e^{\mu_1^-\xi} \to w_m$ (as $\xi \to -\infty$); where $v_1 w_m \hbar'_-(\mu_1^-) = 1$.

By (2.7)-(2.9),we may choose $v_1 = 1, w_m = \frac{1}{\hbar_-(0)}$,in addition to (2.11)-(2.13) we find $(\phi, \phi') \to e^{\mu_1^+\xi}(1, \mu_1^+)$ as $\xi \to +\infty$; by the uniqueness of solution we may choose $Y^+ = (\mu_1^+)^{-1}\phi'$;very similarly, we find $(\phi, \phi') \to \frac{(-\frac{\sqrt{9+48c}-3-16c}{8\beta}, \frac{r}{\beta})}{\hbar_-(0)}$ as $\xi \to -\infty$ and choose $Z^- = (-\frac{\sqrt{9+48c}-3-16c}{8\beta})^{-1}\phi$, in virtue of (1.35)of [11], we compute

$$\frac{dD(\lambda)}{d\lambda}\Big|_{\lambda=0} = (-\frac{\sqrt{9+48c}-3-16c}{8\beta}\mu_1^+)^{-1}\int_{-\infty}^{+\infty}\phi\phi' d\xi = (\frac{\sqrt{9+48c}-3-16c}{8\beta})^{-1}\frac{u_0^2}{2(\mu_1^+)} > 0$$

so iii) holds.the proof is little different from that given by Pego R.[11].Now we get the main lemma.

Lemma 1 Let L be given by (2.2),$L : BC \to BC$.Then
(1) $\sigma(L) \setminus \{0\}$ lies entirely in the left half plane.

(2) 0 is a simple eigenvalue.

Proof:by proposition 6 and 7,we know easily (2) holds; by proposition 3,4,6 and 7,we know easily (1) holds;

Therefore we have no difficulties to get the main results.

Theorem 1 If $r^2 < \frac{8\beta c\sqrt{3+16c}}{\sqrt{3}+\sqrt{3+16c}}$, $r < 0$,the oscillatory wave $\phi(\xi)$ is linear stable.

The proof of this theorem is obvious by lemma 1.

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