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# Robust Stability for Descriptor Systems with Time-Varying Delay

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**Abstract**

In this paper, we consider the stability and robust stability for descriptor systems with time-varying delay. First, we attempt to obtain delay-dependent stability conditions via linear matrix inequalities (LMIs). In the derivation of delay-dependent stability conditions, we define an appropriate Lyapunov-Krasovskii functional and use a free weighting matrix method and Jensen's inequality, which are known to give less conservative stability conditions. In addition to such stability conditions, we obtain a robust stability condition for uncertain descriptor system with time-varying delay. Finally, we give some examples to give the advantage of our conditions.

**Mathematics Subject Classification:** 93E20

**Keywords:** Delay systems, discrete-time systems, uncertain systems, linear matrix inequality

# 1 Introduction

A descriptor system describes a natural representation for physical systems. In general, the descriptor representation consists of differential and algebraic equations, and hence it is a generalized representation of the state-space system. In fact, descriptor systems can be found in electrical circuits, moving robots and many other practical systems which are modeled with additional algebraic constraints. The descriptor system is also referred to as singular system, implicit system, generalized state-space system, differential-algebraic system, or semistate system. System analysis and control design of descriptor systems have been extensively investigated in the past years due to their potential representation([4], [5], [6], [8], [11]).

An important characteristic of descriptor systems is the possible impulse modes, which are harmful to physical systems and are undesirable in system control. In [4], [16], such descriptor system behaviors are described and notion of regularity, non-impulse, stability and stabilization are given. In [1] and [9], quadratic stability for descriptor systems was considered. Robust stability for descriptor systems was analyzed in [11]. Its discrete-time system counterpart was investigated in [15].

When we consider the control design of practical systems, time-delay often appears in many situations. When a time-delay is small, it can be ignored. If it is large, however, it may cause instability in the system. In general, the dynamic behavior of continuous-time descriptor systems with delays is more complicated than that of system without any time-delay because the continuous time-delay system is infinite dimensional. To overcome such a difficulty, Lyapunov-Krasovskii functional approach has recently been taken to the stability of time-delay systems. This approach gives sufficient stability conditions, and useful techniques to reduce the conservatism in the stability conditions have recently been proposed. Main results in this approach can be classified in two types: (i) delay-independent methods and (ii) delay-dependent methods. The delay-independent methods do not consider the size of the delay while the delay-dependent methods take care of it. Generally speaking, the delay-independent methods can be applied for system with any large time-delay. The delay-dependent method is considered to be less conservative than the delay-independent one, especially for a small size time-delay. In [5], [12], [14], [18], stability and robust stability for descriptor time-delay systems were considered. $H_\infty$ control for descriptor time-delay systems was studied in [6], [17]. The results have been extended to a class of discrete-time descriptor delay systems in [2], [3], [10] where discrete-time version of Lyapunov functional approach was taken to analyze the problems. In most papers mentioned above, time-delay is assumed to be constant. However, actual time delays in practical systems are usually time-varying and unknown. Hence, the results for constant delay systems cannot be implemented, and time-varying delay case is required in many practical situations.

In this paper, we consider the stability and robust stability for continuous-time descriptor systems with time-varying delay. Time-delay is assumed to be unknown but its upper bound and derivative are assumed to be known. We attempt to obtain stability conditions for such descriptor time-delay systems. We use free weighting method to obtain less conservative stability conditions. We also obtain a robust stability condition for an uncertain descriptor time-delay system. Our conditions are given in terms of linear matrix inequalities(LMIs). Finally, we give some numerical examples to illustrate our results and to show their effectiveness.

## 2 Descriptor System and Preliminary Results

In this section, we describe a class of linear descriptor systems with time-delay under consideration. We also give some definitions and useful lemmas for descriptor time-delay systems. Consider the following system:

$$\begin{aligned} \tilde{E}\dot{x}(t) &= \tilde{A}\tilde{x}(t) + \tilde{A}_d\tilde{x}(t-h(t)), \\ \tilde{x}(t) &= \tilde{\phi}(t),\ t \in [-h_M,\ 0] \end{aligned} \tag{1}$$

where $x(t) \in \Re^n$ is the state. $\phi(t)$ is some given initial continuous function. $\tilde{E},\ \tilde{A}$ and $\tilde{A}_d$ are constant system matrices with appropriate dimensions. $\tilde{E}$ has rank $\tilde{E} = n_1 \leq n$. $h(t)$ is assumed to be unknown but known to be a time-varying delay, which satisfies $0 < h(t) \leq h_M$ where $h_M$ is a known constant and its derivative satisfies $0 \leq \dot{h}(t) \leq d < 1$ where $d$ is a known constant.

It is known([4]) that there exist invertible matrices $M$ and $N$ such that

$$M\tilde{E}N = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} := E. \tag{2}$$

Then, by the transformation $x = N^{-1}\tilde{x}$ the system (1) can be described by the following system:

$$\begin{aligned} Ex(t) &= Ax(t) + A_dx(t-h(t)), \\ x(t) &= \phi(t),\ t \in [-h_M,\ 0] \end{aligned} \tag{3}$$

where $x = [x_1^T\ x_2^T]^T,\ x_1 \in \Re^{n_1},\ x_2 \in \Re^{n_2}$ and $n_1 + n_2 = n$. $E$ is as in (2) and

$$\begin{aligned} A &= M\tilde{A}N := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\ A_d &= M\tilde{A}_dN := \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix} \end{aligned}$$

where $A_{11},\ A_{d11} \in \Re^{n_1 \times n_1},\ A_{22},\ A_{d22} \in \Re^{n_2 \times n_2}$ and other matrices are of appropriate dimensions.

**Definition 2.1** *([4]) (i) The pair $(E,\ A)$ is said to be regular if det$(sE - A)$ is not identically zero.*
*(ii) The pair $(E,\ A)$ is said to be impulse-free if deg(det$(sE - A)) =$ rank$E$ is not identically zero.*

The descriptor time-delay system (3) may have an impulsive solution, but the regularity and non-impulse of $(E,\ A)$ guarantee the existence and uniqueness of impulse-free solution to (3) on $[0,\ \infty)$([14]).

**Definition 2.2** *([5], [14]) The descriptor time-delay system (3) is said to be regular and impulse-free if the pair $(E,\ A)$ is regular and impulse-free. The descriptor time-delay system (3) is said to be asymptotically stable if for any $\varepsilon > 0$ there exists a scalar $\delta(\varepsilon) > 0$ such that for any compatible initial condition $\phi(t)$ with $\sup_{-h\leq t\leq 0}||\phi(t)|| < \delta(\varepsilon)$ the solution $x(t)$ of (3) satisfies $||x(t)|| < \varepsilon$ for $t \geq 0$ and $\lim_{t\to 0} x(t) = 0$. The descriptor time-delay system (3) is said to be admissible if it is regular, impulse-free and asymptotically stable.*

**Lemma 2.3** *([4]) The descriptor system*

$$E\dot{x}(t) = Ax(t),$$

*where*

$$(E,\ A) = \left( \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right),$$

*is regular and impulse-free if and only if $A_{22}$ is invertible.*

**Lemma 2.4** *([5]) If a functional $V : C_n[-h, 0] \longrightarrow \Re$ is continuous and $x(t, \phi)$ is a solution to (3), we define*

$$\dot{V}(\phi) = \lim_{\tau\to 0^+} \sup \frac{1}{\tau}(V(x(t+\tau, \phi) - V(\phi))).$$

*Denote the system parameters of (3) as*

$$(E,\ A,\ A_d) = \left( \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix} \right).$$

*Assume that the descriptor system (3) is regular and impulse-free, $A_{22}$ is invertible and $\rho(A_{22}^{-1}A_{d22}) < 1$. Then, the system (3) is asymptotically stable if there exist positive numbers $\alpha,\ \mu,\ \nu$ and a continuous functional $V : C_n[-h, 0] \longrightarrow \Re$ such that*

$$\mu||\phi_1(0)||^2 \leq V(\phi) \leq \nu||\phi||^2,$$
$$\dot{V}(x_t) \leq -\alpha||x_t||^2$$

*where $x_t = x(t+\theta)$ with $\theta \in [-h_M,\ 0]$ and $\phi = [\phi_1^T,\ \phi_2^T]^T$ with $\phi_1 \in \Re^{n_1}$.*

**Lemma 2.5** *([7]) For any matrix $M > 0$, scalar $\gamma > 0$ and vector function $\omega : [0,\ \gamma] \to \Re^n$ such that the integrations concerned are well defined, the following inequality holds*

$$\left(\int_0^t \omega(s)ds\right)^T M \left(\int_0^t \omega(s)ds\right) \leq \gamma \int_0^t \omega(s)M\omega(s)ds.$$

**Lemma 2.6** *([13]) Given matrices $Q = Q^T,\ H,\ E$ and $R = R^T > 0$ with appropriate dimensions*

$$Q + HF(t)E + E^T F^T(t) H^T < 0$$

*for all $F(t)$ satisfying $F^T(t)F(t) \le R$ if and only if there exists a scalar $\varepsilon > 0$ such that*

$$Q + \frac{1}{\varepsilon} HH^T + \varepsilon E^T R E < 0.$$

# 3 Delay-Dependent Stability

This section investigates the admissibility of the descriptor time-delay system (3). The following theorem gives a delay-dependent admissibility condition for the descriptor time-delay system (3).

**Theorem 3.1** *The descriptor time-delay system (3) is admissible if there exist matrices*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix},\ P_{11} > 0,\ Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} > 0,\ R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} > 0,$$
$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} > 0,\ Y = [\, Y_1 \quad 0\,],\ W = [\, W_1 \quad 0\,],\ U = [\, U_1 \quad 0\,]$$

*such that*

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & U^T & -h_M Y_1 \\ * & \Lambda_{22} & -U^T + \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \end{bmatrix} & -h_M W_1 \\ * & * & -R - \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \end{bmatrix} & -h_M U_1 \\ * & * & * & -h_M Z_{11} \end{bmatrix} < 0 \tag{4}$$

*where*

$$\begin{aligned} \Lambda_{11} &= A^T P^T + PA + Y + Y^T + Q + R + h_M A^T Z A, \\ \Lambda_{12} &= PA_d - Y + W^T + h_M A^T Z A_d, \\ \Lambda_{22} &= -(1-d)Q - W - W^T + h_M A_d^T Z A_d - \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{5}$$

**Proof:** We first show the regularity and non-impulse of the descriptor system (3). Suppose a condition (4) holds. Considering $(2,2)$-block of $\Lambda_{11}$, we have

$$P_{22}A_{22} + A_{22}^T P_{22}^T < 0$$

since $Q_{22} > 0$ and $R_{22} > 0$. This implies that $A_{22}$ is invertible. Then, by Definition 2.2 and Lemma 2.3, we conclude that the system (3) is regular and impulse-free.

Next, we prove the asymptotic stability of the descriptor system (3). If a condition (4) holds, we have

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ * & \Lambda_{22} \end{bmatrix} < 0.$$

Pre- and post-multiplying the above inequality by

$$\begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and its transpose, respectively, we obtain

$$\begin{bmatrix} A_{22}P_{22} + P_{22}^T A_{22}^T + Q_{22} + R_{22} & P_{22}A_{d22} \\ * & -(1-d)Q_{22} \end{bmatrix} < 0,$$

which implies that

$$\begin{bmatrix} A_{22}^T P_{22} + P_{22}^T A_{22} + Q_{22} & P_{22}^T A_{d22} \\ * & -Q_{22} \end{bmatrix} < 0.$$

It follows from $(1,1)$-block that $A_{22}$ is invertible. Then, pre- and post-multiplying the above inequality by

$$\begin{bmatrix} -A_{d22}^T A_{22}^{-T} & I \end{bmatrix}$$

and its transpose, respectively, we get

$$(A_{22}^{-1}A_{d22})^T Q_{22}(A_{22}^{-1}A_{d22}) - Q_{22} < 0.$$

This ensures that

$$\rho(A_{22}^{-1}A_{d22}) < 1.$$

Now, let us choose a delay-dependent Lyapunov-Krasovskii functional as

$$V(x_t) = V_1(x) + V_2(x_t) + V_3(x_t)$$

where

$$\begin{aligned} V_1(x) &= h_M x^T(t)PEx(t), \\ V_2(x_t) &= h_M \int_{t-h(t)}^{t} x^T(s)Qx(s)ds + h_M \int_{t-h_M}^{t} x^T(s)Rx(s)ds, \\ V_3(x_t) &= h_M \int_{-h_M}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)E^T ZE\dot{x}(s)dsd\theta, \end{aligned}$$

and $PE = E^T P^T \geq 0$ with $0 < P_{11} \in \Re^{n_1 \times n_1}$, $x_t = x(t+\theta)$, $\theta \in [-h_M,\ 0]$, and $Q > 0,\ R > 0,\ Z > 0$ to be determined. Differentiating $V_i(x_t),\ i = 1, 2, 3$

with respect to $t$, we get

$$\begin{aligned}
\dot{V}_1(x_t) = {} & h_M x^T(t)(A^T P^T + PA)x(t) + 2h_M x^T(t)PA_d x(t-\tau(t)),\\
\dot{V}_2(x_t) = {} & h_M x^T(t)(Q+R)x(t) - h_M(1-d)x^T(t-h(t))Qx(t-h(t))\\
& -h_M x^T(t-h_M)Rx(t-h_M),\\
\dot{V}_3(x_t) = {} & h_M^2 \dot{x}^T(t)E^T ZE\dot{x}(t) - h_M \int_{t-h_M}^{t} \dot{x}^T(s)E^T ZE\dot{x}(s)ds\\
= {} & h_M^2 (Ax(t) + A_d x(t-h(t)))^T Z(Ax(t) + A_d x(t-h(t)))\\
& -h_M \int_{t-h_M}^{t} \dot{x}_1^T(s)Z_{11}\dot{x}_1(s)ds.
\end{aligned}$$

Hence, adding the following zero quantity to $\dot{V}(x_t)$,

$$\begin{aligned}
& 2h_M[x^T(t)Y_1 + x^T(t-h(t))W_1 + x^T(t-h_M)U_1]\\
& \qquad \times[x_1(t) - x_1(t-t(h)) - \int_{t-h(t)}^{t} \dot{x}_1(s)ds] = 0,
\end{aligned}$$

we obtain

$$\begin{aligned}
\dot{V}(x_t) = {} & \frac{h_M}{h(t)}\int_{t-h(t)}^{t} \xi^T(t,s)\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & U^T & -h(t)Y_1\\ * & \Lambda_{22} + \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0\\ 0 & 0\end{bmatrix} & -U^T & -h(t)W_1\\ * & * & -R & -h(t)U_1\\ * & * & * & -h(t)Z_{11}\end{bmatrix}\xi(t,s)ds\\
& -h_M \int_{t-h_M}^{t-h(t)} \dot{x}_1^T(s)Z_{11}\dot{x}_{11}(s)ds
\end{aligned}$$

where $\xi(t,s) = [x^T(t) \;\; x^T(t-h(t)) \;\; x^T(t-h_M) \;\; \dot{x}^T(s)]^T$. Using Lemma 2.5, we get

$$\begin{aligned}
& -h_M \int_{t-h_M}^{t-h(t)} \dot{x}_1^T(s)Z_{11}\dot{x}_1(s)ds\\
\leq {} & -(h_M - h(t))\int_{t-h_M}^{t-h(t)} \dot{x}_1^T(s)Z_{11}\dot{x}_1(s)ds\\
\leq {} & -\left(\int_{t-h_M}^{t-h(t)} \dot{x}_1^T(s)ds\right) Z_{11} \left(\int_{t-h_M}^{t-h(t)} \dot{x}_1(s)ds\right)\\
= {} & -(x_1^T(t-h(t)) - x_1^T(t-h_M))Z_{11}(x_1(t-h(t)) - x_1(t-h_M)).
\end{aligned}$$

Taking this into account in $\dot{V}(x_t)$, we finally obtain

$$\dot{V}(x_t) \leq \frac{h_M}{h(t)}\int_{t-h(t)}^{t} \xi^T(t,s)\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & U^T & -h(t)Y_1\\ * & \Lambda_{22} & -U^T + \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0\\ 0 & 0\end{bmatrix} & -h(t)W_1\\ * & * & -R - \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0\\ 0 & 0\end{bmatrix} & -h(t)U_1\\ * & * & * & -h(t)Z_{11}\end{bmatrix}\xi(t,s)ds. \tag{6}$$

If a condition (4) is satisfied, by Schur complement formula, we get

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & U^T \\ * & \Lambda_{22} & -U^T + \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \end{bmatrix} \\ * & * & -R - \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} + h(t)\begin{bmatrix} Y_1 \\ W_1 \\ U_1 \end{bmatrix} Z_{11} \begin{bmatrix} Y_1^T \\ W_1^T \\ U_1^T \end{bmatrix}^T$$
$$\leq \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & U^T \\ * & \Lambda_{22} & -U^T + \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \end{bmatrix} \\ * & * & -R - \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} + h_M\begin{bmatrix} Y_1 \\ W_1 \\ U_1 \end{bmatrix} Z_{11} \begin{bmatrix} Y_1^T \\ W_1^T \\ U_1^T \end{bmatrix}^T < 0.$$

The most right-hand-side of the above inequality is negative definite, and it follows from (6) that

$$\dot{V}(x_t) < -\alpha||x_t||.$$

By lemma 2.4, we conclude that the system (3) is asymptotically stable. This is the end of proof.

**Theorem 3.2** *The descriptor time-delay system (3) is admissible if there exist matrices L,*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0,\ Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} > 0,\ R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} > 0,$$
$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} > 0,\ Y = [\,Y_1 \quad 0\,],\ W = [\,W_1 \quad 0\,],\ U = [\,U_1 \quad 0\,]$$

*such that*

$$\begin{bmatrix} \tilde{\Lambda}_{11} & \tilde{\Lambda}_{12} & U^T & -h_M Y_1 \\ * & \tilde{\Lambda}_{22} & -U^T + \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \end{bmatrix} & -h_M W_1 \\ * & * & -R - \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \end{bmatrix} & -h_M U_1 \\ * & * & * & -h_M Z_{11} \end{bmatrix} < 0$$

*where $S \in \Re^{n_2 \times n}$ is any matrix with full column rank and satisfies $SE = 0$ and*

$$\tilde{\Lambda}_{11} = A^T(E^TP + LS)^T + (E^TP + LS)A + Y + Y^T + Q + R + h_M A^T Z A,$$
$$\tilde{\Lambda}_{12} = (E^TP + LS)A_d - Y + W^T + h_M A^T Z A_d,$$
$$\tilde{\Lambda}_{22} = -(1-d)Q - W - W^T + h_M A_d^T Z A_d - \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

**Proof:** Let $\mathcal{P} = E^TP + LS$. Then, we have

$$\begin{aligned} \mathcal{P}E &= (E^TP + LS)E \\ &= E^TPE + LSE \\ &= E^TPE \\ &= E^T(E^TP + LS)^T \\ &= E^T\mathcal{P}^T. \end{aligned}$$

Therefore, this $\mathcal{P}$ satisfies a necessary condition. Replacing $P$ in (4) by $E^T P + LS$, we obtain the desired result.

The time-invariant delay case is similarly obtained by assuming $x(t - h(t)) = x(t - h_M)$ and letting $R = 0$ and $U_1 = 0$. The result coincides with that of [18].

**Corollary 3.3** *For a constant delay with $d = 0$, the descriptor time-delay system (3) is admissible if there exist matrices*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix},\ P_{11} > 0,\ Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} > 0,\ Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} > 0,$$
$$Y = [\, Y_1 \quad 0 \,],\ W = [\, W_1 \quad 0 \,],$$

*such that*

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & -h_M Y_1 \\ * & \hat{\Lambda}_{22} & -h_M W_1 \\ * & * & -h_M Z_{11} \end{bmatrix} < 0$$

*where $\Lambda_{12}$ is given as in (5), and*

$$\tilde{\Lambda}_{11} = A^T P^T + PA + Y + Y^T + Q + h_M A^T Z A,$$
$$\tilde{\Lambda}_{22} = -(1-d)Q - W - W^T + h_M A_d^T Z A_d.$$

# 4 Robust Stability

We extend the admissibility result to the robust admissibility for the following uncertain descriptor time-delay system.

$$E\dot{x}(t) \quad = \quad (A + \Delta A)x(t) + (A_d + \Delta A_d)x(t - h(t)). \tag{7}$$

The time varying uncertainties $\Delta A$ and $\Delta A_d$ are of the form

$$[\, \Delta A \quad \Delta A_d \,] = HF(t)\, [\, E_1 \quad E_d \,]$$

where $F(t) \in \Re^{l \times j}$ is an unknown time varying matrix satisfying $F^T(t)F(t) \leq I$ and $H,\ E_1$ and $E_d$ are known constant matrices of appropriate dimensions.

**Definition 4.1** *The system (7) is said to be robustly stable if it is asymptotically stable for all admissible uncertainties $\Delta A$ and $\Delta A_d$. The system (7) is said to be robustly admissible if it is regular, impulse-free and robustly stable.*

A robust admissibility condition is given in the following theorem.

**Theorem 4.2** *The descriptor time-delay system (7) is robustly admissible if there exist matrices*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix},\ P_{11} > 0,\ Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} > 0,\ R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} > 0,$$
$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} > 0,\ Y = [\, Y_1 \quad 0 \,],\ W = [\, W_1 \quad 0 \,],\ U = [\, U_1 \quad 0 \,]$$

*and a scalar $\varepsilon > 0$ such that*

$$\begin{bmatrix} \Theta + \varepsilon \bar{E}^T \bar{E} & \bar{H} \\ * & -\varepsilon I \end{bmatrix} < 0$$

*where*

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} & U^T & -h_M Y_1 & h_M A^T Z \\ * & \Theta_{22} & -U^T + \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \end{bmatrix} & -h_M W_1 & h_M A_d^T Z \\ * & * & -R - \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \end{bmatrix} & -h_M U_1 & 0 \\ * & * & * & -h_M Z_{11} & 0 \\ * & * & * & * & -h_M Z \end{bmatrix},$$
$$\Theta_{11} = A^T P^T + PA + Y + Y^T + Q + R,$$
$$\Theta_{12} = PA_d - Y + W^T,$$
$$\Theta_{22} = -(1-d)Q - W - W^T - \frac{1}{h_M}\begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \end{bmatrix},$$
$$\bar{H} = [\, H^T P^T \quad 0 \quad 0 \quad 0 \quad h_M H^T Z \,]^T,$$
$$\bar{E} = [\, E_1 \quad E_d \quad 0 \quad 0 \quad 0 \,].$$

**Proof:** Replacing $A$ and $A_d$ by $A + HF(t)E_1$ and $A_d + HF(t)E_d$, respectively, in a condition (4) and applying Schur complement, we have

$$\Theta + \bar{H}F(t)\bar{E} + \bar{E}^T F^T(t)\bar{H}^T < 0.$$

We can show by Lemma 2.6 that the above inequality holds if and only if there exists a scalar $\varepsilon > 0$ such that

$$\Theta + \frac{1}{\varepsilon}\bar{H}\bar{H}^T + \varepsilon \bar{E}^T \bar{E} < 0.$$

**Remark 4.3** *Similar to Theorem 3.2, Theorem 4.2 can be restated with more general $P$. In addition, the results in Theorems 3.1, 3.2, 4.2 and Corollary 3.3 can be easily extended to a class of descriptor systems with multiple delays by considering a Lyapunov-Krasovskii functional and free weighting matrices that correspond to multiple delays.*

# 5 Examples

We consider the admissibility of the descriptor system (3) with

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} 0.5 & 0 \\ -1 & -1 \end{bmatrix},\ A_d = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},\ H = 0,\ E_1 = E_d = 0.$$

For the constant delay with $d = 0$, Xu et al. [14] do not show the admissibility for any $h$. Zhong and Yang [17], Fridman [5], and Fridman and Shaked [6] guarantee the admissibility of the system for $h_M = 0.2485,\ h_M = 1.0000$ and $h_M = 1.1612$, respectively. Corollary 3.3 gives the admissibility for $h_M = 1.2011$. In the time-varying delay case with $d = 0.5$, Theorem 3.1 shows that the system is admissible for any time-delay $h(t)$ satisfying $h(t) \leq 1.0512$.

Next, we consider the robust admissibility of the descriptor system (7) with:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix},\ A_d = \begin{bmatrix} -1 & 0.1 \\ 1 & 0 \end{bmatrix},\ H = I,\ E_1 = E_d = 0.2I.$$

For the constant delay with $d = 0$, Zhu et al. [18] gives the maximum upper bound of $h_M = 0.7076$ for robust admissibility. On the other hand, Theorem 4.2 guarantees it for $h_M = 1.0182$. In the time-varying delay case with $d = 0.5$, Theorem 4.2 shows that the system is robustly admissible for any time-delay $h(t)$ satisfying $h(t) \leq 0.8547$. When $d = 0.8$, it is shown by Theorem 4.2 that the system is robustly admissible for any $h(t) \leq 0.3178$. All the examples for constant delay give the better results than others in the literature. Theorem 4.2 extends to robust stability.

# 6 Conclusions

The paper has discussed the admissibility of a descriptor system with time-varying delay. The time delay was assumed to be unknown and time-varying. First, we have given stability conditions for such a descriptor time-delay system. Our conditions are delay-dependent and less conservative than other results in the literature. Then, we have extended the results to robust admissibility for an uncertain descriptor time-delay system. Finally, we have given some numerical examples to give the advantages of our results.

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