Applied Mathematical Sciences, Vol. 4, 2010, no. 20, 991 - 996

General Solutions of a Class of Axisymmetric Deformation Problems Based on Nonlinear Elasticity[1]

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Abstract

This paper studies the general solutions of a system of partial differential equations, which can describe the finite deformation of an incompressible hyperelastic cylindrical tube composed of the known neo-Hookean material. We first formulate the mathematical model based on the theory of nonlinear elasticity, and then reduce the partial differential equations to a third order nonlinear ordinary differential equation by using the boundary conditions. Finally, we successfully obtain the general solutions of the problem.

Mathematics Subject Classiifcation: 34A05; 34B15; 74B20

Keywords: partial differential equation; general solution; incompressible hyperelastic material; cylindrical tube, pre-stressed state

1 Formulation of mathematical model

The purpose of this paper is to study the problem of axially symmetric deformation of an incompressible hyperelastic cylindrical tube in the pre-stressed state, namely, the tube is subjected to a uniform prescribed axial stretch $\lambda_3 > 0$ and the lateral surface of the tube is traction-free.

Let R, Θ, Z and r, θ, z be systems of cylindrical coordinates in the undeformed and the deformed state of the tube, respectively. Under the assumption of axially symmetric deformation, the deformed configuration is given by

$$r = r(R, Z), 0 < B < R \leq A; \theta = \Theta, z = \lambda_3 Z \tag{1}$$

[1]Supported by the National Natural Science Foundation of China (10872045) and the Science Foundation of China Postdoctor (20070421049).

where $r = r(R, Z)$ is an undetermined deformation function, A and B are radii of the inner and the outer surfaces of the undeformed tube. The deformation gradient is given by [1,2,3]

$$\mathbf{F} = r_R \mathbf{e}_r \otimes \mathbf{E}_R + r_Z \mathbf{e}_r \otimes \mathbf{E}_Z + (r/R)\mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \lambda_3 \mathbf{e}_z \otimes \mathbf{E}_Z \tag{2}$$

In this paper, $(\bullet)_x$ denotes the partial derivative of $\bullet$ with respect to the variable x.

For incompressible hyper-elastic materials, the incompressibility condition requires that $det\mathbf{F} = 1$, i.e., $\frac{\partial r}{\partial R}\frac{r\lambda_3}{R} = 1$, so we have

$$r(R, Z) = (R^2/\lambda_3 + f(Z))^{1/2} \tag{3}$$

where $f(Z)$ is an undetermined function with respect to Z. Obviously, the deformation function $r(R, Z)$ can be completely described by the form of $f(Z)$.

In the absence of body force, the differential equations which describe the finite deformation of the tube are given by [1,3]

$$(S_{rR})_R + (S_{rZ})_Z + R^{-1}(S_{rR} - S_{\theta\Theta}) = 0 \tag{4}$$

$$(S_{zR})_R + (S_{zZ})_Z + R^{-1}S_{zR} = 0 \tag{5}$$

where

$$S_{rR} = \mu r_R - \frac{\lambda_3 r}{R}p, \quad S_{rZ} = \mu r_Z, \quad S_{\theta\Theta} = \mu\frac{r}{R} - \frac{R}{r}p,$$
$$S_{zR} = \frac{r_Z r}{R}p, \quad S_{zZ} = \mu\lambda_3 - \frac{r_R r}{R}p \tag{6}$$

are the nonlinear components of the Piola-Kirchhoff stress tensor $\mathbf{S}$ corresponding to the known neo-Hookean material [1,2].

Since the lateral surface of the tube is traction-free, the boundary conditions are given by

$$S_{rR}(B) = S_{rR}(A) = 0, \quad S_{zR}(B) = S_{zR}(A) = 0 \tag{7}$$

In sum, the mathematical model that describes the finite deformation of an incompressible hyperelastic cylindrical tube composed of the known neo-Hookean material in the pre-stressed state is composed of Eqs.(3)∼(7).

Note. For some special deformation configurations of incompressible (or compressible) hyperelastic cylindrical tube, many significant investigations have been made, such as [4]∼[7].

2 Solutions

From Eq.(3), it is easy to show that the following expressions are valid

$$r_{RR}(R,Z)=\frac{1}{\lambda_3}\frac{f(Z)}{r^3},\quad r_{ZZ}(R,Z)=\frac{1}{2}f''(Z)r^{-1}-\frac{1}{4}f'^2(Z)r^{-3} \tag{8}$$

Moreover, substituting Eq.(6) into Eq.(4) and using the incompressibility condition $r_R r\lambda_3/R=1$, we obtain

$$p_R(R,Z)=\frac{\mu}{\lambda_3}\frac{R}{r}[r_{RR}+r_{ZZ}+R^{-2}(r_R R-r)] \tag{9}$$

Substituting Eq.(8) into Eq.(9) and integrating the resulting equation with respect to R from R to A, we have

$$\begin{aligned}p(R,Z)=&-\frac{\mu}{\lambda_3}\int_R^A[r_{RR}+r_{ZZ}+R^{-2}(r_R R-r)]\frac{R}{r}dR\\ =&-\frac{1}{2}\mu\left(-\frac{f(Z)}{\lambda_3 r_A^2}+\frac{1}{2}f''(Z)\ln r_A^2+\frac{1}{4}f'^2(Z)\frac{1}{r_A^2}+\frac{1}{\lambda_3}\ln r_A^2\right.\\ &\left.+\frac{f(Z)}{\lambda_3 r^2}-\frac{1}{2}f''(Z)\ln r^2-\frac{1}{4}f'^2(Z)\frac{1}{r^2}-\frac{1}{\lambda_3}\ln r^2\right)+\frac{\mu}{\lambda_3}\ln\frac{A}{R}\end{aligned} \tag{10}$$

where $r_A(A,Z)=(A^2/\lambda_3+f(Z))^{1/2}$.
Further, multiplying both sides of Eq.(5) by R, and then integrating the resulting equation from B to A, it yields

$$\int_B^A (S_{zZ})_Z R dR=0 \tag{11}$$

upon using the boundary conditions (7). Substituting Eq.$(6)_5$ into Eq.(11) leads to

$$\begin{aligned}&\left(\frac{f(Z)}{\lambda_3 r_A^4}f'(Z)+\frac{1}{2}f'''(Z)\ln r_A^2+f'(Z)f''(Z)\frac{1}{r_A^2}-\frac{1}{4}f'^3(Z)\frac{1}{r_A^4}\right)\frac{A^2-B^2}{\lambda_3}+\\ &\left(\frac{f(Z)}{\lambda_3 r_A^2}f'(Z)-\frac{1}{2}f'''(Z)(r_A^2\ln r_A^2-r_A^2)-f'(Z)f''(Z)\ln r_A^2-\frac{1}{4}f'^3(Z)\frac{1}{r_A^2}\right)-\\ &\left(\frac{f(Z)}{\lambda_3 r_B^2}f'(Z)-\frac{1}{2}f'''(Z)(r_B^2\ln r_B^2-r_B^2)-f'(Z)f''(Z)\ln r_B^2-\frac{1}{4}f'^3(Z)\frac{1}{r_B^2}\right)=0\end{aligned} \tag{12}$$

where $r_A(A,Z)=(A^2/\lambda_3+f(Z))^{1/2}$. Obviously, Eq.(12) is a third order nonlinear ordinary differential equation.

Integrating Eq.(12) with respect to Z, we have

$$\left(-\frac{f(Z)}{\lambda_3 r_A^2}+\frac{1}{2}f''(Z)\ln r_A^2+\frac{1}{4}f'^2(Z)\frac{1}{r_A^2}+\frac{1}{\lambda_3}\ln r_A^2\right)\frac{A^2-B^2}{\lambda_3}$$
$$-\frac{A^2}{\lambda_3^2}\ln r_A^2-\frac{1}{2}f''(Z)\left(r_A^2\ln r_A^2-r_A^2\right)-\frac{1}{4}f'^2(Z)\ln r_A^2+\frac{B^2}{\lambda_3^2}\ln r_B^2$$
$$+\frac{1}{2}f''(Z)\left(r_B^2\ln r_B^2-r_B^2\right)+\frac{1}{4}f'^2(Z)\ln r_B^2=D_1 \tag{13}$$

where D_1 is an integral constant.
Multiplying both sides of Eq.(13) by $f'(Z)$, and then integrating it with respect to Z, we have

$$\left(-\frac{2}{\lambda_3}f(Z)+\frac{2A^2}{\lambda_3^2}\ln r_A^2+\frac{1}{4}f'^2(Z)\ln r_A^2+\frac{1}{\lambda_3}f(Z)\ln r_A^2\right)\frac{A^2-B^2}{\lambda_3}$$
$$-\frac{A^2}{\lambda_3^2}f(Z)\ln r_A^2+\frac{A^2}{\lambda_3^2}f(Z)-\frac{A^4}{\lambda_3^3}\ln r_A^2-\frac{1}{4}f'^2(Z)\left(r_A^2\ln r_A^2-r_A^2\right)$$
$$+\frac{B^2}{\lambda_3^2}f(Z)\ln r_B^2-\frac{B^2}{\lambda_3^2}f(Z)+\frac{B^4}{\lambda_3^3}\ln r_B^2+\frac{1}{4}f'^2(Z)\left(r_B^2\ln r_B^2-r_B^2\right)$$
$$=D_1 f(Z)+D_2 \tag{14}$$

where D_2 is also an integral constant. The simplest form of Eq.(14) maybe written as

$$-\frac{A^2}{\lambda_3^2}f(Z)+\frac{B^2}{\lambda_3^2}f(Z)+\frac{A^4}{\lambda_3^3}\ln r_A^2-\frac{2A^2B^2}{\lambda_3^3}\ln r_A^2+\frac{B^4}{\lambda_3^3}\ln r_B^2$$
$$-\frac{B^2}{\lambda_3^2}f(Z)\ln r_A^2+\frac{B^2}{\lambda_3^2}f(Z)\ln r_B^2-D_1 f(Z)-D_2$$
$$=\frac{1}{4}f'^2(Z)\left(r_B^2\ln\frac{r_A^2}{r_B^2}+\frac{B^2-A^2}{\lambda_3}\right) \tag{15}$$

It is not difficult to show that the implicit solution of $f(Z)$ is given by

$$\int_{f(0)}^{f(Z)}\left(\frac{G(f(w),\lambda_3)}{H(f(w),\lambda_3)}\right)^{1/2}df(w)=Z \tag{16}$$

where

$$G(f(Z),\lambda_3)=\frac{1}{4}\left(r_B^2\ln\frac{r_A^2}{r_B^2}+\frac{B^2-A^2}{\lambda_3}\right),$$
$$H(f(Z),\lambda_3)=-\frac{A^2}{\lambda_3^2}f(Z)+\frac{B^2}{\lambda_3^2}f(Z)+\frac{A^4}{\lambda_3^3}\ln r_A^2-\frac{2A^2B^2}{\lambda_3^3}\ln r_A^2+\frac{B^4}{\lambda_3^3}\ln r_B^2$$

$$-\frac{B^2}{\lambda_3^2} f(Z) \ln r_A^2 + \frac{B^2}{\lambda_3^2} f(Z) \ln r_B^2 - D_1 f(Z) - D_2$$

Remark. Since the deformation is axially symmetric, the function $f(Z)$ must be a symmetric function with respect to $Z = 0$. If some boundary conditions are imposed, the integral constants in the general solution (16) will be determined, in other words, for the given material and structure parameters, the corresponding solution (3) will completely describes the finite deformation of the incompressible incompressible hyperelastic cylindrical tube.

3 Conclusions

In this paper, we formulate the finite deformation problem of an incompressible incompressible hyperelastic cylindrical tube in the pre-stressed state as a class of boundary value problems of a system of nonlinear partial differential equations. Moreover, we successfully obtain the general solutions of the problem.

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Received: October, 2009