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On a Study of a Plate Partially Recessed

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Abstract

In this paper we propose a study of the boundary value problem modeling in the context of linear elasticity, the deflection of a rectangular plate embedded on two opposite sides, the two other sides are free, and subjected to a given density of forces. The question that arises is that of the approximation of the solution of this problem. The basic idea is that if the solution is quite regular, then its rotational is solution of stokes problem, this result is well known in the case of Dirichlet problem for bilaplacien.

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1 Introduction

As with all equations of order four, the finite element approximation conform, if it has the advantage of giving a continuous approximation of the solution and its derivatives, however, requires a large amount of calculations. There are, of course, other approximation methods such as the mixed methods and finite element methods not conform. What we propose here is a method that a priority has the advantage of a conform method but needs a lower cost.If the solution is quite regular, then its rotational is solution of stokes problem.
In section 2 we recall the description of the problem. In section 3 we give the

variational formulation of problem and we study the regularity of the solution of this problem and in section 4 we show that the rotational of the solution is solution of a Stokes-type system, and we show that latter admits a variational formulation. In section 5 we show how the solution can be obtained from its rotational.The numerical results are obtained using Lagrange finite element [1], [7] for the rotational solution and for the solution. we compare the results obtained with those given by the conform Method using the Argyris finite element.

2 Description of the problem

We study the problem of deformation of a thin homogeneous isotropic plate. In its non-deformed state, the plate is considered as two-dimensional environment, occupies a region Ω the open square $]0,1[\times]0,1[$ of $\mathbb{R}^2$ of border $\Gamma=\partial\Omega$; $\Gamma=\Gamma_V\cup\Gamma_H$ where $\Gamma_V=\{0,1\}\times]0,1[=\Gamma_2\cup\Gamma_4$ and $\Gamma_H=]0,1[\times\{0,1\}=\Gamma_1\cup\Gamma_3$. It is assumed that the plate is subjected to a transverse load F, $u=u(x,y)$ deflection of the plate then verifies:

$$-\Delta^2 u = F \quad \text{in} \quad \Omega, \tag{1}$$

$$u=\frac{\partial u}{\partial \eta}=0 \quad \text{on} \quad \Gamma_V, \tag{2}$$

$$\Delta u-(1-\nu)\frac{\partial^2 u}{\partial \tau^2}=h \quad \text{on} \quad \Gamma_H, \tag{3}$$

$$(1-\nu)\frac{\partial^3 u}{\partial\eta\partial\tau^2}-\frac{\partial(\Delta u)}{\partial\eta}=g \quad \text{on} \quad \Gamma_H, \tag{4}$$

ν is the poisson coefficient.

3 Variational formulation of the problem

We introduce the functional space $V=\left\{v\in H^2(\Omega);\ v=\ \frac{\partial v}{\partial\eta}=0 \ \text{ on } \Gamma_V\right\}$
we consider:
The bilinear form $a(u,v)$ is set to $V\times V$ by:
$a(u,v)=\int_\Omega\,(\Delta u\ \Delta v+(1-\nu)(2\ \frac{\partial^2 u}{\partial x\partial y}\ \frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2}-\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2}))d\Omega$
The linear form $L(.)$ is set to V by:
$L(v)=\int_\Omega F.v\ d\Omega-\int_{\Gamma_H} g.v\ d\sigma+\int_{\Gamma_H} h.\frac{\partial v}{\partial\eta}\ d\sigma.$

Theorem 3.1 *For F,h and g given in $L^2(\Omega),H^{\frac{3}{2}}(\Gamma_H)$ and $H^{\frac{1}{2}}(\Gamma_H)$ respectively, the variational problem is:*

$$\begin{cases} Find\ u\ in\ V such\ that \\ a(u,v)=L(v),\forall v\in V\end{cases}$$

admits a unique solution.

Proof. The bilinear form $a(.,\ .)$ is continuous coercive on $V \times V$.
The linear form $L(.)$ is continuous on V.
It follows from the Lax Milgram theorem that the problem admits a unique variational solution u in V.
We will need farther regularity conditions on the solution u. which can show that u is in $H^4(\Omega)$

4 Transition to a system of Stokes type

We set $\overrightarrow{\varphi} = \overrightarrow{rot}u$ then for u in $H^4,(\Omega)$ $\overrightarrow{\varphi}$ belongs to $H^3(\Omega)$.
We take $\overrightarrow{f}$ in $H^1(\Omega)$ such that $rot\overrightarrow{f} = F$
Using the identity $\Delta^2 = -rot(\Delta\overrightarrow{rot})$
Equation (1) of the problem becomes:
$-rot(\Delta\ \overrightarrow{rot}u) = rot\overrightarrow{f}$ in Ω
$rot(-\Delta\ \overrightarrow{rot}u - \overrightarrow{f}) = 0$ in Ω
$(-\Delta\ \overrightarrow{rot}u - \overrightarrow{f}) \in L^2(\Omega)$ then there exists a function $(-p)$ of $H^1(\Omega)$ such that:
$-\Delta\ \overrightarrow{rot}u - \overrightarrow{f} = -\overrightarrow{\nabla}p$ in Ω that is $-\Delta\overrightarrow{\varphi} + \overrightarrow{\nabla}p = \overrightarrow{f}$
$\overrightarrow{\varphi} = \overrightarrow{rot}u$ then $div\ \overrightarrow{\varphi} = div(\overrightarrow{rot}u) = 0$ in Ω

Boundary condition on Γ_V :
$\overrightarrow{\varphi}/_{\Gamma_V} = \overrightarrow{rot}u/_{\Gamma_V} = (\overrightarrow{rot}u \cdot \overrightarrow{\eta})\overrightarrow{\eta} + (\overrightarrow{rot}u \cdot \overrightarrow{\tau})\overrightarrow{\tau} = \frac{\partial u}{\partial \tau}\overrightarrow{\eta} - \frac{\partial u}{\partial \eta}\overrightarrow{\tau} = 0$

Boundary condition on Γ_H :
Equation (3) of the problem gives:
$\frac{\partial \varphi_1}{\partial y} - \nu\frac{\partial \varphi_2}{\partial x} = h$ on Γ_H
Equation (4) of the problem gives:
$(\frac{\partial^2 \varphi_1}{\partial y^2} + (2-\nu)\frac{\partial^2 \varphi_1}{\partial x^2})\eta_y = g$ on Γ_H
Thus we obtain the system:

$$(\mathcal{P}_1)\begin{cases} -\Delta\overrightarrow{\varphi} + \overrightarrow{\nabla}p = \overrightarrow{f} & \text{in } \Omega \\ div\ \overrightarrow{\varphi} = 0 & \text{in } \Omega \\ \overrightarrow{\varphi} = 0 & \text{on } \Gamma_V \\ \frac{\partial \varphi_1}{\partial y} - \nu\frac{\partial \varphi_2}{\partial x} = h & \text{on } \Gamma_H \\ (\frac{\partial^2 \varphi_1}{\partial y^2} + (2-\nu)\frac{\partial^2 \varphi_1}{\partial x^2})\eta_y = g & \text{on } \Gamma_H \end{cases}$$

Variational formulation of the problem
We introduce the space V defined by:
$V = \left\{ \overrightarrow{\psi} \in H^1(\Omega)\ ,\ div\overrightarrow{\psi} = 0 \text{ in } \Omega\ ,\ \overrightarrow{\psi} = \overrightarrow{0} \text{ on } \Gamma_V \right\}$
The variational problem is:

Find $\overrightarrow{\varphi} \in V$ such that:
$a(\overrightarrow{\varphi}, \overrightarrow{\psi}) = l(\overrightarrow{\psi}), \qquad \forall \overrightarrow{\psi} \in V$
where the bilinear form $a(.,.)$ is defined on $V \times V$ by:
$a(\overrightarrow{\varphi}, \overrightarrow{\psi}) = \sum_{i=1}^{2} \int_\Omega \overrightarrow{\nabla}\varphi_i \overrightarrow{\nabla}\psi_i d\Omega - \nu \int_\Omega \left(\frac{\partial\varphi_1}{\partial x} \cdot \frac{\partial\psi_1}{\partial x} + \frac{\partial\varphi_2}{\partial y} \cdot \frac{\partial\psi 2}{\partial y} + \frac{\partial\varphi_1}{\partial y} \cdot \frac{\partial\psi_2}{\partial x} + \frac{\partial\varphi_2}{\partial x} \cdot \frac{\partial\psi_1}{\partial y} \right) d\Omega$
The form $l(\cdot)$ is defined on V by:
$l(\overrightarrow{\psi}) = \int_\Omega \overrightarrow{f}.\overrightarrow{\psi} d\Omega + \int_{\Gamma_H} h \ \psi_1 \eta_y d\sigma$

Theorem 4.1 *For all $\overrightarrow{f}$, h and g given in $H^1(\Omega)$, $H^{\frac{3}{2}}(\Gamma_H)$ and $H^{\frac{1}{2}}(\Gamma_H)$ respectively, the variational problem admits a unique solution in V.*

Proof. Simply check the hypotheses of the Lax Milgram theorem for $\overrightarrow{f}$ in $H^1(\Omega)$ and h in $H^{\frac{3}{2}}(\Gamma_H)$ then the application: $\overrightarrow{\psi} \longrightarrow l(\overrightarrow{\psi}) = \int_\Omega \overrightarrow{f}.\overrightarrow{\psi} d\Omega$ $+ \int_{\Gamma_H} h \ \psi_1 \eta_y d\sigma$ is a continuous linear form on V. The bilinear form $a(\cdot, \cdot)$ is continuous coercive on $V \times V$

5 Mixed formulation

To use an approximation of the problem $(\mathcal{P}_1)$ by mixed finite element Lagrange, we give its mixed variational formulation.
we introduce the spaces:
$X = \left\{ \overrightarrow{v} \in H^1(\Omega) \ , \ \ \overrightarrow{v} = \overrightarrow{0} \text{ on } \Gamma_V \right\}$
$M = \left\{ q \in L^2(\Omega) \ , \ \ \int_\Omega q d\Omega = 0 \ \right\} = L_0^2(\Omega)$
we Consider the bilinear form $a(\cdot, \cdot)$ defined on $M \times M$ by:
$a(\overrightarrow{\varphi}, \overrightarrow{\psi}) = \sum_{i=1}^{2} \int_\Omega \overrightarrow{\nabla}\varphi_i \overrightarrow{\nabla}\psi_i d\Omega - \nu \int_\Omega \left(\frac{\partial\varphi_1}{\partial x} \cdot \frac{\partial\psi_1}{\partial x} + \frac{\partial\varphi_2}{\partial y} \cdot \frac{\partial\psi 2}{\partial y} + \frac{\partial\varphi_1}{\partial y} \cdot \frac{\partial\psi_2}{\partial x} + \frac{\partial\varphi_2}{\partial x} \cdot \frac{\partial\psi_1}{\partial y} \right)$
The bilinear form $b(\cdot,\cdot)$ defined on $M \times M$ by:
$b(\overrightarrow{\varphi}, q) = -\int_\Omega q div(\overrightarrow{\varphi}) d\Omega$
The form $l(\cdot)$ is defined on X by:
$l(\overrightarrow{\psi}) = \int_\Omega \overrightarrow{f}.\overrightarrow{\psi} d\Omega + \int_{\Gamma_H} h \ \psi_1 \eta_y d\sigma$
The associated mixed variational problem is written as follows:

$$\left\{ \begin{array}{c} \text{find } (\overrightarrow{\varphi}, p) \in X \times M \ \text{ such as} \\ a(\overrightarrow{\varphi}, \overrightarrow{\psi}) + b(\overrightarrow{\varphi}, p) = l(\overrightarrow{\psi}) \ , \forall \overrightarrow{\psi} \in X \\ b(\overrightarrow{\varphi}, q) = 0 \ , \ \forall q \in M \end{array} \right.$$

Theorem 5.1 *The mixed variational problem admits a unique solution $(\overrightarrow{\varphi}, p) \in X \times M$*

Proof. The bilinear form $a(\cdot, \cdot)$ is continuous coercive on $X \times X$.. The bilinear form $b(\cdot, \cdot)$ is continuous on $X \times M$. For $\overrightarrow{f}$ and h given in $L^2(\Omega)$ and $H^{\frac{3}{2}}(\Gamma_H)$ respectively, the form $l(\cdot)$ is continuous on X. The condition of compatibility between the spaces X and M is verified, i.e. there exists a constant $\beta \succ 0$, such that $\sup\limits_{v\in X}\frac{|b(v,\ q)|}{\|v\|_X} \geq \beta \|q\|_M$ Then the mixed variational problem admits a unique solution $(\overrightarrow{\varphi}, p) \in X \times M$

Determination of the plate deflection u **.**

Problems verified by u:

After determining $\overrightarrow{\varphi} = \overrightarrow{rot}\, u$, solution of the problem $(\mathcal{P}_1)$ we determine the deflection u of the plate by integrating one derivative of u, or in solving boundary value problems of order two.

Integration:

we have the condition $u = 0$ on Γ_V then integrating the derivative of u with x.

Let $(x_0, y_0) \in \{0\} \times]0,\ 1[\subset \Gamma_V$ then

$u\,(x,\ y) - u\,(x_0,\ y_0) = \int_{x_0}^{x} \frac{\partial u}{\partial x}(s,\ y)ds = -\int_{x_0}^{x} \varphi_2(s,\ y)ds$

so $u\,(x,\ y) = -\int_{x_0}^{x} \varphi_2(s,\ y)ds$

Second-order problem:

we have: $-\Delta\ u = -(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = -(-\frac{\partial \varphi_2}{\partial x} + \frac{\partial \varphi_1}{\partial y}) = rot\overrightarrow{\varphi}$ in Ω

Moreover $u = 0$ on Γ_V.

It remains to find the boundary conditions on Γ_H.

Two cases occur:

$1°$ case the Dirichlet Problem:

We have $\overrightarrow{\varphi} = \overrightarrow{rot}\, u$ in Ω then $-\frac{\partial u}{\partial x} = \varphi_2$ in Ω therefore in particular on Γ_H i.e. $-\frac{\partial u}{\partial x} = \varphi_2$ on Γ_H which can be integrated according to the variable x.

we obtain on Γ_1 :

$-\frac{\partial u}{\partial x}(x, 0) = \varphi_2(x, 0)$

$-\int_0^x \frac{\partial u}{\partial x}(t, 0)dt = \int_0^x \varphi_2(t, 0)dt$ for $0 \leq x \leq 1$

$-(u(x, 0) - u(0, 0)) = \int_0^x \varphi_2(t, 0)dt$

We know that $u \in H^4(\Omega)$ therefore $u \in C^0(\overline{\Omega})$ and the values $u(x, 0)$ and $u(0, 0)$ make sense. Moreover $u = 0$ on Γ_V then $u(0, 0) = 0$

where: $u(x, 0) = -\int_0^x \varphi_2(t, 0)dt$

We note $u\,|_{\Gamma_1} = h_1$

In the same way on the Γ_3, we get:

$u(x, 1) = -\int_0^x \varphi_2(t, 1)dt$ We note $u\,|_{\Gamma_3} = h_3$

The conditions of compatibility are necessarily verified. Hence the problem:

$$(\mathcal{P}_2)\begin{cases} -\Delta u = rot\overrightarrow{\varphi} & in\ \Omega \\ u = 0 & on\ \Gamma_V \\ u = h_1 & on\ \Gamma_1 \\ u = h_3 & on\ \Gamma_3 \end{cases}$$

we note by $(\mathcal{F})$ the function defined by:

$$\mathcal{F} = \begin{cases} h_1 = -\int_0^x \varphi_2(t,0)dt & on\ \Gamma_1 \\ h_2 = 0 & on\ \Gamma_2 \\ h_3 = -\int_0^x \varphi_2(t,1)dt & on\ \Gamma_3 \\ h_4 = 0 & on\ \Gamma_4 \end{cases}$$

We can then write $u\,|_{\Gamma_i} = h_i\ ,\ \ i = 1, ..., 4$
There is an increase $\phi \in H^4(\Omega)$ as $\phi\,|_{\Gamma_i} = h_i\ ,\ \ i = 1, ..., 4$
We set $U = u - \phi$ and we have the following homogeneous problem:

$$(\mathcal{P}_3)\begin{cases} -\Delta U = rot\overrightarrow{\varphi}\ + \Delta\phi & in\ \Omega \\ U = 0 & on\ \Gamma \end{cases}$$

2° case mixed problem (Dirichlet-Neumann):
We have $\overrightarrow{\varphi} = \overrightarrow{rot}\,u$ in Ω
thus particular on Γ_H : $\frac{\partial u}{\partial y} = \varphi_1$
We keep the condition $u = 0$ on Γ_V
We get the problem:

$$(\mathcal{P}_4)\begin{cases} -\Delta u = rot\overrightarrow{\varphi} & in\ \Omega \\ u = 0 & on\ \Gamma_V \\ \frac{\partial u}{\partial \eta} = \varphi_1 \eta_y & on\ \Gamma_H \end{cases}$$

Variational formulation of these two problems:
For the first case we set $V = H_0^1(\Omega)$ which combines the variational problem:

$$\begin{cases} Find\ \ U \in H_0^1(\Omega) \text{ such that} \\ a(U,v) = l_1(v)\ ,\ \forall v \in H_0^1(\Omega) \end{cases}$$

where $a(.,.)$ is the bilinear continuous coercive on $H_0^1(\Omega) \times H_0^1(\Omega)$ defined by:
$a(U,v) = \int_\Omega \overrightarrow{\nabla} U\ \overrightarrow{\nabla} v\ d\Omega$
and $l_1(.)$is the continuous linear form on $H_0^1(\Omega)$ defined by:
$l_1(v) = \int_\Omega rot\overrightarrow{\varphi}\ .\ v\ d\Omega\ - \int_\Omega \overrightarrow{\nabla}\phi\ \overrightarrow{\nabla} v\ d\Omega$ all assumptions of Lax-Milgram are checked and it was the existence and uniqueness in $H_0^1(\Omega)$.
The same for the seconde where we set:
$W = \{v \in H^1(\Omega)\ ,\ v = 0\ \ sur\ \ \Gamma_V\ \}$
which combines the variational problem:

$$\begin{cases} Find\ \ u \in W \text{ such as} \\ a(u,v) = l_2(v)\ ,\ \forall v \in W \end{cases}$$

where $a(.,.)$ is coercive continuous bilinear form on $W \times W$ defined by:
$a(u,v) = \int_\Omega \overrightarrow{\nabla} u\ \overrightarrow{\nabla} v\ d\Omega$
and $l_2(.)$ is the continuous linear form on W defined by:
$l_2(v) = \int_\Omega rot\overrightarrow{\varphi}\ .\ v\ d\Omega\ + \int_{\Gamma_H} \phi_1\ v\ \eta_y d\sigma$
all assumptions of Lax-Milgram are checked and it was the existence and uniqueness in W.

6 Conclusion

We tested three related problems for several chosen deflections. We notice that we have the convergence of the solution calculated towards the exact solution, curves are close, we get a good estimate of the solution. The order of Stokes system is significantly less than the order of the bilaplacien system. The problem studied is particular as it concerns the problem in the open square domain. As further works we will study more general geometries. For example in a circular domain.

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