

A Note on the Sumudu Transforms and Differential Equations

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Abstract. In this work we introduce some relationship between Sumudu and Laplace transforms, further; for the comparison purpose, we apply both transforms to solve differential equations to see the differences and similarities. Finally, we provide some examples regarding to second order differential equations with non constant coefficients as special case.

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1. INTRODUCTION

In the literature there are numerous integral transforms and widely used in physics, astronomy as well as in engineering. The integral transform method is also an efficient method to solve the differential equations. In [9], the integral transform was applied to partial differential equations with non-homogenous forcing term and having singular variable data. Recently, Watugala introduced a new transform and named as Sumudu transform which is defined by the following formula

$$F(u) = S[f(t); u] = \frac{1}{u} \int_0^{\infty} e^{-\left(\frac{t}{u}\right)} f(t) dt, \quad u \in (-\tau_1, \tau_2)$$

and applied this new transform to the solution of ordinary differential equations and control engineering problems, see [11], [8]. In [2], some fundamental

properties of the Sumudu transform were established. In [6], this new transform was applied to the one-dimensional neutron transport equation. In fact one can easily show that there is a strong relationship between double Sumudu and double Laplace transforms see, [7]. Further in [5], the Sumudu transform was extended to the distributions and some of their properties were studied.

In this study, our purpose is to show the applicability of this interesting new transform and its efficiency in solving the linear ordinary differential equations with constant and non constant coefficients. In the next we compare two solutions by using the two different methods. The following definition was given in [4].

Definition 1. The space W of test functions of exponential decay is the space of complex valued functions $\phi(t)$ satisfying the following properties:

- (i) $\phi(t)$ is infinitely differentiable; i.e., $\phi(t) \in C^\infty(\mathbb{R}^n)$.
- (ii) $\phi(t)$ and its derivatives of all orders vanish at infinity faster than the reciprocal of the exponential of order $1/\omega$; that is

$$|e^{t/\omega} D^k \phi(t)| < M, \forall 1/\omega, k.$$

Then a function $f(t)$ is said to be of exponential growth if and only if $f(t)$ together with all its derivatives grow more slowly than the exponential function of order $1/\omega$; that is, there exists a real constant $1/\omega$ and M such that $|D^k \phi(t)| < M e^{t/\omega}$. A linear continuous functional over the space W of test functions is called a distribution of exponential growth and this dual space of W is denoted W' .

Let us find The Sumudu Transform of the function

$$f(t) = (t^\eta)_+ = H(t)t^\eta$$

where $\eta \neq -1, -2, -3, \dots$

Since $(t^\eta)_+ \in W'$, we have

$$S[f(t)] = \int_0^\infty (tv)^\eta e^{-t} dt = v^\eta \Gamma(\eta + 1).$$

Similarly, if we consider the regular distribution $f(t) = H(t) \ln(t)$ then the Sumudu transform with respect to t becomes

$$F(v) = S[f(t)] = \int_0^\infty e^{-t} \ln(vt) dt = \ln(v) - \gamma$$

where $H(t)$ is a Heaviside function and Laplace transform for the same function is given by

$$L[f(t)] = \int_0^{\infty} (t)^\eta e^{-st} dt = \frac{1}{s^{\eta+1}} \Gamma(\eta + 1).$$

We can use the above result to evaluate the Sumudu transform of the singular distribution $\text{pf} \left[\frac{H(t)}{t} \right]$. Because

$$\frac{\partial}{\partial t} [H(t) \ln(t)] = \text{pf} \left[\frac{H(t)}{t} \right].$$

Thus Sumudu transform of above equation is given by

$$S \left[\frac{d}{dt} [H(t) \ln(t)] \right] = \left[\text{pf} \left[\frac{H(t)}{t} \right] \right] = \frac{1}{v} [\ln(v) - \gamma] = \frac{1}{v} F(v),$$

where s, v are complex variables. For more information see [4]. The next theorem very useful in study of differential equations having non constant coefficient.

Theorem 1. *If Sumudu transform of the function $f(t)$ given by $S[f(t)] = F(u)$, then*

(I)

$$S[tf'(t)] = u^2 \frac{d}{du} \left[\frac{F(u) - f(0)}{u} \right] + u \left[\frac{F(u) - f(0)}{u} \right]$$

(II)

$$S[t^2 f''(t)] = u^2 \frac{d}{du} \left[\frac{F(u) - f(0) - f'(0)}{u^2} \right] + u \left[\frac{F(u) - f(0) - f'(0)}{u^2} \right]$$

(III)

$$\begin{aligned} S[t^2 f''(t)] &= u^4 \frac{d^2}{du^2} \left[\frac{F(u) - f(0) - f'(0)}{u^2} \right] + 4u^3 \frac{d}{du} \left[\frac{F(u) - f(0) - f'(0)}{u^2} \right] \\ &\quad + 2u^2 \left[\frac{F(u) - f(0) - f'(0)}{u^2} \right]. \end{aligned}$$

Proof. To prove (I) we use the following formula

$$S[tf(t)] = u^2 \frac{d}{du} F(u) + uF(u)$$

given by Asiru in [1]. We have

$$\begin{aligned} \frac{d}{du} \left[\frac{F(u) - f(0)}{u} \right] &= \frac{d}{du} \int_0^\infty \frac{1}{u} e^{-\frac{t}{u}} f'(t) dt = \int_0^\infty \frac{d}{du} \left[\frac{1}{u} e^{-\frac{t}{u}} \right] f'(t) dt \\ &= \frac{1}{u^3} \int_0^\infty e^{-\frac{t}{u}} t f'(t) dt - \frac{1}{u^2} \int_0^\infty e^{-\frac{t}{u}} f'(t) dt \\ &= \frac{1}{u^2} S [t f'(t)] - \frac{1}{u} S [f'(t)], \end{aligned}$$

then we have

$$S [t f'(t)] = u^2 \frac{d}{du} \left[\frac{F(u) - f(0)}{u} \right] + u \left[\frac{F(u) - f(0)}{u} \right].$$

The proof of (II) and (III) are similar to the proof of (I). ■

Now, we apply the above theorem to find Sumudu transform for Bessel function J_0 in the following equation

$$t J_0''(t) + J_0'(t) + t J_0(t) = 0 \quad \text{for } t > 0.$$

Let $f = J_0 H$. Then, except at $t = 0$, we have

$$(1.1) \quad t f''(t) + f'(t) + t f(t) = 0 \quad \text{for } t > 0.$$

By using Sumudu transform for equation (1.1) and the theorem (1), we have

$$(u^2 + 1) F'(u) + u F(u) = 0$$

Thus this a differential equation for unknown function F . Now it can be solve by writing it in the form of

$$\frac{d}{du} \left(\sqrt{u^2 + 1} F(u) \right) = 0,$$

to give

$$F(u) = \frac{C}{\sqrt{u^2 + 1}}$$

for some constant C . For the determination of C we refer to [10] with $\lambda = 0$ then we have $\lim_{t \rightarrow 0^+} [f(t)] = 1$ and therefore $\lim_{u \rightarrow \infty} u F(u) = 1$. Since

$$u F(u) = \frac{u C}{\sqrt{u^2 + 1}} \rightarrow 1 \text{ as } u \rightarrow \infty,$$

where we obtain $C = 1$. Thus Sumudu transform of Bessel function of first kind is given by

$$S [J_0(t) H(t)] = \frac{1}{\sqrt{u^2 + 1}}.$$

Laplace transform for the same function is given by

$$L [J_0(t)H(t)] = \frac{1}{\sqrt{s^2 + 1}}.$$

Now, if we let $f(t) = \frac{H(t)}{\sqrt{t}}$. Then we note that the generalized derivative of f is not a locally integrable function and sometimes we only consider the finite part of its derivative $-\frac{H(t)}{2t^{\frac{3}{2}}}$.

Sumudu transform of f given by

$$S [f(t)] = \frac{\sqrt{\pi u}}{u}.$$

Note also that the domain of $S [df]$ contains the interval $(0, \infty)$ and we have

$$S [df] (u) = \frac{1}{u} S [f(t)] (u) = \frac{\sqrt{\pi u}}{u^2}.$$

The following proposition was proved in [10].

Proposition 1. *Let f be n times differentiable on $(0, \infty)$ and let $f(t) = 0$ for $t < 0$. Suppose that $f^{(n)} \in L_{loc}$. Then $f^{(k)} \in L_{loc}$ for $0 \leq k \leq n - 1$, $dom(Sf) \subset dom(Sf^{(n)})$ and, for any polynomial P of degree n ,*

$$(1.2) \quad P(u)S(y)(u) = S(f)(u) + M_P(u)\varphi(y, n)$$

for $u \in dom(Sf)$.

In particular

$$(1.3) \quad (Sf^{(n)})(u) = \frac{1}{u^n}(Sf)(u) - \left(\frac{1}{u^n}, \frac{1}{u^{n-1}}, \dots, \frac{1}{u} \right) \varphi(f; n)$$

where $\varphi(f; n)$ is written as a column vector. For $n = 2$ we have

$$(1.4) \quad (Sf'')(u) = \frac{1}{u^2}(Sf)(u) - \frac{1}{u^2}f(0+) - \frac{1}{u}f'(0+).$$

In particular case if we consider

$$y(t) = \frac{1}{k} \sin(kt)$$

then clearly

$$y'' + k^2y = 0$$

and in the operator form we write

$$(D^2 + k^2) f = 0.$$

Since $\text{dom}(Sf)$ contains $(0, \infty)$ then on using the Eq(1.2) with $n = 2$ and $P(x) = x^2 + k^2$, for $u > 0$,

$$0 = \left(\frac{1}{u^2} + k^2 \right) S(f) - \begin{pmatrix} 1 & 1 \\ u & u^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since

$$\varphi(y, 2) = (f(0), f'(0)) = (0, 1).$$

Thus we can obtain the same result without using definitions or transforms table as

$$U(t) = S \left[\frac{1}{k} \sin(kt) H(t) \right] = \frac{ku}{k^2 u^2 + 1}$$

then the inverse given by

$$\frac{1}{k} \sin(kt) H(t) = S^{-1} \left[\frac{ku}{k^2 u^2 + 1} \right]$$

let $U = gH$, where $g = \frac{1}{k} \sin(kt)$ and H is Heaviside function

$$\begin{aligned} DU &= gDH + g'H = g\delta + g'H \\ &= g\delta(0) + g'H = g'H \end{aligned}$$

then

$$\begin{aligned} D^2U &= g'DH + g''H = g'\delta + g''H \\ g'(0)\delta + g''H &= \delta + g''H \end{aligned}$$

therefore

$$P(D)U = D^2U + k^2U = \delta + (g'' + k^2g)H = \delta.$$

Theorem 2. Let f be Sumudu transformable and satisfy $f(t) = 0$ for $t < 0$. Then $\lim_{s \rightarrow \infty} (S[f](u)) = 0$.

Proof. Let $f_0(t) = f(t)[1 - H(t - 1)]$, $f_1(t) = f(t)H(t - 1)$. Since f_0 vanishes outside $[0, 1]$, we have $S[f](u) = \int_0^1 e^{-\frac{t}{u}} f(t) dt$ for any $\frac{1}{u}$. Moreover, $f = f_0 + f_1$, $\text{dom}(S(f_1)) = \text{dom}(S(f))$ and $S[f](u) = \int_0^1 e^{-\frac{t}{u}} f(t) dt + S(f_1)(u)$ for all $\frac{1}{u} \in \text{dom}(S(f))$. Let $\frac{1}{u_0} \in \text{dom}(S(f))$ and apply $|S(f)(u)| \leq Ae^{-\frac{c}{u}}$ to f_1 we conclude that there is a constant A such that

$$|S(f)(u)| \leq \int_0^1 e^{-\frac{t}{u}} |f(t)| dt + Ae^{-\frac{1}{u}} \quad \text{for all } \frac{1}{u} \geq \frac{1}{u_0}$$

as $\frac{1}{u} \rightarrow \infty$ the second term on the right clearly tends to zero. The same applies to the first term. ■

In the next proposition we discuss the inverse Sumudu transform of the functions $S^{-1}\left(\frac{u}{(u^2 + 1)^m}\right)$, and $S^{-1}\left(\frac{u}{(u^2 - 1)^m}\right)$ where u is complex variables and m is positive integers.

Proposition 2. Let $S^{-1}\left(\frac{u}{(u^2 + 1)^m}\right)$, and $S^{-1}\left(\frac{u}{(u^2 - 1)^m}\right)$ then the inverse Sumudu transforms exist and are given by

$$(1.5) \quad S^{-1}\left(\frac{u}{(u^2 + 1)^m}\right) = \frac{(-1)^{m-1} t^{2m-1} H(t)}{2^{m-1} (m-1)!} \left(\frac{1}{t} \frac{d}{dt}\right)^{m-1} \left(\frac{\sin t}{t}\right)$$

and

$$(1.6) \quad S^{-1}\left(\frac{u}{(u^2 - 1)^m}\right) = \frac{t^{2m-1} H(t)}{2^{m-1} (m-1)!} \left(\frac{1}{t} \frac{d}{dt}\right)^{m-1} \left(\frac{\sinh t}{t}\right).$$

Proof. We shall give the proof of eq(1.5) only. The proof of eq(1.6) is similar. For each $y > 0$ the Sumudu transform of the function $(\sin(\sqrt{y}t/\sqrt{y})) H(t)$ is $u/(u^2 + y)$, so that we may write,

$$\int_0^\infty e^{-\left(\frac{t}{u}\right)} \frac{\sin \sqrt{y}t}{\sqrt{y}} dt = \frac{u}{(u^2 + y)}$$

by differentiating this equation (m) times with respect to y gives

$$\int_0^\infty e^{-\left(\frac{t}{u}\right)} \left(\frac{\partial}{\partial y}\right)^m \frac{\sin \sqrt{y}t}{\sqrt{y}} dt = \frac{(-1)^m m!u}{(u^2 + y)^{m+1}}.$$

We now compute both sides of this equation at $y = 1$. With regard to the left side we let $\alpha = \sqrt{y}t$. Then

$$\frac{\partial}{\partial \alpha} = \frac{\partial \alpha}{\partial y} \frac{\partial}{\partial \alpha} = \frac{t^2}{2} \left(\frac{1}{\alpha} \frac{\partial}{\partial \alpha}\right).$$

Thus

$$\begin{aligned} \left[\left(\frac{\partial}{\partial y}\right)^m \frac{\sin \sqrt{y}t}{\sqrt{y}}\right]_{y=1} &= \left[\frac{t^{2m}}{2^m} \left(\frac{1}{\alpha} \frac{\partial}{\partial \alpha}\right)^m \frac{t \sin \alpha}{\alpha}\right]_{\alpha=t} \\ &= \frac{t^{2m+1}}{2^m} \left(\frac{1}{t} \frac{\partial}{\partial t}\right)^m \frac{\sin t}{t}. \end{aligned}$$

So we have proved that Sumudu transform

$$S \left[\frac{t^{2m+1}}{2^m} \left(\frac{1}{t} \frac{\partial}{\partial t}\right)^m \frac{\sin t}{t} H(t) \right] = \frac{(-1)^m m!u}{(u^2 + 1)^{m+1}}.$$

Replacing m by $m - 1$ we obtain the equation (1.5). ■

In particular $m = 3$ in eq(1.5) we have

$$S^{-1} \left(\frac{u}{(u^2 + 1)^3} \right) = \frac{t^5 H(t)}{8} \left(\frac{1}{t} \frac{d}{dt} \right)^2 \left(\frac{\sin t}{t} \right).$$

The computing shows that

$$S^{-1} \left(\frac{u}{(u^2 + 1)^3} \right) = \frac{1}{8} (3 \sin t - 3t \cos t - t^2 \sin t) H(t).$$

Example 1. Solve the differential equation

$$(1.7) \quad \frac{d^2 y}{dt^2} - 3 \frac{dy}{dt} + 2y = \delta + \frac{d^2 \delta(t-1)}{dt^2} + \sin(t)H(t)$$

where $H(t)$ is Heaviside function and δ is Dirac delta function. On using Sumudu transform for eq (1.7), we have

$$(1.8) \quad Y(u) = \frac{e^{\frac{1}{u}}}{u(1-3u+2u^2)} + \frac{(2u^3+u)}{(u^2+1)(1-3u+2u^2)}$$

in order to obtain the solution of differential equation we apply the inverse transform, but before we apply the inverse we replace the complex variable u by the complex viable $\frac{1}{s}$ thus the equation (1.8) becomes

$$(1.9) \quad Y \left(\frac{1}{s} \right) = \frac{s^3 e^s}{(s^2 - 3s + 2)} + \frac{s(s^2 + 2)}{(s^2 + 1)(s^2 - 3s + 2)}$$

then by using Cauchy residue theorem we obtain the solution in the form of

$$y(t) = \left(-\frac{3}{2}e^t + \frac{6}{5}e^{2t} + \frac{1}{10}(3 \cos(t) + \sin(t)) \right) H(t) \\ + \delta(t+1) + (4e^{2t+2} - e^{t+1}) H(t+1).$$

Example 2. Consider the non constant coefficient differential equation in the form of

$$(1.10) \quad \begin{aligned} tf''(t) - tf'(t) + f(t) &= 2, \\ f(0) &= 2, \quad f'(0) = -1. \end{aligned}$$

First solution by Laplace transform: By using Laplace transform and apply the initial condition we have

$$F'(s) + \left(\frac{2}{s} \right) F(s) = -\frac{2}{s^2(s-1)}$$

then we obtain the solution

$$F(s) = -\frac{\ln(s-1)}{s^2} + \frac{c}{s^2} \text{ where } c \text{ is constant}$$

by taking inverse Laplace transform we have

$$(1.11) \quad f(t) = 2 - 2i\pi t + ct.$$

Second solution by Sumudu transform: On using Sumudu transform for Eq(1.10) we have

$$F'(u) - \frac{F(u)}{u} = -\frac{2}{u}$$

by applying the same technique that was used with the same problem in case of Laplace transform, then we have the following solution

$$(1.12) \quad F(u) = 2 + au$$

where a is a constant, by using inverse Sumudu transform for Eq(1.12) with respect to u we obtain the solution in the form of

$$(1.13) \quad f(t) = 2 + at,$$

we note that, if we compare two Eqs(1.11) and (1.13), we see that the solution which is given by Laplace transform in complex domain and given by Sumudu transform in real domain. Thus this leads us to consider that if the solution exists by Sumudu transform then the solution also exists by Laplace transform that is

If Sumudu transform exists \Rightarrow Laplace transform also exists.

The converse of this statement need not necessarily correct.

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REFERENCES

- [1] Asiru, M. A. Application of the Sumudu transform to discrete dynamic systems, International Journal of Mathematical Education in Science and Technology, 34(6)(2003), pp. 944-949.
- [2] Belgacem, F. B. M. Boundary value problem with indefinite weight and applications, International Journal of Problems of Nonlinear Analysis in Engineering Systems, 10(2)(1999), pp. 51-58.
- [3] Belgacem, F. B. M., A. A Karaballi, and L. S. Kalla, Analytical investigations of the Sumudu transform and applications to integral production equations, Mathematical Problems in Engineering 3(2003), pp. 103-118.
- [4] Eltayeb, H., A. Kılıçman & B. Fisher. A new integral transform and associated Distributions, Integral Transforms and Special Functions, doi: 10.1080/10652460903335061.
- [5] Eltayeb, H. and A. Kılıçman. On Some Application of A new Integral Transform, International Journal of Mathematical Analysis, 4(3)(2010), pp. 123-132.

- [6] Kadem, A. Solving the one-dimensional neutron transport equation using Chebyshev polynomials and the Sumudu transform, *Analele Universitatii din Oradea. Fascicola Matematica* Vol. XII, (2005), pp. 153–171.
- [7] Kılıçman, A. & H. Eltayeb. A note On Integral Transforms and Partial Differential Equations, *Applied Mathematical Sciences*, 4(3)(2010), pp. 109–118.
- [8] Kılıçman, A. and H. E. Gadain. An Application of Double Laplace Transform and Double Sumudu Transform, *Lobachevskii Journal of Mathematics*, 2009, Vol. 30, No. 3, pp. 214–223.
- [9] Kılıçman, A. & H. Eltayeb. On the Partial Differential Equations with Non-Constant Coefficients and Convolution Method, *European Journal of Pure and Applied Mathematics*, to appear.
- [10] Kılıçman, A. and H. Eltayeb, A new Integral Transform and Differential Equations, submitted for publication.
- [11] Watugala, G.K. Sumudu transform: a new integral transform to solve differential equations and control engineering problems, *International Journal of Mathematical Education in Science and Technology*, 24(1)(1993), pp. 35–43.
- [12] Watugala, G.K. The Sumudu transform for functions of two variables, *Mathematical Engineering in Industry*, 8(4)(2002), pp. 293–302.

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