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Characterizations of Instuitionistic Fuzzy Ideals of Γ-Rings

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Abstract. In this paper, we study the notion of intuitionistic fuzzy ideals of Γ–rings and study some of its properties.

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1. Introduction

The notion of a fuzzy set was introduced by L.A.Zadeh [11], and since then this concept has been applied to various algebraic structures. The idea of "Intuitionistic Fuzzy Set" was first introduced by K.T.Atanassov [1] as a generalization of the notion of fuzzy set. N.Nobusawa [9] introduced the notion of a Γ - ring, as more general than a ring. W.E.Barnes[2] weakened slightly the conditions in the definition of the Γ - rings in the sense of Nobusawa. W.E.Barnes [2], S.Kyuno [6,7] and J.Luh [8] studied the structure of Γ - rings and obtained various generalizations analogous to corresponding parts in ring theory. Y.B.Jun and C.Y.Lee [4] introduced the concept of fuzzy ideals in the theory of Γ- rings. In this paper, we introduce the notion of intuitionistic fuzzy ideals in Γ - rings and study some of its properties.

2. **Preliminaries**

In this section we include some elementary concepts that are necessary for this paper.

Definition 2.1[2]. If $M = \{x, y, z, \cdots\}$ and $\Gamma = \{\alpha, \beta, \gamma \cdots\}$ be two additive abelian groups and for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, the following conditions are satisfied:

1. $x\alpha y \in M$,
2. $(x+y)\alpha z = x\alpha z + y\alpha z, x(\alpha+\beta)y = x\alpha y + x\beta y, x\alpha(y+z) = x\alpha y + x\alpha z$,
3. $(x\alpha y)\beta z = x\alpha(y\beta z)$, then M is called a Γ - ring.

If these conditions are strengthened to

(1′) $x\alpha y \in M$, $\alpha x\beta \in \Gamma$,
(2′) $(x+y)\alpha z = x\alpha z + y\alpha z, x(\alpha+\beta)y = x\alpha y + x\beta y, x\alpha(y+z) = x\alpha y + x\alpha z$,
(3′) $(x\alpha y)\beta z = x(\alpha y\beta)z = x\alpha(y\beta z)$,
(4′) $x\alpha y = 0$ for all $x, y \in M$ implies $\alpha = 0$,

we then have a Γ - ring in the sense of Nobusawa [9]. As indicated in [9], an example of a Γ - ring is obtained by letting X and Y be Abelian groups, $M = Hom(X, Y)$, $\Gamma = Hom(Y, X)$ and $x\alpha y$ be the usual composite map. (While Nobusawa does not explicitly require that M and Γ be Abelian groups, it appears clear that this is intended.) We may note that it follows from (1) - (3) that $0\alpha y = x0y = x\alpha 0 = 0$, for all $x, y \in M$ and all $\alpha \in \Gamma$.

Definition 2.2[2]. A subset A of a Γ - ring M is a left (resp. right) ideal of M if A is an additive subgroup of M such that $M\Gamma A \subseteq A$ (resp. $A\Gamma M \subseteq A$), where $M\Gamma A = \big\{x\alpha y \mid x \in M, \alpha \in \Gamma, y \in A\big\}$ and $A\Gamma M = \big\{y\alpha x \mid y \in A, \alpha \in \Gamma, x \in M\big\}$. If A is both a left and a right ideal, then A is a two sided ideal or simply an ideal of M.

Definition 2.3[10]. A fuzzy set A in M is a function $A : M \to [0, 1]$.

Definition 2.4[10]. Let μ be a fuzzy set in a Γ - ring M. For any $t \in [0, 1]$, the set $U(\mu, t) = \big\{x \in M | \mu(x) \geq t\big\}$ is called a level set of μ.

Definition 2.5[10]. A fuzzy set μ in a Γ - ring M is called a fuzzy left (resp. right) ideal of M, if it satisfies:
(i) $\mu(x - y) \geq \mu(x) \ \wedge \ \mu(y)$,
(ii)$\mu(x\alpha y) \geq \mu(y)$ $\big($resp. $\mu(x\alpha y) \geq \mu(x)\big)$, for all $x, y \in M$ and $\alpha \in \Gamma$.
If μ is both a fuzzy left and right ideal of M, then μ is called a fuzzy ideal of M.

Definition 2.6[1]. Let X be a nonempty fixed set. An intuitionistic fuzzy set A in X is an object having the form $A = \big\{ \langle\ x, \mu_A(x), \nu_A(x)\ \rangle / x \in X\big\}$, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for every $x \in X$.

Notation. For the sake of simplicity, we shall denote the intuitionistic fuzzy set (IFS in short) $A = \{ \langle x, \mu_A(x), \nu_A(x)\rangle / x \in X \}$ by $A = \langle \mu_A, \nu_A\rangle$.

Definition 2.7[1]. Let X be a non empty set and let $A = \langle \mu_A, \nu_A\rangle$ and $B = \langle \mu_B, \nu_B\rangle$ be IFSs in X. Then

1. $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
2. $A = B$ iff $A \subset B$ and $B \subset A$.
3. $A^c = \langle \nu_A, \mu_A\rangle$.
4. $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.
5. $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.
6. $\Box A = (\mu_A, 1 - \mu_A)$, $\Diamond A = (1 - \nu_A, \nu_A)$.

3. Intuitionistic fuzzy ideals

In what follows, let M denote a Γ - ring unless otherwise specified.

Definition 3.1. An IFS $A = \langle \mu_A, \nu_A\rangle$ in M is called an intuitionistic fuzzy left (resp. right) ideal of a Γ - ring M if
(i) $\mu_A(x-y) \geq \{\mu_A(x) \wedge \mu_A(y)\}$ and $\mu_A(x\alpha y) \geq \mu_A(y)$(resp.$\mu_A(x\alpha y) \geq \mu_A(x)$),
(ii)$\nu_A(x - y) \leq \{\nu_A(x) \vee \nu_A(y)\}$ and $\nu_A(x\alpha y) \leq \nu_A(y)$(resp.$\nu_A(x\alpha y) \leq \nu_A(x)$),
for all $x, y \in M$ and $\alpha \in \Gamma$.

Theorem 3.2. If A is an intuitionistic fuzzy left (resp. right) ideal of a Γ - ring M, then the IFS $\Box A = <\mu_A$, 1 - $\mu_A>$ is an intuitionistic fuzzy left (resp. right) ideal of M.

Proof. Let A be an intuitionistic fuzzy left (resp. right) ideal of a Γ - ring M. Then

μ_A(x - y) $\geq \{\mu_A$(x) $\wedge$ μ_A(y)$\}$ and μ_A(x α y)$\geq$ μ_A(y) (resp.μ_A(xαy) $\geq$ μ_A(x)),
ν_A(x - y) $\leq \{\nu_A$(x) $\vee$ ν_A(y)$\}$ and ν_A(x α y)$\leq$ ν_A(y) (resp. ν_A(xαy) $\leq$ ν_A(x)),

for all x , y $\in$ M and $\alpha \in \Gamma$.
Let B = $\Box$A. Then $\mu_A = \mu_B$ and 1 - $\mu_A = \nu_B$.
Now μ_B(x - y) = μ_A(x - y) $\geq \{ \mu_A$(x) $\wedge$ μ_A(y) $\} = \{ \mu_B$(x) $\wedge$ μ_B(y) $\}$ and
μ_B(xαy) = μ_A(xαy) $\geq$ μ_A(y) = μ_B(y) (resp. μ_B(xαy) $\geq$ μ_B(x)).
Also ν_B(x - y) = 1 - μ_A(x - y) $\leq$ 1 - (μ_A(x) $\wedge$ μ_A(y))
$= (1 - \mu_A(x)) \vee (1 - \mu_A(y)) = \nu_B(x) \vee \nu_B(y)$.
Hence ν_B(x - y) $\leq \{ \nu_B$(x) $\vee$ ν_B(y) $\}$.
Also ν_B(xαy) = 1 - μ_A(xαy) $\leq$ 1 - μ_A(y) = ν_B(y)
(resp. ν_B(xαy) = 1 - μ_A(xαy) $\leq$ 1 - μ_A(x) = ν_B(x)).
Hence $\Box$A is an intuitionistic fuzzy left (resp. right) ideal of a Γ - ring M.

Theorem 3.3. If the IFS A = $<\mu_A$, $\nu_A>$ is an intuitionistic fuzzy left (resp. right) ideal of a Γ - ring M, then μ_A and 1 - ν_A are fuzzy left (resp. right) ideals of a Γ - ring M.

Proof. Let A be an intuitionistic fuzzy left (resp. right) ideal of a Γ - ring M. Then we have

$\mu_A(x-y) \geq \{\mu_A(x) \wedge \mu_A(y)\}$ and $\mu_A(x\alpha y) \geq \mu_A(y)$ $(resp.\ \mu_A(x\alpha y) \geq \mu_A(x))$,

$\nu_A(x-y) \leq \{\nu_A(x) \vee \nu_A(y)\}$ and $\nu_A(x\alpha y) \leq \nu_A(y)$ $\big(resp.\ \nu_A(x\alpha y) \leq \nu_A(x)\big)$, for all x, y ∈ M and $\alpha \in \Gamma$.

(i) Let B = <x, μ_A> be a fuzzy set. Then $\mu_B = \mu_A$.

Now $\mu_B(x-y) = \mu_A(x-y) \geq \{\mu_A(x) \wedge \mu_A(y)\} = \{\mu_B(x) \wedge \mu_B(y)\}$,

$\mu_B(x\alpha y) = \mu_A(x\alpha y) \geq \mu_A(y) = \mu_B(y)$ $\big(resp.\ \mu_B(x\alpha y) \geq \mu_B(x)\big)$.

Hence <x, μ_A> is a fuzzy left (resp. right) ideal of a Γ - ring M.

(ii) Let B = <x, 1-ν_A> be a fuzzy set. Then $\mu_B = 1 - \nu_A$.

Now $\mu_B(x-y) = 1 - \nu_A(x-y) \geq 1 - \{\nu_A(x) \vee \nu_A(y)\}$

$\geq \{(1-\nu_A(x)) \wedge (1-\nu_A(y))\} = \mu_B(x) \wedge \mu_B(y)$.

$\mu_B(x\alpha y) = 1 - \nu_A(x\alpha y) \geq 1 - \nu_A(y) = \mu_B(y)$ $\big(resp.\ \mu_B(x\alpha y) \geq \mu_B(x)\big)$.

Hence <x, $1-\nu_A$> is a fuzzy left (resp. right) ideal of a Γ - ring M.

Theorem 3.4. If the intuitionistic fuzzy set A = <μ_A,ν_A> is an intuitionistic fuzzy left (resp. right) ideal of a Γ - ring M and if μ_A(x-y) = 1 and ν_A(x-y) = 0 for all x , y ∈ M, then μ_A(x) = μ_A(y) and ν_A(x) = ν_A(y).

Proof. Let A be an intuitionistic fuzzy left (resp. right) ideal of a Γ - ring M, μ_A(x - y) = 1 and ν_A(x - y) = 0. Let x, y ∈ A.

Then $\mu_A(x) = \mu_A(x-y+y) \geq \{\mu_A(x-y) \wedge \mu_A(y)\} = \{1 \wedge \mu_A(y)\} = \mu_A(y)$.

Similarly

$\mu_A(y) = \mu_A(x-(x-y)) \geq \{\mu_A(x) \wedge \mu_A(x-y)\} = \{\mu_A(x) \wedge 1\} = \mu_A(x)$.

Hence $\mu_A(x) = \mu_A(y)$.

Also $\nu_A(x) = \nu_A(x-y+y) \leq \{\nu_A(x-y) \vee \nu_A(y)\} = \{0 \vee \nu_A(y)\} = \nu_A(y)$.

Similarly $\nu_A(y) = \nu_A(x-(x-y)) \leq \{\nu_A(x) \vee \nu_A(x-y)\} = \{\nu_A(x) \vee 0\} = \nu_A(x)$.

Hence $\nu_A(x) = \nu_A(y)$.

Definition 3.5. Let A = <μ_A,ν_A> and B = <μ_B,ν_B> be two intuitionistic fuzzy subsets of a Γ - ring M. Then the product AΓB is defined by

$$\mu_{A\Gamma B}(x) = \begin{cases} \bigvee_{x=y\alpha z} \{\mu_A(y) \wedge \mu_B(z)\} & if\ x = y\alpha z, \\ 0 & Otherwise, \end{cases}$$

$$\nu_{A\Gamma B}(x) = \begin{cases} \bigwedge_{x=y\alpha z} \{\nu_A(y) \vee \nu_B(z)\} & if\ x = y\alpha z, \\ 1 & Otherwise. \end{cases}$$

Definition 3.6 [5]. Let A = <μ_A,ν_A> and B = <μ_B,ν_B> be two intuitionistic fuzzy subsets of a Γ - ring M. Then the intuitionistic sum of A and B is defined to be the intuitionistic fuzzy set A⊕B = <$\mu_{A\oplus B}, \nu_{A\oplus B}$> in M given by

$$\mu_{A\oplus B}(x) = \begin{cases} \bigvee\limits_{x=y+z} \{\mu_{\mathrm{A}}(y) \wedge \mu_{\mathrm{B}}(z)\} & if\, x = y+z, \\ 0 & Otherwise, \end{cases}$$

$$\nu_{A\oplus B}(x) = \begin{cases} \bigwedge\limits_{x=y+z} \{\nu_{\mathrm{A}}(y) \vee \nu_{\mathrm{B}}(z)\} & if\, x = y+z, \\ 1 & Otherwise. \end{cases}$$

Theorem 3.7. If A = $<\mu_A,\nu_A>$ and B = $<\mu_B,\nu_B>$ are intuitionistic fuzzy left (resp. right) ideals of M, then the intuitionistic sum A⊕B = $<\mu_{A\oplus B}, \nu_{A\oplus B}>$ is an intuitionistic fuzzy left (resp. right) ideal of M.

Proof. For any x , y ∈ M, we have

$\mu_{A\oplus B}(x) \wedge \mu_{A\oplus B}(y) = \bigvee \Big\{\mu_A(a) \wedge \mu_B(b) : x = a+b\Big\} \wedge \bigvee \Big\{\mu_A(c) \wedge \mu_B(d) : y = c+d\Big\}$

$= \bigvee \Big\{ \Big(\mu_A(a) \wedge \mu_B(b)\Big) \wedge \Big(\mu_A(c) \wedge \mu_B(d)\Big) : x = a+b, y = c+d\Big\}$

$= \bigvee \Big\{ \Big(\mu_A(a) \wedge \mu_B(b)\Big) \wedge \Big(\mu_A(-c) \wedge \mu_B(-d)\Big) : x = a+b, -y = -c-d\Big\}$

$= \bigvee \Big\{ \Big(\mu_A(a) \wedge \mu_A(-c)\Big) \wedge \Big(\mu_B(b) \wedge \mu_B(-d)\Big) : x = a+b, -y = -c-d\Big\}$

$\leq \bigvee \Big\{ \Big(\mu_A(a-c) \wedge \mu_B(b-d)\Big) : x-y = (a-c)+(b-d)\Big\}$

$= \mu_{A\oplus B}(x-y),$

$\nu_{A\oplus B}(x) \vee \nu_{A\oplus B}(y) = \bigwedge \Big\{\nu_A(a) \vee \nu_B(b) : x = a+b\Big\} \vee \bigwedge \Big\{\nu_A(c) \vee \nu_B(d) : y = c+d\Big\}$

$= \bigwedge \Big\{ \Big(\nu_A(a) \vee \nu_B(b)\Big) \vee \Big(\nu_A(c) \vee \nu_B(d)\Big) : x = a+b, y = c+d\Big\}$

$= \bigwedge \Big\{ \Big(\nu_A(a) \vee \nu_B(b)\Big) \vee \Big(\nu_A(-c) \vee \nu_B(-d)\Big) : x = a+b, -y = -c-d\Big\}$

$= \bigwedge \Big\{ \Big(\nu_A(a) \vee \nu_A(-c)\Big) \vee \Big(\nu_B(b) \vee \nu_B(-d)\Big) : x = a+b, -y = -c-d\Big\}$

$\geq \bigwedge \Big\{ \Big(\nu_A(a-c) \vee \nu_B(b-d)\Big) : x-y = (a-c)+(b-d)\Big\}$

$= \nu_{A\oplus B}(x-y).$

Also, we have

$\mu_{A\oplus B}(x) = \bigvee \Big\{\mu_A(a) \wedge \mu_B(b) : x = a+b\Big\}$

$\leq \bigvee \Big\{\mu_A(a\alpha y) \wedge \mu_B(b\alpha y) : x\alpha y = a\alpha y + b\alpha y\Big\}$

$\leq \bigvee \Big\{\mu_A(u) \wedge \mu_B(v) : x\alpha y = u+v\Big\} = \mu_{A\oplus B}(x\alpha y),$

$\nu_{A\oplus B}(x) = \bigwedge \Big\{\nu_A(a) \vee \nu_B(b) : x = a+b\Big\}$

$\geq \bigwedge \Big\{\nu_A(a\alpha y) \vee \nu_B(b\alpha y) : x\alpha y = a\alpha y + b\alpha y\Big\}$

$$\geq \bigwedge \Big\{\nu_A(u) \vee \nu_B(v) : x\alpha y = u + v\Big\} = \nu_{A\oplus B}(x\alpha y).$$

Hence $\mu_{A\oplus B}(x\alpha y) \geq \mu_{A\oplus B}(x)$ and $\nu_{A\oplus B}(x\alpha y) \leq \nu_{A\oplus B}(x)$.

Similarly, we get $\mu_{A\oplus B}(x\alpha y) \geq \mu_{A\oplus B}(y)$ and $\nu_{A\oplus B}(x\alpha y) \leq \nu_{A\oplus B}(y)$.

Therefore A⊕B is an intuitionistic fuzzy left (resp. right) ideal of M.

Theorem 3.8. If $A = \langle\mu_A, \nu_A\rangle$ and $B = \langle\mu_B, \nu_B\rangle$ are two intuitionistic fuzzy left (resp. right) ideals of M, then $A\bigcap B$ is an intuitionistic fuzzy left (resp. right) ideal of M. If A is an intuitionistic fuzzy right ideal and B is an intuitionistic fuzzy left ideal, then $A\Gamma B \subseteq A\bigcap B$.

Proof. Suppose A and B are intuitionistic fuzzy left (resp. right) ideals of M and let $x, y \in M \text{ and } \alpha \in \Gamma$.Then

$$\begin{aligned}
\mu_{A\cap B}(x-y) = \mu_A(x-y) \wedge \mu_B(x-y) &\geq \Big[\mu_A(x) \wedge \mu_A(y)\Big] \wedge \Big[\mu_B(x) \wedge \mu_B(y)\Big] \\
&= \Big[\mu_A(x) \wedge \mu_B(x)\Big] \wedge \Big[\mu_A(y) \wedge \mu_B(y)\Big] \\
&= \mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y), \\
\nu_{A\cap B}(x-y) = \nu_A(x-y) \vee \nu_B(x-y) &\leq \Big[\nu_A(x) \vee \nu_A(y)\Big] \vee \Big[\nu_B(x) \vee \nu_B(y)\Big] \\
&= \Big[\nu_A(x) \vee \nu_B(x)\Big] \vee \Big[\nu_A(y) \vee \nu_B(y)\Big] \\
&= \nu_{A\cap B}(x) \vee \nu_{A\cap B}(y).
\end{aligned}$$

Also we have, $\mu_A(x\alpha y) \geq \mu_A(y) \text{ and } \nu_A(x\alpha y) \leq \nu_A(y)$, $\mu_B(x\alpha y) \geq \mu_B(y)$ and $\nu_B(x\alpha y) \leq \nu_B(y)$.(resp. $\mu_A(x\alpha y) \geq \mu_A(x) \text{ and } \nu_A(x\alpha y) \leq \nu_A(x)$, $\mu_B(x\alpha y) \geq \mu_B(x)$ and $\nu_B(x\alpha y) \leq \nu_B(x)$.)

Now

$$\begin{aligned}
\mu_{A\cap B}(x\alpha y) &= \mu_A(x\alpha y) \wedge \mu_B(x\alpha y) \\
&\geq \mu_A(y) \wedge \mu_B(y) \\
&= \mu_{A\cap B}(y), \\
\nu_{A\cap B}(x\alpha y) &= \nu_A(x\alpha y) \vee \nu_B(x\alpha y) \\
&\leq \nu_A(y) \vee \nu_B(y) \\
&= \nu_{A\cap B}(y).
\end{aligned}$$

Hence $A\bigcap B$ is an intuitionistic fuzzy left (resp. right) ideal of M.

To prove the second part, if $\mu_{A\Gamma B}(x) = 0$ and $\nu_{A\Gamma B}(x) = 1$, there is nothing to show.

Suppose $A\Gamma B(x) \neq (0, 1)$.

From the definition of $A\Gamma B$,

$$\mu_A(x) = \mu_A(y\alpha z) \geq \mu_A(y) \text{ and } \nu_A(x) = \nu_A(y\alpha z) \leq \nu_A(y).$$

Since A is an intuitionistic fuzzy right ideal and B is an intuitionistic fuzzy left ideal, we have

$$\mu_A(x) = \mu_A(y\alpha z) \geq \mu_A(y) \text{ and } \nu_A(x) = \nu_A(y\alpha z) \leq \nu_A(y),$$
$$\mu_B(x) = \mu_B(y\alpha z) \geq \mu_B(z) \text{ and } \nu_B(x) = \nu_B(y\alpha z) \leq \nu_B(z).$$

Hence by Definition 3.5,

$$\mu_{A\Gamma B}(x) = \bigvee_{x=y\alpha z}\Big\{\mu_A(y) \wedge \mu_B(z)\Big\} \leq \mu_A(x) \wedge \mu_B(x) = \mu_{A\cap B}(x),$$

$$\nu_{A\Gamma B}(x) = \bigwedge_{x=y\alpha z}\Big\{\nu_A(y) \vee \nu_B(z)\Big\} \geq \nu_A(x) \vee \nu_B(x) = \nu_{A\cap B}(x),$$

which means that $A\Gamma B \subseteq A \cap B$.

Definition 3.9[2]. Let I be an ideal of a Γ - ring M . If for each a+I, b+I in the factor group M/I and each α∈Γ, we define (a + I)α(b + I) = aαb + I then M/I is a Γ - ring which we shall call the Γ - residue class ring of M with respect to I.

Theorem 3.10. Let I be an ideal of a Γ - ring M. If A is an intuitionistic fuzzy left (resp. right) ideal of M, then the IFS $\tilde{A}$ of M/I defined by

$$\mu_{\tilde{A}}(a+I) = \bigvee_{x\in I}\mu_A(a+x) \text{ and } \nu_{\tilde{A}}(a+I) = \bigwedge_{x\in I}\nu_A(a+x)$$

is an intuitionistic fuzzy left (resp. right) ideal of the Γ - residue class ring M/I of M with respect to I.

Proof. Let $a, b \in M$ be such that $a + I = b + I$.
Then $b = a + y$ for some $y \in I$, and so

$$\mu_{\tilde{A}}\big(b+I\big) = \bigvee_{x\in I}\mu_A\big(b+x\big) = \bigvee_{x\in I}\mu_A\big(a+y+x\big) = \bigvee_{x+y=z\in I}\mu_A\big(a+z\big) = \mu_{\tilde{A}}\big(a+I\big),$$

$$\nu_{\tilde{A}}\big(b+I\big) = \bigwedge_{x\in I}\nu_A\big(b+x\big) = \bigwedge_{x\in I}\nu_A\big(a+y+x\big) = \bigwedge_{x+y=z\in I}\nu_A\big(a+z\big) = \nu_{\tilde{A}}\big(a+I\big).$$

Hence $\tilde{A}$ is well defined.
For any $x + I, y + I \in M/I$ and $\alpha \in \Gamma$, we have

$$\begin{aligned}\mu_{\tilde{A}}\Big((x+I)-(y+I)\Big) &= \mu_{\tilde{A}}\Big((x-y)+I\Big) = \bigvee_{z\in I}\mu_A\Big((x-y)+z\Big)\\ &= \bigvee_{z=u-v\in I}\mu_A\Big((x-y)+(u-v)\Big) = \bigvee_{u,v\in I}\mu_A\Big((x+u)-(y+v)\Big)\\ &\geq \bigvee_{u,v\in I}\Big(\mu_A(x+u)\wedge\mu_A(y+v)\Big)\\ &= \Big(\bigvee_{u\in I}\mu_A(x+u)\Big)\wedge\Big(\bigvee_{v\in I}\mu_A(y+v)\Big) = \mu_{\tilde{A}}(x+I)\wedge\mu_{\tilde{A}}(y+I)\end{aligned}$$

$$\begin{aligned}\nu_{\tilde{A}}\Big((x+I)-(y+I)\Big) &= \nu_{\tilde{A}}\Big((x-y)+I\Big) = \bigwedge_{z\in I}\nu_A\Big((x-y)+z\Big)\\ &= \bigwedge_{z=u-v\in I}\nu_A\Big((x-y)+(u-v)\Big) = \bigwedge_{u,v\in I}\nu_A\Big((x+u)-(y+v)\Big)\\ &\leq \bigwedge_{u,v\in I}\Big(\nu_A(x+u)\vee\nu_A(y+v)\Big)\end{aligned}$$

$$= \Big(\bigwedge_{u\in I}\nu_A(x+u)\Big) \vee \Big(\bigwedge_{v\in I}\nu_A(y+v)\Big) = \nu_{\tilde{A}}(x+I) \vee \nu_{\tilde{A}}(y+I)$$

$$\mu_{\tilde{A}}\Big((x+I)\alpha(y+I)\Big) = \mu_{\tilde{A}}\Big((x\alpha y)+I\Big) = \bigvee_{z\in I}\mu_A\Big((x\alpha y)+z\Big)$$
$$\geq \bigvee_{z\in I}\mu_A(x\alpha y + x\alpha z) \text{ because } x\alpha z \in I$$
$$= \bigvee_{z\in I}\mu_A\Big(x\alpha(y+z)\Big) \geq \bigvee_{z\in I}\mu_A(y+z) = \mu_{\tilde{A}}(y+I)$$

and

$$\nu_{\tilde{A}}\Big((x+I)\alpha(y+I)\Big) = \nu_{\tilde{A}}\Big((x\alpha y)+I\Big) = \bigwedge_{z\in I}\nu_A\Big((x\alpha y)+z\Big)$$
$$\leq \bigwedge_{z\in I}\nu_A(x\alpha y + x\alpha z) \text{ because } x\alpha z \in I$$
$$= \bigwedge_{z\in I}\nu_A\Big(x\alpha(y+z)\Big) \leq \bigwedge_{z\in I}\nu_A(y+z) = \nu_{\tilde{A}}(y+I).$$

Similarly $\mu_{\tilde{A}}\big\{(x+I)\alpha(y+I)\big\} \geq \mu_{\tilde{A}}(x+I)$ and $\nu_{\tilde{A}}\big\{(x+I)\alpha(y+I)\big\} \leq \nu_{\tilde{A}}(x+I)$. Hence $\tilde{A}$ is an intuitionistic fuzzy left (resp. right) ideal of M/I.

Theorem 3.11. Let I be an ideal of a Γ - ring M. Then there exists a one-to-one correspondence between the set of all intuitionistic fuzzy left ideals A of M such that $\mu_A(0) = \mu_A(\text{u})$, $\nu_A(0) = \nu_A(\text{u})$, for all u ∈ I and the set of all intuitionistic fuzzy left ideals $\tilde{A}$ of M/I.

Proof. Let A be an intuitionistic fuzzy left ideal of M.

We know that $\tilde{A}$ defined by $\mu_{\tilde{A}}(a+I) = \bigvee_{x\in I}\mu_A(a+x)$ and $\nu_{\tilde{A}}(a+I) = \bigwedge_{x\in I}\nu_A(a+x)$ is an intuitionistic fuzzy left ideal of M/I.

Since $\mu_A(0) = \mu_A(\text{u})$, $\nu_A(0) = \nu_A(\text{u})$, for all u ∈ I, we get

$$\mu_A(a+u) \geq \{\mu_A(a) \wedge \mu_A(u)\} = \mu_A(a),$$
$$\nu_A(a+u) \leq \{\nu_A(a) \vee \nu_A(u)\} = \nu_A(a).$$

Again, $\mu_A(\text{a}) = \mu_A(\text{a + u - u}) \geq \{\ \mu_A(\text{a + u}) \wedge \mu_A(\text{u})\ \} = \mu_A(\text{a + u})$ and

$\nu_A(\text{a}) = \nu_A(\text{a + u - u}) \leq \{\nu_A(\text{a + u}) \vee \nu_A(\text{u})\ \} = \nu_A(\text{a + u})$.

Hence $\mu_A(\text{a + u}) = \mu_A(\text{a})$ and $\nu_A(\text{a + u}) = \nu_A(\text{a})$, for all u∈I, that is,

$\mu_{\tilde{A}}(\text{a + I}) = \mu_A(\text{a})$ and $\nu_{\tilde{A}}(\text{a + I}) = \nu_A(\text{a})$.

Therefore the correspondence A → $\tilde{A}$ is injective.

Now let $\tilde{A}$ be any intuitionistic fuzzy left ideal of M/I and define an intuitionistic fuzzy set A in M by $\mu_A(\text{a}) = \mu_{\tilde{A}}(\text{a + I})$ and $\nu_A(\text{a}) = \nu_{\tilde{A}}(\text{a + I})$ for all a∈M.

For every x, y∈M and α∈Γ, we have

$$\mu_A(x-y) = \mu_{\tilde{A}}\Big((x-y)+I\Big) = \mu_{\tilde{A}}\Big((x+I)-(y+I)\Big)$$

$$\geq \mu_{\tilde{A}}(x+I) \wedge \mu_{\tilde{A}}(y+I) = \mu_A(x) \wedge \mu_A(y),$$

$$\nu_A(x-y) = \nu_{\tilde{A}}\Big((x-y)+I\Big) = \nu_{\tilde{A}}\Big((x+I)-(y+I)\Big)$$

$$\leq \nu_{\tilde{A}}(x+I) \vee \nu_{\tilde{A}}(y+I) = \nu_A(x) \vee \nu_A(y),$$

$$\mu_A(x\alpha y) = \mu_{\tilde{A}}(x\alpha y+I) = \mu_{\tilde{A}}\Big((x+I)\alpha(y+I)\Big) \geq \mu_{\tilde{A}}(y+I) = \mu_A(y) \text{ and}$$

$$\nu_A(x\alpha y) = \nu_{\tilde{A}}(x\alpha y+I) = \nu_{\tilde{A}}\Big((x+I)\alpha(y+I)\Big) \leq \nu_{\tilde{A}}(y+I) = \nu_A(y).$$

Thus A is an intuitionistic fuzzy left ideal of M.
Note that $\mu_A(z) = \mu_{\tilde{A}}(z+I) = \mu_{\tilde{A}}(I)$ and $\nu_A(z) = \nu_{\tilde{A}}(z+I) = \nu_{\tilde{A}}(I)$ for all $z \in I$, which shows that $\mu_A(z) = \mu_A(0)$ and $\nu_A(z) = \nu_A(0)$ for all $z \in I$.
This completes the proof.

Definition 3.12 [2]. A function f : M → N, where M and N are Γ - rings, is said to be a Γ - homomorphism if f(a+b) = f(a) + f(b), f(aαb) = f(a)αf(b), for all a,b ∈ M and α ∈ Γ.

Definition 3.13 [2]. Let f : X → Y be a mapping of Γ - rings and A be an intuitionistic fuzzy set of Y. Then the map $f^{-1}(A)$ is the pre-image of A under f, if $\mu_{f^{-1}(A)}(x) = \mu_A\big(f(x)\big)$ and $\nu_{f^{-1}(A)}(x) = \nu_A\big(f(x)\big)$, for all x ∈ X.

Theorem 3.14 Let f: X→Y be a homomorphism of Γ - rings. If A is an intuitionistic fuzzy left (resp. right) ideal of Y, then f^{-1}(A) is an intuitionistic fuzzy left (resp. right) ideal of X.
Proof. For any x , y ∈ X , we have

$$\mu_{f^{-1}(A)}(x-y) = \mu_A\big(f(x-y)\big) = \mu_A\big(f(x)-f(y)\big)$$

$$\geq \Big\{\mu_A\big(f(x)\big) \wedge \mu_A\big(f(y)\big)\Big\} = \Big\{\mu_{f^{-1}(A)}(x) \wedge \mu_{f^{-1}(A)}(y)\Big\}.$$

and $\mu_{f^{-1}(A)}(x\alpha y) = \mu_A\big(f(x\alpha y)\big) = \mu_A\big(f(x)\alpha f(y)\big) \geq \mu_A\big(f(y)\big) = \mu_{f^{-1}(A)}(y)$

$\Big(resp.\ \mu_{f^{-1}(A)}(x\alpha y) = \mu_A\big(f(x\alpha y)\big) = \mu_A\big(f(x)\alpha f(y)\big) \geq \mu_A\big(f(x)\big) = \mu_{f^{-1}(A)}(x)\Big).$

Similarly

$$\nu_{f^{-1}(A)}(x-y) = \nu_A\big(f(x-y)\big) = \nu_A\big(f(x)-f(y)\big) \leq \Big\{\nu_A\big(f(x)\big) \vee \nu_A\big(f(y)\big)\Big\}$$

$$= \Big\{\nu_{f^{-1}(A)}(x) \vee \nu_{f^{-1}(A)}(y)\Big\}.$$

and $\nu_{f^{-1}(A)}(x\alpha y) = \nu_A\big(f(x\alpha y)\big) = \nu_A\big(f(x)\alpha f(y)\big) \leq \nu_A\big(f(y)\big) = \nu_{f^{-1}(A)}(y)$.

$\Big(resp.\ \nu_{f^{-1}(A)}(x\alpha y) = \nu_A\big(f(x\alpha y)\big) = \nu_A\big(f(x)\alpha f(y)\big) \leq \nu_A\big(f(x)\big) = \nu_{f^{-1}(A)}(x)\Big).$

Hence f^{-1}(A) is an intuitionistic fuzzy left (resp. right) ideal of X.

Theorem 3.15. Let f : X → Y be an epimorphism of Γ - rings and let A be an IFS in Y. If f^{-1}(A) is an intuitionistic fuzzy left (resp. right) ideal of X, then A is an intuitionistic fuzzy left (resp. right) ideal of Y.

Proof. If y_1 , $y_2 \in$Y, then there exists x_1 , $x_2 \in$X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Therefore $\mu_A\big(y_1 - y_2\big) = \mu_A\big(f(x_1) - f(x_2)\big) = \mu_{f^{-1}(A)}\big(x_1 - x_2\big)$

$$\geq \Big\{\mu_{f^{-1}(A)}(x_1) \wedge \mu_{f^{-1}(A)}(x_2)\Big\}$$
$$= \Big\{\mu_A\big(f(x_1)\big) \wedge \mu_A\big(f(x_2)\big)\Big\}$$
$$= \Big\{\mu_A(y_1) \wedge \mu_A(y_2)\Big\}.$$

That is $\mu_A(y_1 \text{ - } y_2) \geq \Big\{\mu_A(y_1) \wedge \mu_A(y_2)\Big\}$.

$\mu_A(y_1 \alpha y_2\ ) = \mu_A\big(\mathrm{f}(x_1)\alpha\mathrm{f}(x_2)\big) = \mu_{f^{-1}(A)}\big(x_1 \alpha x_2\big)$

$$\geq \mu_{f^{-1}(A)}(x_2) = \mu_A\big(\mathrm{f}(x_2)\big) = \mu_A(y_2).$$

$\big($resp. $\mu_A(y_1 \alpha y_2) = \mu_A\big(f(x_1)\alpha f(x_2)\big) = \mu_{f^{-1}(A)}(x_1 \alpha x_2)$

$$\geq \mu_{f^{-1}(A)}(x_1) = \mu_A\big(f(x_1)\big) = \mu_A(y_1)\big).$$

Also $\nu_A(y_1 - y_2) = \nu_A\big(f(x_1) - f(x_2)\big) = \nu_{f^{-1}(A)}(x_1 - x_2)$

$$\leq \Big\{\nu_{f^{-1}(A)}(x_1) \vee \nu_{f^{-1}(A)}(x_2)\Big\}$$
$$= \Big\{\nu_A\big(f(x_1)\big) \vee \nu_A\big(f(x_2)\big)\Big\}$$
$$= \Big\{\nu_A(y_1) \vee \nu_A(y_2)\Big\}.$$

That is $\nu_A(y_1 \text{ - } y_2) \leq \Big\{\nu_A(y_1) \bigvee \nu_A(y_2)\Big\}$.

$\nu_A(y_1 \alpha y_2) = \nu_A\big(\mathrm{f}(x_1)\alpha\mathrm{f}(x_2)\big) = \nu_{f^{-1}(A)}(x_1 \alpha x_2) \leq \nu_{f^{-1}(A)}(x_2) = \nu_A\big(\mathrm{f}(x_2)\big) = \nu_A(y_2)$.

$\big($resp.$\nu_A(y_1 \alpha y_2) = \nu_A\big(\mathrm{f}(x_1)\alpha\mathrm{f}(x_2)\big) = \nu_{f^{-1}(A)}(x_1 \alpha x_2) \leq \nu_{f^{-1}(A)}(x_1) = \nu_A\big(\mathrm{f}(x_1)\big) = \nu_A(y_1)\big)$.

Hence A is an intuitionistic fuzzy left (resp. right) ideal of Y.

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