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Explicit Analytic Solution for the Nonlinear Ion Sound Waves Equation

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Abstract

In this article, we use an efficient analytical method called homotopy analysis method (HAM) to derive an approximate solution of nonlinear ion sound waves equation. Actually, we solved Korteweg-de Vries equation arises in a one-dimensional macroscopic plasma model describing the weakly nonlinear evolution of ion sound speed by the HAM. Unlike the perturbation method, the HAM does not require the addition of a small physically parameter to the differential equation. It is applicable to strongly and weakly nonlinear problems. Moreover, the HAM involves an auxiliary parameter, which renders the convergence parameter of series solutions controllable, and increases the convergence, and increases the convergence significantly. This article depicts that the HAM is an efficient and powerful method for solving nonlinear differential equations. Its performance is considerable and the solution of the equation is the same as the numerical methods solution.

Keywords: Homotopy Analysis Method, Plasma Physics, Nonlinear Ion Sound Waves Equation, Korteweg-de Vries Equation

1 Introduction

Modeling of natural phenomena in most sciences yields nonlinear differential equations the exact solutions of which are usually rare. Therefore, analytical methods are strongly needed. For instance, one analytical method, called perturbation, involves creating a small physically parameter in the problem, however, finding this parameter is impossible in most cases [1, 2]. Generally speaking, one simple solution for controlling convergence and increasing it does not exist in all analytical methods.

In 1992, Liao [3] presented Homotopy Analysis Method (HAM) based on fundamental concept of homotopy in topology [4-9]. In this method, we do not need to apply the small parameter and unlike all other analytic techniques, the HAM provides us with a simple way to adjust and control the convergence region of approximate series solutions. HAM has been successfully applied to solve many types of nonlinear problems [10-14].

In this work, the basic idea of HAM is described, and then we apply it to the nonlinear ion sound waves equation. In [15,16] the authors use the Korteweg-de Vries equations and show how this equation arises in a one-dimensional macroscopic plasma model describing the weakly nonlinear evolution of ion sound speed. The importance of this equation in plasma physics is mentioned in Section 2.

The paper is organized as follows: In Section 2 we briefly describe the Korteweg-de Vries equation and Historically show the application and importance of it in plasma physics. Basic Idea of HAM is introduced in Section 3. In Section 4, using HAM, we propose a suitable formulation of nonlinear ion sound waves equation for HAM and find a solution which has a performance like numerical methods. In Section 5 we conclude the method and the proposed solution.

2 Ion Sound Wave Model

Two of the most important properties characteristic of a plasma are nonlinearity and dispersion. We begin in this section by discussing a classic nonlinear partial differential equation, known as the Korteweg-de Vries equation, which aries in a variety of physical situation, including problems relevant to plasma physics. The Korteweg-de Vries equation is given by

$$\frac{\partial U}{\partial \tau} + a \cdot U \frac{\partial U}{\partial \zeta} + b \frac{\partial^3 U}{\partial \zeta^3} = 0, \tag{1}$$

where ζ and τ are independent variables and a and b are real, nonzero constants. Equation (1) is nonlinear through the convective term $U\frac{\partial U}{\partial \zeta}$, and dispersive through the term $\frac{\partial^3 U}{\partial \zeta^3}$. Historically, Equation (1) was first derived by Korteweg and de Vries [17] in relation to the problem of long surface waves in water in a channel of constant depth. Subsequently, Gardner and Morikawa [18] derived Equation (1) from a cold-plasma hydromagnetic model describing the long-time behavior of disturbances propagating perpendicular to a magnetic field with velocity near the Alfvén velocity. As demonstrated by Kruskal and Zabusky [19-21], Equation (1) can also describe one-dimensional acoustic waves in anharmonic crystals. Moreover, as a further example from plasma physics, Washimi and Taniuti [22] have shown that Equation (1) gives a weakly nonlinear description of one-dimensional ion sound wave disturbances traveling through near the ion sound speed. In view of these many diverse applications of the Korteweg-de Vries equation, it is apparent that some generalizations are in order. In this regard, Su and Gardner [23] have shown that Equation (1) arises in a broad class of weakly nonlinear dispersive systems, just as Burgers'equation [24] a broad class of weakly nonlinear dissipative systems.

For present purposes, it is sufficient to derive the Korteweg-de Vries equation for one specific problem of interest in plasma physics, and keep in mind the many applications [23] of Equation (1). In the rest of this section we show in detail how Equation (1) arises in a one-dimensional macroscopic plasma model describing the weakly nonlinear evolution of ion sound speed.

We now derive the Korteweg-de Vries equation for the case of ion sound wave disturbance moving with Mach number (defined relative to the ion sound speed) slightly greater than unity in a uniform, magnetic field-free, plasma background. The ions are assumed cold and nondrifting relative to the electrons ($T_i \ll T_e$), and a one-dimensional macroscopic description is used. Moreover, electron interia effects are neglected ($M_e \to 0$) and the isothermal equation of state, $P_e = n_e k_B T_e \, (T_e = const)$, is adopted for the electrons. We then find

$$0 = e \cdot \frac{\partial \Phi}{\partial x'} - \frac{K_B T_e}{n_e} \cdot \frac{\partial n_e}{\partial x'}, \tag{2}$$

where $-e$ is the charge on the electron, $n_e(x', t')$ is the electron density, and $\Phi(x', t')$ is the electrostatic potential ($E = -\partial\Phi/\partial x'$). Equation (2) may be integrated to give $n_e = n_0 exp(e\Phi/k_B T_e)$, where n_0 is the uniform background electron density. Poisson's equation becomes

$$\partial^2\Phi/\partial x'^2 = 4\pi e n_0[exp(e\Phi/k_B T_e)] - n_i \tag{3}$$

For the ions, we have

$$\frac{\partial n_i}{\partial t'} + \frac{\partial n_i \cdot v_i}{\partial x'} = 0, \tag{4}$$

$$\frac{\partial v_i}{\partial t'} + v_i \cdot \frac{\partial v_i}{\partial x'} = -\frac{e}{m_i} \cdot \frac{\partial \Phi}{\partial x'}, \tag{5}$$

where $n_i(x', t')$ is the ion density, $v_i(x', t')$ the ion mean velocity, and e and m_i the ion charge and mass, respectively. It is convenient to introduce the dimensionless quantities (x, t, ϕ, n, v) where

$$x \equiv \frac{x'}{(k_B T_e / 4\pi n_0 e^2)^{1/2}},$$

$$t \equiv t' (\tfrac{4_0 e^2}{m_i})^{1/2},$$

$$\phi \equiv \tfrac{e\Phi}{k_B T_e},$$

$$n \equiv \tfrac{n_i}{n_0},$$

$$v \equiv \tfrac{v_i}{(k_B T_e / m_i)^{1/2}}$$

Equations (3)-(5) may then be written in the dimensionless form

$$\frac{\partial n}{\partial t} + \frac{\partial (n \cdot \nu)}{\partial x} = 0 \tag{6}$$

$$\frac{\partial v}{\partial t} + v \cdot \frac{\partial (v)}{\partial x} = -\frac{\partial \phi}{\partial x} \tag{7}$$

$$\frac{\partial^2 \phi}{\partial x^2} = e^{\phi} - n \tag{8}$$

where n is the ion density, v is the ion velocity, and ϕ is the electrostatic potential. For a travelling-wave solution, we assume the n, v and ϕ are functions of $\zeta := x - M \cdot t$, where M is the mach number. The appropriate boundary conditions for a solitary-wave solution are:

$$|\zeta| \to \infty \Rightarrow = \begin{cases} n \to 1 \\ v \to 0 \\ \phi \to 0 \\ n' \to 0 \\ v' \to 0 \\ \phi' \to 0 \end{cases}, \tag{9}$$

where a dash denotes differentiation with respect to ζ.

Equations (6) and (7) may be integrated once to give:

$$n = \frac{M}{M-v}, (M-v)^2 = M^2 - 2\cdot\phi, \tag{10}$$

where we have used $n \to 1$, $v \to 0$ and $\phi \to 0$ as $|\zeta| \to \infty$. Substitution of Equation (10) into Equation (8) gives:

$$\phi'' = e^{\phi} - \frac{M}{\sqrt{M^2 - 2\cdot\phi}}. \tag{11}$$

One more integration gives:

$$\frac{1}{2}\phi'^2 = -V(\phi) := e^{\phi} + M\sqrt{M^2 - 2\cdot\phi} - (M^2+1) \tag{12}$$

where we have used $\phi \to 0$ and $\phi' \to 0$ as $|\zeta| \to \infty$. Although Equation

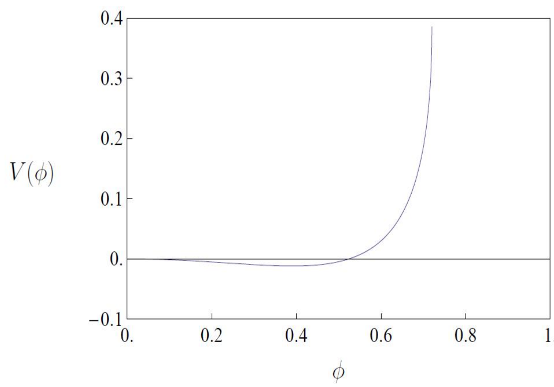

Figure 1: $v(\phi)$, as defined by (12), for $M = 1.2$ so that $a = 0.5244$

(12) does not have an analytic solution, we can deduce some properties of the solution by considering the graph of V as a function of ϕ. (Figure 1 shows the graph of $V(\phi)$) when $M = 1.2$. Notice that curve terminates at $\phi = \frac{M^2}{2} = 0.72$ where $V(\frac{M^2}{2}) = 0.3856$.)

We observe that $V(0) = 0, V'(0) = 0$ and $V"(\frac{M^2}{2}) = -1 + M^{-2}$ so that $V"(0) < 0$ provided $M > 1$. Thus, provided $M > 1$ and $V(\frac{M^2}{2}) > 0$, $V(a) = 0$ with a somewhere in the interval $0 < a < \frac{M^2}{2}$. But $V(\frac{M^2}{2}) > 0$ implies that $M < M_{max}$ where M_{max} is the root of $V(\frac{M^2}{2}) = 0$, i.e.

$$e^{\frac{M^2}{2}} - M^2 - 1 = 0 \tag{13}$$

Equation (13) gives $M_{max} = 1.585$ (to $3D$). The condition $V(a) = 0$ can be rearranged to give:

$$M^2 = \frac{(e^a - 1)^2}{2 \cdot (e^a - a - 1)}. \qquad (14)$$

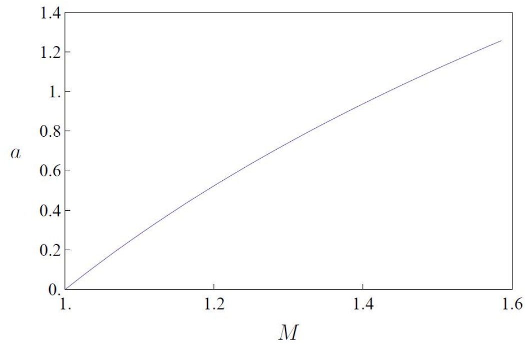

Figure 2: a, as a function of M as computed from (14)

When $M = 1$, $a = 0$. When $M = M_{max}$,$a = \frac{M_{max}^2}{2} = 1.256$ (to $3D$). (Figure 2 shows a as a function of M, for $1 < M \leq M_{max}$. Notice that the curve terminates at $M = M_{max}$). It follows that, if $1 < M \leq M_{max}$, (12) has a solitary-wave solution of amplitude a, where a and M are related by Equation (14). Without loss of generality we can take the the crest of the wave to be located at $\zeta = 0$ so that $\phi(0) = a$ and then the solitary wave is symmetric about $\zeta = 0$. (As an example, Figure 3 shows the graph of $\phi(\zeta)$ when $M = 1.2$. This was plotted by solving Equation (12) numerically).

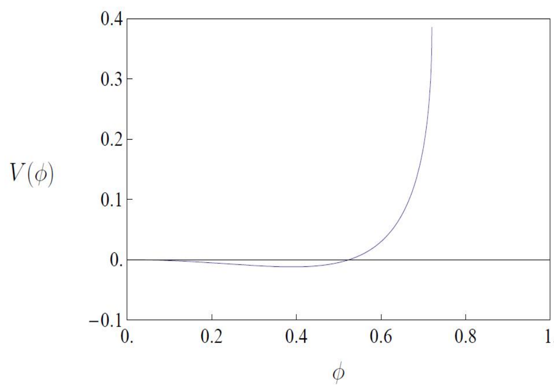

Figure 3: $\phi(\zeta)$ for $M = 1.2$ so that $a = 0.5244$ (Numerical computation)

3 Basic idea of HAM

Let us consider the following differential equation:

$$N[u(\tau)] = 0, \tag{15}$$

where N is a nonlinear operator, τ denotes in dependent variable, $u(\tau)$ is an unknown function that is the solution of the equation. We define the function

$$\lim_{p\to 0} \varphi(\tau;p) = u_0(\tau), \tag{16}$$

where, $p \in [0,1]$ and $u_0(\tau)$ is the initial guess which satisfies the initial or boundary condition and is

$$\lim_{p\to 1} \varphi(\tau;p) = u(\tau). \tag{17}$$

By means of generalizing the traditional homotopy method, Liao [3] constructs the so-called *zero-order deformation* equation:

$$(1-p)\cdot L[\varphi(\tau;p) - u_0(\tau)] = p\cdot \hbar \cdot H(\tau)\cdot N[\varphi(\tau;p)], \tag{18}$$

where $\hbar$ is the auxiliary parameter which increases the results convergence, $H(\tau) \neq 0$ is an auxiliary function and L is an auxiliary linear operator, p increases from 0 to 1, the solution $\varphi(\tau;p)$ changes between the initial guess $u_0(\tau;p)$ and solution $u(\tau)$. Expanding $\varphi(\tau;p)$ in Taylor series with respect to p, we have:

$$\varphi(\tau;p) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau)\cdot p^m, \tag{19}$$

where

$$u_m(\tau) = \frac{1}{m!}\cdot \frac{\partial^m \varphi(\tau;p)}{\partial p^m}\,|_{p=0}\;. \tag{20}$$

if the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series Equation (19) converges at $p = 1$, and then we have:

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau), \tag{21}$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao [7]. It is clear that if the auxiliary parameter is $\hbar = -1$ and auxiliary function is determined to be $H(\tau) = 1$, Equation (18) will be:

$$(1-p)\cdot L[\varphi(\tau;p)-u_0(\tau)]+p\cdot N[\varphi(\tau;p)]=0. \tag{22}$$

this statement is commonly used in HPM procedure. Indeed, in HPM we solve the nonlinear differential equation by separating every Taylor expansion term.

Now we define the vector of $\overrightarrow{u}_m$ as follows $\overrightarrow{u}_m=\{\overrightarrow{u}_1,\overrightarrow{u}_2,\ldots,\overrightarrow{u}_n\}$.

According to the definition Equation (20), the governing equation and the corresponding initial condition of $u_m(\tau)$ can be deduced from *zero-order deformation* Equation (18). Differentiating Equation (18) for m-times with respect to the embedding parameter p and setting $p=0$ and finally dividing by $m!$, we will have the so-called *m-th order deformation* equation in the following from

$$L[u_m(\tau)-x_m\cdot u_{m-1}(\tau)]=p\cdot\hbar\cdot H(\tau)\cdot R_m(\overrightarrow{u}_{m-1}), \tag{23}$$

where

$$R_m(\overrightarrow{u}_{m-1})=\frac{1}{(m-1)!}\cdot\frac{\partial^{m-1}N[\varphi(\tau;p)]}{\partial p^{m-1}}\,|_{p=0} \tag{24}$$

and

$$x_m=\begin{cases}0 & m\le 1\\ 1 & m>1\end{cases} \tag{25}$$

So by applying inverse linear operator to both sides of the linear equation, Equation (23), we can easily solve the equation and compute the generation constant by applying the initial or boundary condition.

4 The problem as formulated for the HAM

$$\phi(0)=a,\quad \phi'(0)=0,\quad \phi(\infty)=0 \tag{26}$$

We choose the initial approximation.

$$\phi_0(\zeta)=a\cdot e^{-\zeta^2}\,|_{a=0.5244} \tag{27}$$

and the linear operator for equation (11)

$$\ell[\varphi(\zeta;q)]=\frac{\partial^2\varphi(\zeta,q)}{\partial\zeta^2} \tag{28}$$

We change equations Equation (11) and Equation (28) to nonlinear form:

$$N[\varphi(\zeta,;q)]=\phi''(\zeta)-e^{\phi(\zeta)}+\frac{M}{\sqrt{M^2-2\cdot\phi(\zeta)}} \tag{29}$$

assuming $H(\tau)=1$, we use above definition to construct the *zero-order deformation* equations.

$$(1-q)\cdot \ell[\varphi(\zeta;q)-\phi_0(\zeta)] = q\hbar N[\varphi(\zeta;q)] \tag{30}$$

Obviously it is observed that:

$$q=0 \Rightarrow \varphi(\zeta;0)=\phi_0(\zeta) \tag{31}$$

and:

$$q=1 \Rightarrow \varphi(\zeta;1)=\phi(\zeta) \tag{32}$$

Differentiating the *zero-order deformation* Equation (30) m-times with respect to q, we have:

$$\ell[\phi_m - x_m\phi_{m-1}] = \hbar R_m(\phi_{m-1}) \tag{33}$$

where

$$R_m = R_{m_1} + R_{m_2} + R_{m_3} \tag{34}$$

In (33) for calculating R_m, we expand functions e^{ϕ} and $\frac{1}{\sqrt{M^2-2\cdot\phi(\zeta)}}$ and use four terms of their Taylor series. In (34) R_{m_1}, R_{m_2} and R_{m_3} are as follow:

$$R_{m_1} = \frac{1}{(m-1)!}\times\frac{\partial^2\phi_{m-1}}{\partial\zeta^2} \tag{35}$$

and

$$R_{m_2} = 0 + \phi_{m-1} + \frac{1}{2}\sum_{k=0}^{m-1}\phi_k\phi_{m-1-k} + \frac{1}{6}\sum_{k=0}^{m-1}\phi_{m-1-k}\sum_{p=0}^{k}\phi_p\times\phi_{k-p} \tag{36}$$

and

$$R_{m_3} = 0 + \frac{\phi_{m-1}}{M^2} + \frac{3}{2M^4}\sum_{k=0}^{m-1}\phi_k\phi_{m-1-k} + \frac{5}{2M^6}\sum_{k=0}^{m-1}\phi_{m-1-k}\sum_{p=0}^{k}\phi_p\times\phi_{k-p} \tag{37}$$

and

$$x_m = \begin{cases} 0 & m\geq 1 \\ 1 & m>1 \end{cases} \tag{38}$$

From Equation (27) and Equation (34), we now successively obtain the $u(\zeta)$ and The Equation (33) is linear and thus can be easily solved, especially by means of symbolic computation software such as Mathematica, Maple, MATLAB and so on. We used 10 terms in evaluating the approximate solution.

$$\phi(\zeta) = \phi_0(\zeta) + \sum_{m=1}^{9} \phi_m(\zeta) \tag{39}$$

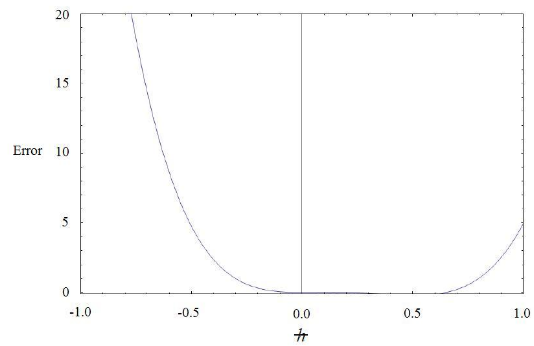

Figure 4: $\hbar$-curve of $\frac{\partial^2(0)}{\partial\zeta^2}$

Note that this series contains the auxiliary parameter $\hbar$, which influence its convergence region and rate. We should therefore focus on the choice of $\hbar$ by plotting of $\hbar$-curve. Figure 4 shows the $\hbar$-curve of $\frac{\partial^2(0)}{\partial\zeta^2}$.

We should select optimal $\hbar$ from the region in which the diagram is quite horizontal. Horizontal region is the optimal $\hbar$ region. Regarding Figure 4 optimal $\hbar$ equals 0.14365176. In this article we have obtained the values of u by applying HAM remarkable method as well as by numerical method and you will see the consequences of these methods in Figure 5. This diagram apparently shows that quite analytic method of HAM is so close to numerical solution with great exactness which is a token of its high accuracy.

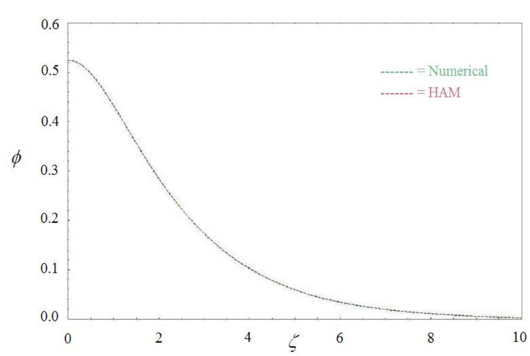

Figure 5: Comparison between HAM and Numerical Method

5 Conclusion

In this paper, we utilized the powerful method of homotopy analysis to obtain the nonlinear ion sound waves equation solution. We achieved a very good approximation with the numerical solution of the considered problem. In addition, this technique is algorithmic and it is easy to implementation by symbolic computation software, such as Maple and Mathematica. Different from all other analytic techniques, it provides us with a simple way to adjust and control the convergence region of approximate series solutions. Unlike perturbation methods, the HAM does not need any small parameter. It shows that the HAM is a very efficient method. We sincerely hope this method can be applied in a wider range.

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References

[1] N. Ali Hasan, Introduction to Perturbation Techniques, John Wiley Sons, New York, 1981.

[2] N. Ali Hasan, Problems in Perturbation, John Wiley Sons, 1985.

[3] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, Ph.D. Thesis, ShanghaiJiao TongUniversity, 1992.

[4] S.J. Liao, An approximate solution technique not depending on small parameters: A special example, Int. J. Nonlinear Mech. 30 (1995) 371-380.

[5] S.J. Liao, A kind of approximate solution technique which does not depend upon small parameters-II An application in fluid mechanics, Int. J. Nonlinear Mech. 32 (5) (1997) 815.

[6] S.J. Liao, An explicit, totally analytic approximate solution of Blasiuse' viscouse flow problems, Int. J. Nonlinear Mech. 34 (4) (1999) 759-778.

[7] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall/CRC Press, Boca Raton (2003).

[8] S.J. Liao,An explicit, totally analytic approximate solution for Blasiuse' viscous flow problems, Appl. Math. Comput. 147 (2004) 499-513.

[9] S.J. Liao, Comparison between the homotopy analysis method and homotopy perturbation method, Appl. Math. Comput. 169 (2005) 1186-1194.

[10] Alireza Doosthoseini, Ahmad Doosthoseini, Ensiyeh Doosthoseini, M. M. Sharpasand, An Analytic Solution for Finding the Natural Frequency of a Nonlinear Oscillating System, Applied Mathematical Sciences, 3 (2009) 1031-1039.

[11] S. Dinarvand, A. Doosthoseini, E. Doosthoseini, M.M. Rashidi, Series solutions for unsteady laminar MHD flow near forward stagnation point of an impulsively rotating and translating sphere in presence of buoyancy forces, Nonlinear Analysis: Real World Applications , Accepted Manuscript (2009).

[12] T. Hayat, M. Khan, M. Ayub, On the explicit analytic solutions of an Oldroyd 6-constant fluid, Int. J. Eng. Sci. 42 (2004), 123-135.

[13] S.J. Liao, I. Pop, Explicit analytic solution for similarity boundary layer equations, Int. J. Heat Mass Transfer 47 (2004) 75-85.

[14] S. Abbasbandy, The application of homotopy analysis method to nonlinear equations arising in heat transfer, Phys. Lett. A 360 (2006) 109-113.

[15] N.A. Krall and A.W. Trivelpiece, Principles of Plasma Physics, McGraw-Hill, New York, 1973.

[16] R.C. Davidson, Methods in Nonlinear Plasma Theory, Academic Press, London, 1972.

[17] D.J. Korteweg and G. de Vries, Phil. Mag. 39, 422, 1895.

[18] C. S. Gardner and G. K. Morikawa, Rep. NYU-9082. Courant Inst. of Math. Sci. New York Univ., New York, 1960.

[19] M.D. Kruskal and N.J. Zabusky, Annu. Rep. MATTQ-21. Plasma Phys. Lab., Princeton Univ., Princeton, New Jersey, 1963.

[20] N.J. Zabusky, Nonlinear Partial Differential Equations, Academic Press, New York, 1967.

[21] N.J. Zabusky, Mathematical Models in Physical Sciences, Printice-Hall, Englewood Cliffs, New Jersey, 1965.

[22] H. Washimi and T. Taniuti, Phys. Rev. Letters 17, 966, 1966.

[23] C. H. Su and C. S. Gardner, J. Math. Phys. 10, 536, 1969.

[24] J. M. Burgers, Proc. Roy. Neth. Acad. Sci. 43, 1, 1940.

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