

Imprecise Centralized Resource Allocation in DEA

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Abstract

The existing DEA model introduced for resource allocation that considered radial reductions of the total consumption of every input and total output production is guaranteed not to decrease defined for exact data. In this paper we improve this model to imprecise data such as, interval, ordinal and fuzzy data, from the dual form of Wang et al. model. We uses a fixed and unified production frontier to determine the efficiencies and resource allocation of decision-making units (DMUs) with interval input and output data. Ordinal preference information and fuzzy data are converted into interval data through the estimation of permissible intervals and α -level sets, respectively, and are incorporated into the interval DEA models.

Keywords: Data envelopment analysis (DEA); Imprecise data; Interval DEA model; Centralized planning

1 Introduction

Data envelopment analysis (DEA) is a mathematical programming for evaluating the relative efficiency of decision making units (DMUs). The first DEA model (CCR model), introduced by Charnes et al. [10], assumed for exact data, after that Cooper et al. [14] introduce the applications of DEA which the data was imprecise. In imprecise data envelopment analysis (IDEA) the data can be ordinal, interval and fuzzy. In dealing with this data the obtained models are usually non-linear. Cooper et al. [16]proposed some methods to convert the nonlinear model to a linear one. Zhu [3, 4] on the other hand shows that the non-linear IDEA can be solved in the standard linear CCR model via identifying a set of exact data from the imprecise input and output data. Despotis

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and Smirlis [5] also studied the problem of IDEA, but improved an alternative approach to deal with imprecise data in DEA. They converted a nonlinear DEA model to a LP equivalent by transforming only on the variables. The resulting efficiency scores were intervals. According to their approach, Wang et al. [2] developed a new pair of interval DEA models that can both overcome some of the shortcomings of the previous interval efficiency models in a simple, rational and effective way. Their new pair of interval DEA models will be developed for interval input and output data rather than for crisp input and output data. The final efficiency score for each DMU will be characterized by an interval bounded by the best lower bound efficiency and the best upper bound efficiency of each DMU, which they refer to as interval efficiency or efficiency interval.

Data envelopment analysis also can be used for the future programming of organizations and the response of the different policies which is related to target setting and resource allocation. The development of the scenario based target setting process will be pursued by demonstrating some of the features of data envelopment analysis in a target setting mode. Previous research by Golany [17], Thanassoulis and Dyson [19] and Athanassopoulos [7, 8, 9] have introduced models for assessing targets and allocating resources based on data envelopment analysis.

Lozano and Villa [1] proposed two phase model for centralized or intraorganizational resource allocation that has the two basic difference with conventional DEA model. First, instead of solving an independent LP model projecting each DMU in turn, all DMUs are simultaneously projected. Second, instead of reducing the inputs of any one DMU, the aim is to reduce the total input consumption of the DMUs. Lozano and Villa model [1], assumed for exact data.

In this paper we propose a model for centralized resource allocation with imprecise data such as ordinal, interval and fuzzy data, based on the dual form of Wang et al. model [2] that uses the production frontier as a benchmark to measure the lower bound efficiency of each DMU. We also uses a fixed and unified production frontier to measure the efficiencies of decision-making units (DMUs) with interval input and output data. Ordinal preference information and fuzzy data are converted into interval data through the estimation of permissible intervals and α -level sets, respectively, and are incorporated into the interval DEA models.

This paper proceeds as follows. In section 2 models and methodology are introduced . In section 3 our model is presented. Numerical example and conclusion are discussed in section 4 and 5 respectively.

2 Models and Methodology

2.1 DEA for Target Setting

Data envelopment analysis also can be used for the future programming of organizations and the response of the different policies which is related to target setting and resource allocation. Previous research by Golany [17], Thanassoulis and Dyson [19] and Athanas-

sopoulos [7, 8, 9] have introduced models for assessing targets an allocating resources based on data envelopment analysis.

2.2 Radial Centralized Resource Allocation

Lozano and Villa [1] established model (1) for resource allocation. Their main goals were first, to reduce total inputs consumption and total outputs production were guaranteed not to decrease. Second, instead of projecting each DMU separately, all of them will usually project to their MPSS position simultaneously.

The proposed model (1) for centralized resource allocation consisted of two phases. In the first phase, a reduction along all input dimensions is considered while, in the second phase, they also concentrate to the reduction of any input and/or expansion of any output. Let $j, r = 1, 2, \dots, n$, be indexes for DMUs; $i = 1, 2; \dots, m$, be index for inputs; $k = 1, 2, \dots, p$, be index for outputs; x_{ij} , amount of input i consumed by DMU $_j$; y_{kj} , quantity of output k produced by DMU $_j$, θ , radial contraction of total input vector; s_i , slack along the input dimension i ; t_k , additional increase along the output dimension k ; $(\lambda_{1r}, \lambda_{2r}, \dots, \lambda_{nr})$ vector is used for projecting DMU r .

The phase I model(1) is:

Model(1)Phase I/Radial/Input-Oriented

$$\begin{aligned} \min \quad & \theta \\ \text{s.t.} \quad & \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} x_{ij} \leq \theta \sum_{j=1}^n x_{ij}, \quad \forall i \\ & \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} y_{kj} \geq \sum_{r=1}^n y_{kr}, \quad \forall k \\ & \sum_{j=1}^n \lambda_{jr} = 1, \quad \forall r \\ & \lambda_{jr} \geq 0, \theta \text{ free.} \end{aligned}$$

Model (1)Phase II/Radial/Input-Oriented

$$\begin{aligned} \max \quad & \sum_{i=1}^m s_i + \sum_{k=1}^p t_k \\ \text{s.t} \quad & \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} x_{ij} = \theta^* \sum_{j=1}^n x_{ij} - s_i, \quad \forall i \\ & \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} y_{kj} = \sum_{r=1}^n y_{kr} + t_k, \quad \forall k \\ & \sum_{j=1}^n \lambda_{jr} = 1, \quad \forall r \\ & \lambda_{jr}, s_i, t_k \geq 0. \end{aligned}$$

Once the phase II model is solved, the corresponding vector $(\lambda_{1r}^*, \lambda_{2r}^*, \dots, \lambda_{nr}^*)$ defines for each DMU r the operating point at which it should aim. The inputs and outputs of each such point can be computed as

$$\hat{x}_{ir} = \sum_{j=1}^n \lambda_{jr}^* x_{ij}, \quad \forall i,$$

$$\hat{y}_{kr} = \sum_{j=1}^n \lambda_{jr}^* y_{kj}, \quad \forall k.$$

Proposition 1 For any DMUr, the operating point onto which it is projected by Model (1) Phase II/Radial/Input-oriented $(\hat{x}_{1r}, \hat{x}_{2r}, \dots, \hat{x}_{mr}, \hat{y}_{1r}, \hat{y}_{2r}, \dots, \hat{y}_{pr})$ is Pareto efficient.

Proof (see Lozano and Villa [1]).

Thus, the radial model proposed jointly projects each of the existing DMUs onto the Pareto efficiency frontier. This fact suggests the comparison with the conventional, separate projection of each DMU onto the efficient frontier

2.3 Interval DEA models based on interval arithmetic

Wang et al. [2] developed a new pair of interval DEA models that result to the best lower bound efficiency and the best upper bound efficiency of each DMU. As they mentioned, their new pair of interval DEA models will be improved for interval input and output data rather than for crisp input and output data.

Let $j = 1, 2, \dots, n$, be indexes for DMUs; $i = 1, 2, \dots, m$, be index for inputs; $k = 1, 2, \dots, p$, be index for outputs; x_{ij} , amount of input i consumed by DMU $_j$; y_{kj} , quantity of output k produced by DMU $_j$. It is clear that θ should also be an interval number, which showed by $[\theta^L, \theta^U]$. We also assume that all the input and output data x_{ij} and y_{rj} ($i = 1, \dots, m; k = 1, \dots, p; j = 1, \dots, n$) are supposed to locate within the upper and lower bounds represented by the intervals $[x_{ij}^L, x_{ij}^U]$ and $[y_{kj}^L, y_{kj}^U]$ where $x_{ij}^L \geq 0$ and $y_{kj}^L \geq 0$. Also j_o is the DMU under decision (usually denoted by DMU_{j_o}); u_k and v_i are the weights assigned to the outputs and inputs; $\theta_{j_o}^U$ stands for the best possible relative efficiency achieved by DMU_{j_o} when all the DMUs are in the state of best production activity, while $\theta_{j_o}^L$ stands for the lower bound of the best possible relative efficiency of DMU_{j_o} . They constitute a possible best relative efficiency interval $[\theta_{j_o}^L, \theta_{j_o}^U]$. In order to avoid the use of different production frontiers to measure the efficiencies of different DMUs, their new pair of interval DEA models developed as follow:

$$\begin{aligned} \max \quad & \theta_{j_o}^U = \sum_{k=1}^p u_k y_{kj_o}^U \\ \text{s.t.} \quad & \sum_{i=1}^m v_i x_{ij_o}^L = 1, \\ & \sum_{k=1}^p u_k y_{kj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n \\ & u_k, v_i \geq 0 \quad \forall i, k. \end{aligned} \quad (2)$$

$$\begin{aligned}
 \max \quad & \theta_{jo}^L = \sum_{k=1}^p u_k y_{kj}^L \\
 \text{s.t.} \quad & \sum_{i=1}^m v_i x_{ij}^U = 1, \\
 & \sum_{k=1}^p u_k y_{kj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n \\
 & u_k, v_i \geq 0 \quad \forall i, k. \quad (3)
 \end{aligned}$$

Model (2) determines the production frontier for all the DMUs and model (3) uses the production frontier as a benchmark to measure the lower bound efficiency of each DMU. In order to avoid the use of different production frontiers to measure the efficiencies of different DMUs, a pair of interval DEA models will be developed. The models are based on the interval arithmetic and always use the same constraint set, which forms a fixed and unified production frontier, for all DMUs as well as for the measures of both the lower and upper bound efficiencies(Wang et al. [2]).

Definition 1 *A DMU, DMU_o , is said to be DEA efficient if its best possible upper bound efficiency $\theta_{jo}^{*U} = 1$; otherwise, it is said to be DEA inefficient if $\theta_{jo}^{*U} < 1$.(see Wang et al., [2]).*

Now, we consider the dual form of Wang et al. models. Model (4) and (5) show the dual form of model (2) and (3) respectively with an extra convexity constraint of $\sum_{j=1}^n \lambda_j = 1$ (this is related to *Variable Returns to Scale* (Cooper et al. [15])).

$$\begin{aligned}
 \min \quad & \theta_{jo}^L \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij}^L \leq \theta_{jo}^L x_{ij}^U, \quad \forall i \\
 & \sum_{j=1}^n \lambda_j y_{kj}^U \geq y_{kj}^L, \quad \forall k \\
 & \sum_{j=1}^n \lambda_j = 1, \\
 & \lambda_j \geq 0 \quad \forall j, \theta_{jo}^L \text{ free.} \quad (4)
 \end{aligned}$$

$$\begin{aligned}
 \min \quad & \theta_{jo}^U \\
 \text{s.t.} \quad & \sum_{j=1}^n \gamma_j x_{ij}^L \leq \theta_{jo}^U x_{ij}^L, \quad \forall i \\
 & \sum_{j=1}^n \gamma_j y_{kj}^U \geq y_{kj}^U, \quad \forall k \\
 & \sum_{j=1}^n \gamma_j = 1,
 \end{aligned}$$

$$\gamma_j \geq 0, \theta_{jo}^U \text{ free.} \quad (5)$$

We call model (5), input-oriented Wang et al. envelopment model. After solving model (5) the corresponding vector $(\gamma_1^*, \gamma_2^*, \dots, \gamma_n^*)$ defines for each DMU_{jo} as the operating point which it should aim. We can compute the inputs and outputs of such a point from the formulation below

$$\hat{x}_{ij} = \sum_{j=1}^n \gamma_j^* x_{ij}^L, \quad \forall i, \quad \hat{y}_{kj} = \sum_{j=1}^n \gamma_j^* y_{kj}^U, \quad \forall k, \quad (6)$$

3 Radial Centralized Resource Allocation with Imprecise Data

In this section we extend model (1) for imprecise data such as ordinal, interval and fuzzy data. The models are based on model (4) and (5) but we also considered the projection of all of the DMUs simultaneously which is related to reduction of total input consumption of the DMUs (Lozano and Villa [1]).

Let $j, r = 1, 2, \dots, n$, be indexes for DMUs; $i = 1, 2, \dots, m$, be index for inputs; $k = 1, 2, \dots, p$, be index for outputs; x_{ij} , amount of input i consumed by DMU_j; y_{kj} , quantity of output k produced by DMU_j. It is obvious that θ should also be an interval number, which we denote by $[\theta^L, \theta^U]$, radial contraction of total input vector; s_i , slack along the input dimension i ; t_k , additional increase along the output dimension k ; $(\lambda_{1r}, \lambda_{2r}, \dots, \lambda_{nr})$ and $(\gamma_{1r}, \gamma_{2r}, \dots, \gamma_{nr})$ vectors for using the projection of DMU r . Without loss of generality, we assume that all the input and output data x_{ij} and y_{kr}, y_{kj} ($i = 1, \dots, m; k = 1, \dots, p; j, r = 1, \dots, n$) can not be exactly obtained due to the existence of uncertainty. They are only known to lie within the upper and lower bounds represented by the intervals $[x_{ij}^L, x_{ij}^U]$, $[y_{kj}^L, y_{kj}^U]$ and $[y_{kr}^L, y_{kr}^U]$, where $x_{ij}^L \geq 0, y_{kj}^L \geq 0$ and $y_{kr}^L \geq 0$.

The phase I model is:

Model(7)Phase I/Radial/Input-Oriented

$$\begin{aligned} \min \quad & \theta^L \\ \text{s.t.} \quad & \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} x_{ij}^L \leq \theta^L \sum_{j=1}^n x_{ij}^U, \quad \forall i \\ & \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} y_{kj}^U \geq \sum_{r=1}^n y_{kr}^L, \quad \forall k \\ & \sum_{j=1}^n \lambda_{jr} = 1, \quad \forall r \\ & \lambda_{jr} \geq 0, \theta^L \text{ free.} \end{aligned}$$

Model (7)Phase II/Radial/Input-Oriented

$$\max \quad \sum_{i=1}^m s_i + \sum_{k=1}^p t_k$$

$$\begin{aligned}
 \text{s.t.} \quad & \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} x_{ij}^L = \theta^{L*} \sum_{j=1}^n x_{ij}^U - s_i, & \forall i \\
 & \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} y_{kj}^U = \sum_{r=1}^n y_{kr}^L + t_k, & \forall k \\
 & \sum_{j=1}^n \lambda_{jr} = 1, & \forall r \\
 & \lambda_{jr}, s_i, t_k \geq 0.
 \end{aligned}$$

Model(8)Phase I/Radial/Input-Oriented

$$\begin{aligned}
 \min \quad & \theta^U \\
 \text{s.t.} \quad & \sum_{r=1}^n \sum_{j=1}^n \gamma_{jr} x_{ij}^L \leq \theta^U \sum_{j=1}^n x_{ij}^L, & \forall i \\
 & \sum_{r=1}^n \sum_{j=1}^n \gamma_{jr} y_{kj}^U \geq \sum_{r=1}^n y_{kr}^U, & \forall k \\
 & \sum_{j=1}^n \gamma_{jr} = 1, & \forall r \\
 & \gamma_{jr} \geq 0, \theta^U \text{ free.}
 \end{aligned}$$

Model (8)Phase II/Radial/Input-Oriented

$$\begin{aligned}
 \max \quad & \sum_{i=1}^m s_i + \sum_{k=1}^p t_k \\
 \text{s.t.} \quad & \sum_{r=1}^n \sum_{j=1}^n \gamma_{jr} x_{ij}^L = \theta^{U*} \sum_{j=1}^n x_{ij}^L - s_i, & \forall i \\
 & \sum_{r=1}^n \sum_{j=1}^n \gamma_{jr} y_{kj}^U = \sum_{r=1}^n y_{kr}^U + t_k, & \forall k \\
 & \sum_{j=1}^n \gamma_{jr} = 1, & \forall r \\
 & \gamma_{jr}, s_i, t_k \geq 0.
 \end{aligned}$$

Once the phase II for the model (8) is solved, the corresponding vector $(\gamma_{1r}^*, \gamma_{2r}^*, \dots, \gamma_{nr}^*)$ defines for each DMU r the operating point which it should aim. The inputs and outputs of each such point can be computed as

$$x'_{ir} = \sum_{j=1}^n \gamma_{jr}^* x_{ij}^L, \quad \forall i, \quad y'_{kr} = \sum_{j=1}^n \gamma_{jr}^* y_{kj}^U, \quad \forall k, \quad (9)$$

By models(7) and (8) introduced in this section we have, first, instead of solving an independent LP model projecting each DMU in turn, all DMUs are simultaneously projected. Second, instead of reducing the inputs of any one DMU, the aim is to reduce the total input consumption of the DMUs. Third, we can have imprecise data such as:

fuzzy, ordinal and interval data. By using formulation (9), we defined exact data as an operating point for all interval data. In Definition 2 Wang et al. [2] determined the efficiency when $\theta_{jo}^{*U} = 1$, according to their definition we defined Proposition 2 and proved it in Appendix A.

Proposition 2 *For any DMUr, the operating point onto which it is projected by Model (8) Phase II/Radial/Input-oriented $(x'_{1r}, x'_{2r}, \dots, x'_{mr}, y'_{1r}, y'_{2r}, \dots, y'_{pr})$ is Pareto efficient.*

It is essential to say that our operating point from model (8) Phase II are introduced with exact data, but in the case of the necessity of interval operating point we also can solve model (7) Phase II (as Wang et al. [2] mentioned that θ_{jo}^L stands for the lower bound of the best possible relative efficiency of DMU_o) and follow the formulations below for computing the lower and upper bound of the operating point.

$$\hat{x}_{ir}^U = \sum_{j=1}^n \lambda_{jr}^* x_{ij}^U, \quad \forall i, \quad \hat{y}_{kr}^L = \sum_{j=1}^n \lambda_{jr}^* y_{kj}^L, \quad \forall k, \quad (10)$$

$$\hat{x}_{ir}^L = \sum_{j=1}^n \gamma_{jr}^* x_{ij}^L, \quad \forall i, \quad \hat{y}_{kr}^U = \sum_{j=1}^n \gamma_{jr}^* y_{kj}^U, \quad \forall k, \quad (11)$$

We should mention that the operating point from formulation (10) are not necessary Pareto efficient.

3.1 Incorporation of ordinal preference information and fuzzy data into the interval DEA models

Wang et al. [2] mentioned that, in real decision-making and evaluation problems, ordinal preference information and/or fuzzy data are appeared. They also, discussed how to transform ordinal preference information and fuzzy data into interval data. We use Wang et al. method [2] to deal with fuzzy and ordinal data.

3.1.1 The transformation of ordinal preference information

Let us take the transformation of ordinal preference information about the output y_{kj} ($j = 1, \dots, n$) (also for y_{kr} , $r = 1, \dots, n$) for example. The ordinal preference information about input and other output data can be converted in the same way.

For weak ordinal preference information $y_{k1} \geq y_{k2} \geq \dots \geq y_{kn}$, we have the following ordinal relationships after scale transformation:

$$1 \geq \check{y}_{k1} \geq \check{y}_{k2} \geq \dots \geq \check{y}_{kn} \geq \sigma_k,$$

where σ_k is a small positive number reflecting the ratio of the possible minimum of $\{y_{kj} | j = 1, \dots, n\}$ to its possible maximum. It can be approximately estimated by the DM. We refer to it as the ratio parameter for convenience. The resultant permissible interval for each \check{y}_{kj} is given by

$$\check{y}_{kj} \in [\sigma_k, 1], \quad j = 1, \dots, n.$$

For strong ordinal preference information $y_{k1} > y_{k2} > \dots > y_{kn}$, we have the following ordinal relationships after scale transformation:

$$1 \geq \check{y}_{k1}, \quad \check{y}_{kj} \geq \chi_k \check{y}_{k,j+1} \quad (j = 1, \dots, n - 1) \quad \text{and} \quad \check{y}_{kn} \geq \sigma_k,$$

where χ_k is a preference intensity parameter satisfying $\chi_k > 1$ provided by the DM and σ_k is the ratio parameter also provided by the DM. The resultant permissible interval for each \hat{y}_{kj} can be derived as follows:

$$\check{y}_{kj} \in [\sigma_k \chi_k^{n-j}, \chi_k^{1-j}], \quad j = 1, \dots, n \quad \text{with} \quad \sigma_k \leq \chi_k^{1-n}.$$

3.1.2 *The transformation of fuzzy data*

In order to extend the usage of interval centralized model (models (7) and (8)) to deal with imprecise data, fuzzy data will be transformed into interval data by using the α -level sets (Zimmermann [6]). Let the inputs \tilde{x}_{ij} and outputs \tilde{y}_{kj} (also for \tilde{y}_{kr}) be fuzzy data with membership functions $\mu_{\tilde{x}_{ij}}$ and $\mu_{\tilde{y}_{kj}}$, respectively, and $S(\tilde{x}_{ij})$ and $S(\tilde{y}_{kj})$ be the support of \tilde{x}_{ij} and \tilde{y}_{kj} , respectively. Then the α -level sets of \tilde{x}_{ij} and \tilde{y}_{kj} can be defined as

$$\begin{aligned} (x_{ij})_\alpha &= \{x_{ij} \in S(\tilde{x}_{ij}) \mid \mu_{\tilde{x}_{ij}}(x_{ij}) \geq \alpha\} \\ &= [\min_{x_{ij}} \{x_{ij} \in S(\tilde{x}_{ij}) \mid \mu_{\tilde{x}_{ij}}(x_{ij}) \geq \alpha\}, \max_{x_{ij}} \{x_{ij} \in S(\tilde{x}_{ij}) \mid \mu_{\tilde{x}_{ij}}(x_{ij}) \geq \alpha\}] \\ &\forall i, j, \\ (y_{kj})_\alpha &= \{y_{kj} \in S(\tilde{y}_{kj}) \mid \mu_{\tilde{y}_{kj}}(y_{kj}) \geq \alpha\} \\ &= [\min_{y_{kj}} \{y_{kj} \in S(\tilde{y}_{kj}) \mid \mu_{\tilde{y}_{kj}}(y_{kj}) \geq \alpha\}, \max_{y_{kj}} \{y_{kj} \in S(\tilde{y}_{kj}) \mid \mu_{\tilde{y}_{kj}}(y_{kj}) \geq \alpha\}] \\ &\forall k, j, \end{aligned}$$

where $0 < \alpha \leq 1$. By setting different levels of confidence, namely $1 - \alpha$, fuzzy data are accordingly transformed into different α -level sets $\{(x_{ij})_\alpha \mid 0 < \alpha \leq 1\}$ and $\{(y_{kj})_\alpha \mid 0 < \alpha \leq 1\}$, which are all intervals. The widest input and output intervals will be $(x_{ij})_0 = \{x_{ij} \in S(\tilde{x}_{ij}) \mid \mu_{\tilde{x}_{ij}}(x_{ij}) > 0\} = [x_{ij}^L, x_{ij}^U]$ and $(y_{kj})_0 = \{y_{kj} \in S(\tilde{y}_{kj}) \mid \mu_{\tilde{y}_{kj}}(y_{kj}) > 0\} = [y_{kj}^L, y_{kj}^U]$ and $x_{ij}^L, x_{ij}^U, y_{kj}^L$ and y_{kj}^U are the lower and upper bounds of fuzzy data \tilde{x}_{ij} and \tilde{y}_{kj} , respectively. The production frontier will obviously be determined by interval data $[x_{ij}^L, x_{ij}^U]$ and $[y_{kj}^L, y_{kj}^U]$ ($i = 1, \dots, m; j = 1, \dots, n; k = 1, \dots, p$). Any α -level sets input and output data $(x_{ij})_\alpha = [(x_{ij})_\alpha^L, (x_{ij})_\alpha^U]$, $(y_{kj})_\alpha = [(y_{kj})_\alpha^L, (y_{kj})_\alpha^U]$ and $(y_{kr})_\alpha = [(y_{kr})_\alpha^L, (y_{kr})_\alpha^U]$ should be measured using the identical production frontier. So, the interval DEA models for fuzzy input and output data will be as follows:

Model(9)Phase I/Radial/Input-Oriented

$$\begin{aligned}
& \min \quad (\theta)_\alpha^L \\
& \text{s.t} \quad \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} (x_{ij})_\alpha^L \leq (\theta)_\alpha^L \sum_{j=1}^n (x_{ij})_\alpha^U, \quad \forall i \\
& \quad \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} (y_{kj})_\alpha^U \geq \sum_{r=1}^n (y_{kr})_\alpha^L, \quad \forall k \\
& \quad \sum_{j=1}^n \lambda_{jr} = 1, \quad \forall r \\
& \quad \lambda_{jr} \geq 0, (\theta)_\alpha^L \text{ free.}
\end{aligned}$$

Model (9)Phase II/Radial/Input-Oriented

$$\begin{aligned}
& \max \quad \sum_{i=1}^m s_i + \sum_{k=1}^p t_k \\
& \text{s.t} \quad \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} (x_{ij})_\alpha^L = (\theta^*)_\alpha^L \sum_{j=1}^n (x_{ij})_\alpha^U - s_i, \quad \forall i \\
& \quad \sum_{r=1}^n \sum_{j=1}^n \lambda_{jr} (y_{kj})_\alpha^U = \sum_{r=1}^n (y_{kr})_\alpha^L + t_k, \quad \forall k \\
& \quad \sum_{j=1}^n \lambda_{jr} = 1, \quad \forall r \\
& \quad \lambda_{jr}, s_i, t_k \geq 0.
\end{aligned}$$

Model(10)Phase I/Radial/Input-Oriented

$$\begin{aligned}
& \min \quad (\theta)_\alpha^U \\
& \text{s.t.} \quad \sum_{r=1}^n \sum_{j=1}^n \gamma_{jr} (x_{ij})_\alpha^L \leq (\theta)_\alpha^U \sum_{j=1}^n (x_{ij})_\alpha^L, \quad \forall i \\
& \quad \sum_{r=1}^n \sum_{j=1}^n \gamma_{jr} (y_{kj})_\alpha^U \geq \sum_{r=1}^n (y_{kr})_\alpha^U, \quad \forall k \\
& \quad \sum_{j=1}^n \gamma_{jr} = 1, \quad \forall r \\
& \quad \gamma_{jr} \geq 0, (\theta)_\alpha^U \text{ free.}
\end{aligned}$$

Model (10)Phase II/Radial/Input-Oriented

$$\begin{aligned}
& \max \quad \sum_{i=1}^m s_i + \sum_{k=1}^p t_k \\
& \text{s.t} \quad \sum_{r=1}^n \sum_{j=1}^n \gamma_{jr} (x_{ij})_\alpha^L = (\theta^*)_\alpha^U \sum_{j=1}^n (x_{ij})_\alpha^L - s_i, \quad \forall i
\end{aligned}$$

$$\sum_{r=1}^n \sum_{j=1}^n \gamma_{jr} (y_{kj})_{\alpha}^U = \sum_{r=1}^n (y_{kr})_{\alpha}^U + t_k, \quad \forall k$$

$$\sum_{j=1}^n \gamma_{jr} = 1, \quad \forall r$$

$$\gamma_{jr}, s_i, t_k \geq 0.$$

where $(\theta)_{\alpha}^U$ and $(\theta)_{\alpha}^L$ are, respectively, the upper and lower bounds of the best possible relative efficiency for DMUs under given α -level sets, which form an efficiency interval denoted by $(\theta)_{\alpha} = [(\theta)_{\alpha}^L, (\theta)_{\alpha}^U]$. Note that we also use one production frontier for every α -level rather than different production frontiers for different α -levels (Wang et al. [2]).

4 Numerical Illustration

In this section we will apply our models to manufacturing industries. The data collected from Wang et al. [2], but according to our subject we just make some changes to the data. Our numerical example included in two parts. In each part we compare our models to the dual form of Wang et al. models.

Example 1. Table 1 presents the data set used by Wang et al. [2]. There are seven manufacturing industries (DMUs) participating in the evaluation, each consuming two inputs (Capital and Labor) and producing one output (Gross output value). We have to mention that we supposed all the manufacturing industries are dependent because the use of independent ones is not consequent with the hypothesis of the existence of a resource allocation scenario. The data are all estimated and are thus imprecise and only known within the prescribed bounds, which are listed in Table 1. Table 2 reports the results from model (8) Phase II and model (5) (we call it Input-Oriented Wang et al. model) , respectively (are solved by LINGO, a powerful professional software package) , when used the interval data from Table 1. We should mention that in Table 2, $\sum_{j=1}^n x_{ij}^L$ and $\sum_{j=1}^n y_{rj}^U$ are the total lower input and total upper output of the existing DMUs.

Note that the I-O Wang et al. model reduces both total inputs with the benefit of significant total output increases. As for the proposed models, the more balanced results are obtained for the Radial model, which has a larger input reduction than I-O Wang et al. model although with a smaller output increase.

Example 2. Table 3 also presents the data set used by Wang et al [2] but we change the data a little bit to be suitable for our reallocating process. For example, there are eight manufacturing enterprise (DMUs). Each manufacturing enterprise manufactures the same type of product, but the qualities are different. Therefore, both the gross output value (GOV) and the product quality (PQ) are considered as outputs. The inputs include Capital and the number of employees (NOE), whose data are known exactly. The data about the gross output values, however, are imprecise due to the unavailability at the

Table 1
Data for seven DMUs

DMU	Inputs		Output
	Capital	Labor	Gross output value
1	[564403, 621755]	[67111, 743281]	[806549, 866063]
2	[614371, 669665]	[685943, 742345]	[917507, 985424]
3	[762203, 798427]	[762207, 805677]	[1117142, 1195562]
4	[862016, 937044]	[779894, 846496]	[1206179, 1261031]
5	[1016898, 1082662]	[799714, 877137]	[1381315, 1462543]
6	[1164350, 1267970]	[807172, 889416]	[1497679, 1652787]
7	[1731916, 1816008]	[818090, 895746]	[1702249, 1812655]

Source: Wang et al. [2].

Table 2
Summery of results from Wang et al. [2] data set

DMU	I-O Wang et al. model			Our approach		
	(\hat{x}_{1j})	(\hat{x}_{2j})	(\hat{y}_{1j})	(x'_{1j})	(x'_{2j})	(y'_{1j})
1	564403	674111	866063	614371	685943	945424
2	614371	685943	985424	1029236	741243	1321432
3	762203	762207	1195562	1164350	807172	1652787
4	830982	751817	1261031	614371	685943	985424
5	1002221	789256	1467812	1164350	807172	1652787
6	1164350	807172	1652787	1164350	807172	1652787
7	1731916	818090	1812655	614371	685943	985424
Total	6670446	5288596	9241334	6365399	5220588	9236065
$\sum_{j=1}^n x_{ij}^L$	6716157	5327131		6716157	5327131	
$\sum_{j=1}^n y_{rj}^U$			9236065			9236065

moment and are thus estimated. Some of them are given as interval numbers and some as triangular fuzzy numbers. The product quality is a qualitative index and is given as strong ordinal preference information that is obtained from the evaluation of customers to their products. The data are presented in Table 3.

Suppose the preference intensity parameter and the ratio parameter about the strong ordinal preference information are given (or estimated) as $\chi_2 = 1.12$ and $\sigma_2 = 0.1$, respectively. By converting ordinal data into interval data described in Section 3, we can derive an interval estimate for the product quality of each DMU, which is shown in the last column of Table 4.

Since the GOV index for DMU2, DMU4 and DMU6 is given in the form of triangular fuzzy number, i.e. $GOV_j = (GOV_j^L, GOV_j^M, GOV_j^U)$

($j = 2, 4, 6$), their membership functions can be expressed as

$$\mu_{GOV_j}(x_j) = \begin{cases} \frac{x_j - GOV_j^L}{GOV_j^M - GOV_j^L}, & GOV_j^L \leq x_j \leq GOV_j^M \\ \frac{GOV_j^U - x_j}{GOV_j^U - GOV_j^M}, & GOV_j^M \leq x_j \leq GOV_j^U, \quad j = 2, 4, 6 \\ 0, & x_j \notin [GOV_j^L, GOV_j^U] \end{cases}$$

where GOV_j^L, GOV_j^M and GOV_j^U are the lower bound, most likely and upper bound values of GOV_j , respectively. For a given α -level, the corresponding α -level sets are given by

$$(GOV_j)_\alpha = \{x_j \in S(GOV_j) | \mu_{GOV_j}(x_j) \geq \alpha\}$$

$$= [(GOV_j)_\alpha^L, (GOV_j)_\alpha^U]$$

$$= [GOV_j^L + \alpha(GOV_j^M - GOV_j^L), GOV_j^U - \alpha(GOV_j^U - GOV_j^M)], j = 2, 4, 6.$$

As for exact data, they can be viewed as a special case of interval data with the lower and upper bounds being equal. Therefore, all the input and output data are now transformed into interval numbers and can be evaluated using interval DEA models. In Table 5 we have compared the results from model (8) Phase II and model (5). Since it is not logical to reallocate product quality between the productive units, in Table 5 we did not considered output 2.

Table 3

Data for eight DMUs with two inputs and two outputs

DMU	Inputs		Outputs	
	C	NOE	GOV	PQ ^a
1	2166	1875	[14548, 14950]	2
2	1455	1342	[12468, 13045, 13584]	7
3	2562	2359	[17896, 18452]	1
4	2346	2018	[14968, 15673, 15900]	3
5	1517	1548	[13980, 14638]	8
6	2034	1760	[14026, 14324, 14582]	6
7	2256	1982	[16542, 17169]	5
8	2465	2245	[17600, 18256]	4

^aOrdinal scale from 1 = best to 8 = worst with the preference intensity parameter $\chi_2 = 1.12$ and the ratio parameter $\sigma_2 = 0.1$

Source: Wang et al. [2].

Table 4

The input–output data for the eight DMUs after the transformation of ordinal preference information.

DMU	Inputs		Outputs	
	C	NOE	GOV	PQ
1	2166	1875	[14548, 14950]	[0.1973823, 0.892857]
2	1455	1342	[12468, 13045, 13584]	[0.1120000, 0.506631]
3	2562	2359	[17896, 18452]	[0.2210681, 1.000000]
4	2346	2018	[14968, 15673, 15900]	[0.1762342, 0.797194]
5	1517	1548	[13980, 14638]	[0.1000000, 0.452349]
6	2034	1760	[14026, 14324, 14582]	[0.1254400, 0.567427]
7	2256	1982	[16542, 17169]	[0.1404928, 0.635518]
8	2465	2245	[17600, 18256]	[0.1573519, 0.711780]

Source: Wang et al. [2].

Table 5

Summary of results for the Wang et al. [2] data set.

	$\alpha = 0$			$\alpha = 0.25$			$\alpha = 0.5$		
	I-O Wang Model	Our approach	Existing	I-O Wang Model	Our approach	Existing	I-O Wang Model	Our approach	Existing
Total									
Input(C)	16172	15962	16801	16179	15962	16801	16186	15512	16801
Input(NOE)	14774	14533	15138	14779	14533	15138	14783	14684	15138
Output (GOV)	127531	127789	127531	127275	127311	127275	127202	127020	127020
	$\alpha = 0.75$			$\alpha = 1$					
Input(C)	16192	15512	16801	16198	15512	16801			
Input(NOE)	14786	14684	15138	14790	14684	15138			
Output (GOV)	126764	126928	126764	126507	126655	126507			

According to the results in Table 5 our approach (model (8) Phase II) has a larger input reduction and the first output (GOV) increase than I-O Wang et al. model (5), except in the case of $\alpha = 0.5$. The operating point will be changed under different

α -level sets, in this case the DM can determine which of the operating point can be chosen. High α means precision of the interval chosen and low α means high confidence in the result. A risk-averse assessor or DM might choose a high alpha because of strong dislike of uncertainty (fuzziness), while a risk-taking assessor or DM might prefer a low alpha because of seeking of risk (Wang et al. [2]).

5 Conclusion

In this paper we have developed DEA model for centralized resource allocation to deal with imprecise data such as interval, ordinal and fuzzy data. Compared with the existing DEA model has been presented for centralized resource allocation for exact data that project all DMUs onto the efficient frontier and consider radial reductions of the total consumption of all the inputs, our model is presented for imprecise data as a new subject of the existing DEA models for setting targets and resource allocation with exact data. To deal with imprecise data and avoiding different efficient frontier we used the dual form of the Wang et al. model [2] which is a interval DEA model computes the best possible relative efficiency and utilize a fixed and unified production frontier as a benchmark to measure the efficiencies of all DMUs, which makes our models more rational and more reliable. Two numerical examples have illustrated the usage and the advantages of our interval DEA model for imprecise data.

Appendix A

Proposition 1 For any DMU r , the operating point onto which it is projected by Model (8) Phase II/Radial/Input-oriented $(x'_{1r}, x'_{2r}, \dots, x'_{mr}, y'_{1r}, y'_{2r}, \dots, y'_{pr})$ is Pareto efficient.

let us assumed that the proposition is false and we will arrive at a contradiction. If $(x'_{1r}, x'_{2r}, \dots, x'_{mr}, y'_{1r}, y'_{2r}, \dots, y'_{pr})$ is not technically efficient then considering model (4), there exist a vector $(\gamma_{1r}, \gamma_{2r}, \dots, \gamma_{nr})$ satisfying $\sum_{j=1}^n \gamma_{jr} = 1$

$$\bar{x}_{ir} = \sum_{j=1}^n \gamma_{jr} x_{ij}^L \leq x'_{ir} \quad \forall i,$$

$$\bar{y}_{kr} = \sum_{j=1}^n \gamma_{jr} y_{kj}^U \geq y'_{kr}, \quad \forall k,$$

such that at least for one input i or one k the previous inequality is strict. Let us assume that it is for input i for which

$$\bar{x}_{ir} = \sum_{j=1}^n \gamma_{jr} x_{ij}^L < x'_{ir}.$$

Then, using in respect of DMU r the vector $(\gamma_{1r}, \gamma_{2r}, \dots, \gamma_{nr})$ instead of optimal one $(\gamma_{1r}^*, \gamma_{2r}^*, \dots, \gamma_{nr}^*)$ would lead to a feasible solution of Model (8)Phase II/Radial/Input-Oriented having an objective function value

$$\sum_{i=1}^m s_i^* + \sum_{k=1}^p t_k^* + \sum_{i=1}^m (x'_{ir} - \bar{x}_{ir}) - \sum_{k=1}^p (\bar{y}_{kr} - y'_{kr}) > \sum_{i=1}^m s_i^* + \sum_{k=1}^p t_k^*.$$

This is higher than the initial optimum, which leads to a contradiction. The same conclusion is obtained if it is for certain output k for which

$$\bar{y}_{kr} = \sum_{j=1}^n \gamma_{jr} y_{kj}^U > y'_{kr}. \quad \blacksquare$$

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