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A New Measure of Probabilistic Entropy and its Properties

Rajneesh Kumar

Department of Mathematics
Kurukshetra University
Kurukshetra, India
rajneesh_kuk@rediffmail.com

Subash Kumar

Department of Mathematics
S.S.M.I.T, Dinanagar, India

Anil Kumar

Department of Mathematics
S.S.C.E.T., Badhani
Pathankot, India

INTRODUCTION

The concept of entropy in communication theory was first introduced by Shannon [13] and it was then realized that entropy is a property of any stochastic system and the concept is now used widely in different disciplines. The tendency of the systems to become more disordered over time is described by the second law of thermodynamics, which states that the entropy of the system cannot spontaneously decrease. Today, information theory is still principally concerned with communications systems, but there are widespread applications in statistics, information processing and computing. A great deal of insight is obtained by considering entropy equivalent to uncertainty, the generalized theory of which has well been explained by Zadeh [15].

The uncertainty associated with probability of outcomes, known as probabilistic uncertainty, is called entropy, since this is the terminology that is well entrenched in the literature. Shannon [13] introduced the concept of information theoretic entropy by associating uncertainty with every probability distribution $P=(p_1, p_2, ..., p_n)$ and found that there is a unique function that can measure the uncertainty, is given by

$$H(P) \quad = -\sum_{i=1}^{n} p_i \text{ In } p_i \tag{1.1}$$

The probabilistic measure of entropy (1.1) possesses a number of interesting properties. Immediately, after Shannon gave his measure, research workers in many fields saw the potential of the application of this expression and a large number of other measures of information theoretic entropies were derived. Renyi [11] defined entropy of order α as:

$$H_\alpha(P)=\frac{1}{1-\alpha} In\left(\sum_{i=1}^{n} p_i^{\alpha} \Big/ \sum_{i=1}^{n} p_i\right), \alpha \neq 1, \alpha > 0 \tag{1.2}$$

which includes Shannon's [13] entropy as a limiting case as $\alpha \to 1$. Zyczkowski [16] explored the relationships between the Shannon's [13] entropy and Renyi's [11] entropies of integer order.

Havrada and Charvat [5] introduced first non-additive entropy, given by:

$$H^{\alpha}(P)=\frac{\left[\sum_{i=1}^{n} p_i^{\alpha}\right]-1}{2^{1-\alpha}-1}, \alpha \neq 1, \alpha > 0 \tag{1.3}$$

Kapur [7] generalized Renyi's [11] measure further to give a measure of entropy of order 'α' and type 'β', viz.,

$$H_{\alpha,\beta}(P)=\frac{1}{1-\alpha} In\left(\sum_{i=1}^{n} p_i^{\alpha+\beta-1} \Big/ \sum_{i=1}^{n} p_i^{\beta}\right), \alpha \neq 1, \alpha > 0, \beta > 0, \alpha+\beta-1 > 0 \tag{1.4}$$

The measure (1.4) reduces to Renyi's [11] measure when β = 1, to Shannon's [13] measure when β = 1, α → 1. When β = 1, α → ∞, it gives the measure

$$H_\infty(P) = -\ln p_{\max}$$

Many other probabilistic measures of entropy have been discussed and derived by Brissaud [1], Chakrabarti [2], Chen [3], Garbaczewski [4], Herremoes [6], Lavenda [8], Nanda and Paul [9], Rao, Yunmei and Wang [10], Sergio [12], Sharma and Taneja [14] etc. The applications of the results obtained by various authors have been provided to various fields of Mathematical Sciences. In section 2, we have introduced a new generalized probabilistic information theoretic measure.

2. A New Generalized Information Theoretic Measure based upon Probability Distributions

In this section, we propose a new generalized information measure for a probability distribution $P=\left\{\left(p_1, p_2, ..., p_n\right), p_i \geq 0, \sum_{i=1}^{n} p_i = 1\right\}$ and study their essential and desirable properties. This generalized entropy depending upon n real parameters $\alpha_1, \alpha_2, ..., \alpha_n$ is given by the following mathematical expression:

$$H^{n}_{\alpha,\alpha_1,...,\alpha_n}(P)=\frac{\sum_{i=1}^{n} p_i^{\alpha+\alpha_1+...+\alpha_n}-1}{2^{1-\alpha-\alpha_1-....-\alpha_n}-1} \tag{2.1}$$

where

$$\alpha+\sum_{i=1}^{n}\alpha_i \neq 1, \alpha+\sum_{i=1}^{n}\alpha_i > 1 \quad \text{and} \quad \alpha \neq 1, \alpha > 0, \alpha_i \geq 0 \tag{2.2}$$

If $\sum_{i=1}^{n}\alpha_i = 0$, then $\alpha \neq 1, \alpha > 0$. Thus, we see that the proposed measure (2.1) becomes

$$H^{n}_{\alpha}(P)=\frac{\sum_{i=1}^{n} p_i^{\alpha}-1}{2^{1-\alpha}-1} \tag{2.3}$$

which is Havrada and Charvat's [5] measure of entropy of order α. The measure (2.3) again reduces to Shannon's [13] measure of entropy as $\alpha \to 1$. Thus, we see that the measure proposed in equation (2.1) is a generalized measure of entropy. Next, we study some important properties of this generalized measure.

The measure (2.1) satisfies the following properties:

(i) It is continuous function of $p_1, p_2, ..., p_n$, so that it changes by a small amount when $p_1, p_2, ..., p_n$ change by small amounts.

(ii) It is permutationally symmetric function of $p_1, p_2, ..., p_n$, that is, it does not change when $p_1, p_2, ..., p_n$ are permuted among themselves.

(iii) $H^{n}_{\alpha,\alpha_1,...,\alpha_n}(P) \geq 0$

(iv) $$H^{n+1}_{\alpha,\alpha_1,...,\alpha_n}(p_1, p_2, ..., p_n, 0)=\frac{\sum_{i=1}^{n} p_i^{\alpha+\alpha_1+....+\alpha_n}-1}{2^{1-\alpha-\alpha_1-....-\alpha_n}-1}$$

$$= H^n_{\alpha,\alpha_1,\ldots,\alpha_n}(P)$$

This property says that entropy does not change by the inclusion of an impossible event with probability zero.

(v) Since $H^n_{\alpha,\alpha_1,\ldots,\alpha_n}(P)$ is an entropy measure, its maximum value must occur. To find the maximum value, we proceed as follows:

Let

$$f(p) = \frac{\sum_{i=1}^{n} p_i^{\alpha+\alpha_1+\ldots.+\alpha_n} - 1}{2^{1-\alpha-\alpha_1-\ldots.-\alpha_n} - 1} - \lambda\left(\sum_{i=1}^{n} p_i - 1\right)$$

Then, we have

$$\frac{\partial f}{\partial p_1} = \frac{(\alpha+\alpha_1+\ldots+\alpha_n)p_1^{\alpha+\alpha_1+\ldots.+\alpha_n-1}}{2^{1-\alpha-\alpha_1-\ldots.-\alpha_n}-1} - \lambda$$

$$\frac{\partial f}{\partial p_2} = \frac{(\alpha+\alpha_1+\ldots+\alpha_n)p_2^{\alpha+\alpha_1+\ldots.+\alpha_n-1}}{2^{1-\alpha-\alpha_1-\ldots.-\alpha_n}-1} - \lambda$$

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$$\frac{\partial f}{\partial p_n} = \frac{(\alpha+\alpha_1+\ldots+\alpha_n)p_n^{\alpha+\alpha_1+\ldots.+\alpha_n-1}}{2^{1-\alpha-\alpha_1-\ldots.-\alpha_n}-1} - \lambda$$

For maximum value, we take

$$\frac{\partial f}{\partial p_1} = \frac{\partial f}{\partial p_2} = \ldots = \frac{\partial f}{\partial p_n} = 0$$

which gives

$$\frac{(\alpha+\alpha_1+\ldots+\alpha_n)p_1^{\alpha+\alpha_1+\ldots.+\alpha_n-1}}{2^{1-\alpha-\alpha_1-\ldots.-\alpha_n}-1} = \frac{(\alpha+\alpha_1+\ldots+\alpha_n)p_2^{\alpha+\alpha_1+\ldots.+\alpha_n-1}}{2^{1-\alpha-\alpha_1-\ldots.-\alpha_n}-1} = \ldots = \frac{(\alpha+\alpha_1+\ldots+\alpha_n)p_n^{\alpha+\alpha_1+\ldots.+\alpha_n-1}}{2^{1-\alpha-\alpha_1-\ldots.-\alpha_n}-1}$$

which is possible only if $p_1 = p = \ldots = p_n$

Thus $\sum_{i=i}^{n} p_i = 1$ gives $p_1 = p = \ldots = p_n = \frac{1}{n}$

Hence, we see that the generalized entropy measure (2.1) possesses maximum value and this value subject to natural constraint $\sum_{i=i}^{n} p_i = 1$ arises when $p_1 = p_2 = \ldots = p_n = \frac{1}{n}$.

This result is most desirable.

(vi) The maximum value is an increasing function of n.

To prove this result, we have

$$f(p) = \frac{n^{1-\alpha-\alpha_1-\ldots-\alpha_n} - 1}{2^{1-\alpha-\alpha_1-\ldots-\alpha_n} - 1}$$

Thus

$$f'(p) = \frac{(1-\alpha-\alpha_1-\ldots-\alpha_n)n^{-\alpha-\alpha_1-\ldots-\alpha_n}}{2^{1-\alpha-\alpha_1-\ldots-\alpha_n} - 1}$$

$$= \frac{(\alpha+\alpha_1+\ldots+\alpha_n-1).2^{\alpha+\alpha_1+\ldots+\alpha_n}}{(2^{\alpha+\alpha_1+\ldots+\alpha_n}-2).n^{\alpha+\alpha_1+\ldots+\alpha_n}} > 0$$

since $\alpha + \sum_{i=1}^{n} \alpha_i \neq 1, \alpha + \sum_{i=1}^{n} \alpha_i > 1$

Hence maximum value is an increasing function of n.

(vii) Recursivity property : To prove that the measure (2.1) is recursive in nature, we consider

$$H^{n-1}_{\alpha,\alpha_1,\ldots,\alpha_n}(p_1+p_2, p_3, p_4, \ldots, p_n) = \frac{\left(p_1+p_2\right)^{\alpha+\alpha_1+\ldots+\alpha_n} + \sum_{i=3}^{n} p_i^{\alpha+\alpha_1+\ldots+\alpha_n} - 1}{2^{1-\alpha-\alpha_1-\ldots-\alpha_n} - 1}$$

$$= \frac{\left(p_1+p_2\right)^{\alpha+\alpha_1+\ldots+\alpha_n} - p_1^{\alpha+\alpha_1+\ldots+\alpha_n} - p_2^{\alpha+\alpha_1+\ldots+\alpha_n}}{2^{1-\alpha-\alpha_1-\ldots-\alpha_n} - 1}$$

$$+ \frac{\sum_{i=1}^{n} p_i^{\alpha+\alpha_1+\ldots+\alpha_n} - 1}{2^{1-\alpha-\alpha_1-\ldots-\alpha_n} - 1}$$

$$= -\left(p_1+p_2\right)^{\alpha+\alpha_1+\ldots.+\alpha_n}\left[\frac{\left(\frac{p_1}{p_1+p_2}\right)^{\alpha+\alpha_1+\ldots.+\alpha_n}+\left(\frac{p_2}{p_1+p_2}\right)^{\alpha+\alpha_1+\ldots.+\alpha_n}-1}{2^{1-\alpha-\alpha_1-\ldots.-\alpha_n}-1}\right]$$

$$+\frac{\sum_{i=1}^{n} p_i^{\alpha+\alpha_1+\ldots.+\alpha_n}-1}{2^{1-\alpha-\alpha_1-\ldots.-\alpha_n}-1}$$

$$= -\left(p_1+p_2\right)^{\alpha+\alpha_1+\ldots.+\alpha_n} H_2\left(\frac{p_1}{p_1+p_2}+\frac{p_2}{p_1+p_2}\right)+H^{n}_{\alpha,\alpha_1,\ldots,\alpha_n}(p_1,p_2,\ldots,p_n)$$

Thus, we have proved that

$$H^{n}_{\alpha,\alpha_1,\ldots,\alpha_n}(p_1,p_2,\ldots,p_n)=H^{n-1}_{\alpha,\alpha_1,\ldots,\alpha_n}(p_1+p_2,p_3,p_4\ldots,p_n)$$

$$+\left(p_1+p_2\right)^{\alpha+\alpha_1+\ldots.+\alpha_n} H_2\left(\frac{p_1}{p_1+p_2}+\frac{p_2}{p_1+p_2}\right)$$

This shows that the measure (2.1) possesses recursivity property.

(vii) Additive property: To show that the measure (2.1) is non-additive, we consider

$$H^{n,m}_{\alpha,\alpha_1,\ldots,\alpha_n}(P\cup Q)=\frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\left(p_i q_j\right)^{\alpha+\alpha_1+\ldots.+\alpha_n}-1}{2^{1-\alpha-\alpha_1-\ldots.-\alpha_n}-1}$$

$$=\left(\frac{\sum_{i=1}^{n} p_i^{\alpha+\alpha_1+\ldots.+\alpha_n}-1}{2^{1-\alpha-\alpha_1-\ldots.-\alpha_n}-1}\right)\left(\sum_{j=1}^{m} q_j^{\alpha+\alpha_1+\ldots.+\alpha_n}-1\right)$$

$$+\left(\frac{\sum_{i=1}^{n} p_i^{\alpha+\alpha_1+\ldots.+\alpha_n}-1}{2^{1-\alpha-\alpha_1-\ldots.-\alpha_n}-1}\right)+\left(\frac{\sum_{j=1}^{m} q_j^{\alpha+\alpha_1+\ldots.+\alpha_n}-1}{2^{1-\alpha-\alpha_1-\ldots.-\alpha_n}-1}\right)$$

$$=\left(2^{1-\alpha-\alpha_1-....-\alpha_{n-1}}\right)\left(\frac{\sum_{i=1}^{n} p_i^{\alpha+\alpha_1+....+\alpha_{n-1}}}{2^{1-\alpha-\alpha_1-....-\alpha_{n-1}}}\right)\left(\frac{\sum_{j=1}^{m} q_j^{\alpha+\alpha_1+....+\alpha_{n-1}}}{2^{1-\alpha-\alpha_1-....-\alpha_{n-1}}}\right)$$

$$+\left(\frac{\sum_{i=1}^{n} p_i^{\alpha+\alpha_1+....+\alpha_{n-1}}}{2^{1-\alpha-\alpha_1-....-\alpha_{n-1}}}\right)+\left(\frac{\sum_{j=1}^{m} q_j^{\alpha+\alpha_1+....+\alpha_{n-1}}}{2^{1-\alpha-\alpha_1-....-\alpha_{n-1}}}\right)$$

$$=\left(2^{1-\alpha-\alpha_1-....-\alpha_{n-1}}\right).H^n(P).H^m(Q)+H^n(P)+H^m(Q)$$

which shows that the generalized entropy (2.1) is non-additive.

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