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Some Fixed Point Theorem in Polish Spaces

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1. Abstract

In this paper we prove fixed point result in generating Polish space (random space which is more general than the other spaces) with implicit relations.

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2. INTRODUCTION

Fixed-point theory is an important branch of non-linear analysis. A point, which is invariant under any transformation, is termed as "Fixed Point" that is for any transformation T on metric space (X, d), x is fixed point of T if T(x) = x.
Probabilistic functional analysis has emerged as one of the important mathematical disciplines in view of its role in analyzing probabilistic models in the applied sciences. The study of fixed points of random operators forms a central topic in this area. The Prague school of probabilistic initiated its study in the 1950s. However, the research in this area flourished after the publication of the survey article of Bharucha-Reid [3]. Since then, many interesting random fixed point results and several applications have appeared in the literature, see, for example the work of Beg and Shahzad [2], Itoh [5], Lin [7], O'Regan [8], Papageorgiou [9], Dhagat et.l.[4], Shahzad and Latif [10], Tan and Yuan [11], Xu [12]. The purpose of this paper is to

establish fixed point result in generating Polish space (random space which is more general than the other spaces).

Let (Ω, $\sum$) be a measurable space with $\sum$ a sigma algebra of subsets of Ω and M a non-empty subset of a metric space $X = (X, d)$. Let 2^M be the family of all non-empty subsets of M and $C(M)$ the family of all nonempty closed subsets of M. A mapping $G : \Omega \to 2^M$ is called measurable if, for each open subset U of M, $G^{-1}(U) \in \sum, where G^{-1}(U) = \{w \in \Omega : G(w) \cap U \neq \phi\}$. A mapping $\xi : \Omega \to M$ is called a measurable selector of a measurable mapping $G : \Omega \to 2^M$ if ξ is measurable and $\xi(w) \in G(w)$ for each $w \in \Omega$. A mapping $T : \Omega \times M \to X$ is said to be a random operator if, for each fixed $x \in M$, $T(., x) : \Omega \to X$ is measurable. A measurable mapping $\xi : \Omega \to M$ is a random fixed point of a random operator $T : \Omega \times M \to X$ if $\xi(w) = T(w, \xi(w))$ for each $w \in \Omega$.

DEFINITION 2.1: Let X be non empty set and $\{d_\alpha : \alpha \in (0,1]\}$ be a family of mappings dα of (Ω×X) × (Ω×X) into R^+, w ϵ Ω be a selector. $(X, d_\alpha : \alpha \in (0,1])$ is called generating Polish space of quasi metric family if it satisfies the following conditions:

2.1.1) $d_\alpha((w,x),(w,y)) = 0 \; \forall \alpha \in (0,1] \Leftrightarrow x = y$

2.1.2) $d_\alpha((w,x),(w,y)) = d_\alpha((w,y),(w,x)) \;\; \forall x, y \in X, w \in \Omega$ *and* $\alpha \in (0,1]$,

2.1.3) *For any* $\alpha \in (0,1]$, *there exists a number* $\mu \in (0, \alpha]$ *such that*

$$d_\alpha((w,x),(w,y)) = d_\mu((w,x),(w,z)) + d_\mu((w,z),(w,y)) \; \forall x, y \in X \text{ and } w \in \Omega \text{ be a selector}.$$

2.1.4) *For any* $x, y \in X, w \in \Omega$, $d_\alpha((w,x),(w,y))$ *is non – increasing and left continuous in* α.

DEFINITION 2.2 : Let $(X, d_\alpha : \alpha \in (0,1])$ be a generating Polish space of quasi metric family and S and T be mappings from Ω×X into X. The mapping S and T are said to be quasi compatible if

$$d_\alpha(ST(w,x_n), TS(w,x_n)) \to 0 \; as \; n \to \infty, \;\; \alpha \in (0,1], w \in \Omega$$

whenever $\{w,x_n\}$ be a sequence in Ω×X such that

$$\lim_{n\to\infty} S(w,x_n) = \lim_{n\to\infty} T(w,x_n) = p \text{ for some } p \in X.$$

DEFINITION 2.3: Let $(X, d_\alpha : \alpha \in (0,1])$ be a generating Polish space of quasi metric family and S and T be mappings from Ω×X into X. The mapping S and

T are said to be compatible of type (A) if: $d_\alpha(TS(w,x_n),SS(w,x_n)=0$ *and* $d_\alpha(ST(w,x_n),TT(w,x_n)=0$

Whenever {w, x_n} be a sequence in Ω×X such that

$$\lim_{n\to\infty} S(w,x_n) = \lim_{n\to\infty} T(w,x_n) = p \text{ for some } p \in X.$$

DEFINITION 2.4: IMPLICIT RELATION: Let $\mathcal{F}$ be the set of all real functions $\mathcal{F}$: $R^4_+ \to R$ such that: (F_1): F is continuous in each coordinate variable,

(F_2): If either $F(u,0,u,v) \le 0$ or $F(u,0,u+v,v) \le 0$ for all $u,v \ge 0$, then there exists a real constant $0 \le h \le 1$ $such\ that\ u \le v.$

3. SOME CONCERNING RESULTS

LEMMA 3.1: Let $(X, d_\alpha : \alpha \in (0,1])$ be a generating Polish space of quasi metric family and S and T be mappings from Ω×X into X. Suppose that

$$\lim_{n\to\infty} S(w,x_n) = \lim_{n\to\infty} T(w,x_n) = p \text{ for some } p \in X.$$

Then we have the following:

3.1.1) $\lim_{n\to\infty} ST(w,x_n) = Tp$ *if T iscontinuous and*

3.1.2) $STp = TSp$ *and* $Sp = Tp$ *if T is continuous*

PROOF 3.1.1: Suppose that

$$\lim_{n\to\infty} S(w,x_n) = \lim_{n\to\infty} T(w,x_n) = p \text{ for some } p \in X.$$

Now, since T is continuous, we have

$$\lim_{n\to\infty} TS(w,x_n) = Tp$$

By 2.1.3 , we have

$$d_\alpha(ST(w,x_n),Tp) = d_\mu(ST(w,x_n),TS(w,x_n)) + d_\mu(TS(w,x_n),Tp); \mu \in (0,\alpha]$$

Since S and T are quasi compatible, we have

$$\lim_{n\to\infty} ST(w,x_n) = Tp$$

PROOF 3.1.2: Since T is continuous,

$$\lim_{n \to \infty} ST(w, x_n) = Tp$$

$Hence\ by\ uniqueness\ of\ \lim it,\ we\ have\ Sp = Tp.$

$Now\ \ again\ \ d_\alpha(STp, TSp) = \lim_{n\to\infty} d_\alpha\ (ST(w, x_n), TS(w, x_n) = 0$

$i.e.\ STp = TSp$

This completes the proof.

LEMMA 3.2: Let $(X,\ d_\alpha : \alpha \in (0, 1])$ be a generating Polish space of quasi metric family and S and T be mappings from Ω×X into X. If S and T are compatible of type (A) for any α ϵ (0, 1] and for μ ϵ (0, α].

Then $STp = TTp = TSp = SSp$

PROOF : Suppose $\{w, x_n\}$ be a sequence in Ω×X defined by $x_n = p$ as $n \to \infty$ and Sp = Tp.

Then we have $\lim_{n\to\infty} S(w, x_n) = \lim_{n\to\infty} T(w, x_n) = Sp$

$Since\ S\ and\ T\ have\ compatible\ of\ type\ (A),\ we\ have$

$d_\alpha(STp, TTp) = \lim_{n\to\infty} d_\alpha\ (ST(w, x_n), TT(w, x_n)) = 0;\ \ \alpha \in (0,1]$

$Hence\ we\ have\ STp = TTp.$

$Similarly,\ we\ have\ TSp = SSp.$

$But\ \ Tp = Sp$

$\Rightarrow TTp = TSp.$

$Therefore\ STp = TTp = TSp = SSP.$

Remark: Quasi compatible pair of maps is compatible of type (A) but converse is not always true.

4. MAIN RESULTS:

THEOREM (4.1): Let $(X,\ d_\alpha : \alpha \in (0,1])$ be a Generating Polish space of quasi metric family and S,T & G are mapping from Ω × X → X are continuous random operator w.r.t. d . Suppose there is some α ϵ (0, 1) such that for x, y ϵ X and w ϵ Ω, we have the following conditions

4.1.1) $S(X) \subseteq G(X)$ *and* $T(X) \subseteq G(X)$

4.1.2) $\phi\{d_\alpha(S(w,x),T(w,y)), d_\alpha(S(w,x),G(w,y)), d_\alpha(G(w,x),T(w,y)),$
$d_\alpha(G(w,x),G(w,y))\} \le 0 \;\forall\, x,y \in X$ *and* $\alpha \in (0,1],$

4.1.3) G *is continuous*

4.1.4) *The pairs* $\{S,G\}$ *and* $\{T,G\}$ *are quasi compatible on* X.

Then S, T *and* G *have common fixed point*.

PROOF :

Let x_0 *be any point of* X

Since $S(X) \subseteq G(X)$ *and* $T(X) \subseteq G(X)$ *and* $SG(X) \subseteq GG(X)$ *and* $TG(X) \subseteq GG(X)$

So there exists x_1 *and* x_2 *in* X *such that*

$GG(w,x_1) = SG(w,x_0)$ *and* $GG(w,x_2) = TG(w,x_1)$

In general

$GG(w,x_{2n+1}) = SG(w,x_{2n})$ *and* $GG(w,x_{2n+2}) = TG(w,x_{2n+1})$

for $n = 0,1,2,3,............$

Let $d_n = d_\alpha(GG(w,x_n), GG(w,x_{n+1}))$

Also we know

$$d_\alpha(GG(w,x_{2n}), GG(w,x_{2n+2})) \le d_\mu(GG(w,x_{2n}), GG(w,x_{2n+1}))$$
$$+\, d_\mu(GG(w,x_{2n+1}), GG(w,x_{2n+2})) \;\forall\, \alpha \in (0,1] \text{ and } \mu \in (0,\alpha].$$

Suppose x_{2n}, x_{2n+1} *satisfy* (4.1.2), *then* $\forall\, \alpha \in (0,1]$

$$\phi\{d_\alpha(SG(w,x_{2n}), TG(w,x_{2n+1})), d_\alpha(SG(w,x_{2n}), GG(w,x_{2n+1})),$$
$$d_\alpha(GG(w,x_{2n}), TG(w,x_{2n+1})), d_\alpha(GG(w,x_{2n}), GG(w,x_{2n+1}))\} \le 0$$

$$\phi\{d_\alpha(GG(w,x_{2n+1}), GG(w,x_{2n+2})), d_\alpha(GG(w,x_{2n+1}), GG(w,x_{2n+1})),$$
$$d_\alpha(GG(w,x_{2n}), GG(w,x_{2n+2})), d_\alpha(GG(w,x_{2n}), GG(w,x_{2n+1}))\} \le 0$$

$$\phi\{d_\alpha(GG(w,x_{2n+1}), GG(w,x_{2n+2})), 0, [d_\mu(GG(w,x_{2n}), GG(w,x_{2n+1}))$$
$$+ d_\mu(GG(w,x_{2n+1}), GG(w,x_{2n+2}))], d_\alpha(GG(w,x_{2n}), GG(w,x_{2n+1})) \le 0$$

Thus from definition of implicit relation 2.4, *we have*

$$d_\alpha(GG(w,x_{2n+1}), GG(w,x_{2n+2})) \le h\{d_\mu(GG(w,x_{2n}), GG(w,x_{2n+1}))\}$$

$$d_{2n+1} \le h\, d_{2n}$$
$$d_{2n+1} \le d_{2n}$$

Similarly $d_{2n} \leq h\, d_{2n-1}$

Thus $\{d_{2n}\}$ *be monotone decreasing and hence converge to zero.*

Therefore $\{GG(w,x_{2n})\}$ *is a Cauchy sequence and converge to Gp and hence to point X.*

Since $\{SG(w,x_{2n})\}$ *and* $\{TG(w,x_{2n})\}$ *are subsequence of* $\{GG(w,x_{2n})\}$ *and so converge to same point p. Now by lemma 3.1 we obtain*

SGp = GSp and Sp = Gp

Similarly TGp = GTp and Tp = Gp

Hence Sp = Tp = Gp.

Also Sp = p = Gp = Tp as Gp = p.

Hence p is common fixed point of S, T and G.

This completes the proof.

THEOREM 4.2: Let $(X,\ d_\alpha : \alpha \in (0,1])$ be a generating Polish space of quasi metric family and S, T and G be mappings from Ω×X into X satisfying

4.2.1) $S(X) \subseteq G(X)\ and\ T(X) \subseteq G(X)$

4.2.2) $\phi\{d_\alpha(S(w,x),T(w,y)), d_\alpha(S(w,x),G(w,y)), d_\alpha(G(w,x),T(w,y)),$

$$d_\alpha(G(w,x),G(w,y))\} \leq 0\ \ \forall\, x,y \in X\ and\ \alpha \in (0,1],$$

4.2.3) $G\ is\ continuous$

4.2.4) $The\ pairs\ \{S,G\}\ and\ \{T,G\}\ are\ compatible\ of\ type\ (A).$

$Then\ S,T\ \ and\ G\ \ have\ \ common\ fixed\ po\,\text{int}.$

PROOF :Similar to the proof of the theorem 4.1 by using lemma 3.2.

COROLLARY 4.3: Let $(X,\ d_\alpha : \alpha \in (0,1])$ be a Generating Polish space of quasi metric family and S,T & G are mapping from Ω × X → X are continuous random operator w.r.t. d . Suppose there is some α ϵ (0, 1) such that for x, y ϵ X and w ϵ Ω, we have the following conditions

4.3.1) $S(X) \subseteq G(X)\ and\ T(X) \subseteq G(X)$

4.3.2) $\phi\{d_\alpha(S(w,x),T(w,y)), d_\alpha(S(w,x),G(w,y)), d_\alpha(G(w,y),T(w,y)),$

$$d_\alpha(G(w,x),G(w,y))\} \leq 0\ \ \forall\, x,y \in X\ and\ \alpha \in (0,1],$$:

4.3.3) $G\ is\ continuous$

4.3.4) $The\ pairs\ \{S,G\}\ and\ \{T,G\}\ are\ quasi\ compatible\ on\ X.$

$Then\ S,T\ \ and\ G\ have\ common\ fixed\ po\,\text{int}.$

PROOF:

$Let\ x_0\ be\ any\ po\text{int}\ of\ X$

$Since\ S(X) \subseteq G(X)\ and\ T(X) \subseteq G(X)\ \ and\ SG(X) \subseteq GG(X)\ \ and\ TG(X) \subseteq GG(X)$

$So\ there\ exists\ x_1\ and\ x_2\ in\ X\ such\ that$

$GG(w, x_1) = SG(w, x_0)\ \ and\ GG(w, x_2) = TG(w, x_1)$

$In\ general$

$GG(w, x_{2n+1}) = SG(w, x_{2n})\ \ and\ GG(w, x_{2n+2}) = TG(w, x_{2n+1})$

$for\ n = 0, 1,\ 2, 3, ...$

$Let\ d_n = d_\alpha\ (GG(w, x_n), GG(w, x_{n+1})).$

$Suppose\ x_{2n}, x_{2n+1}\ satisfy (4.3.2), then\ \forall\ \alpha \in (0, 1]$

$$\phi\{d_\alpha\ (SG(w, x_{2n}), TG(w, x_{2n+1}))\ , d_\alpha\ (SG(w, x_{2n}), GG(w, x_{2n+1})),$$
$$d_\alpha\ (GG(w, x_{2n+1}), TG(w, x_{2n+1})), d_\alpha\ (GG(w, x_{2n}), GG(w, x_{2n+1}))\} \le 0$$
$$\phi\{d_\alpha (GG(w, x_{2n+1}), GG(w, x_{2n+2})), d_\alpha\ (GG(w, x_{2n+1}), GG(w, x_{2n+1})),$$
$$d_\alpha\ (GG(w, x_{2n+1}), GG(w, x_{2n+2})), d_\alpha\ (GG(w, x_{2n}), GG(w, x_{2n+1}))\} \le 0$$
$$\phi\{d_\alpha (GG(w, x_{2n+1}), GG(w, x_{2n+2})), 0, d_\alpha\ (GG(w, x_{2n+1}), GG(w, x_{2n+2})),$$
$$d_\alpha\ (GG(w, x_{2n}), GG(w, x_{2n+1}))\} \le 0$$

$Thus\ from\ definition\ of\ \ implicit\ \ relation\ \ \ 2.4, we\ have$

$$d_\alpha\ (GG(w, x_{2n+1}), GG(w, x_{2n+2})) \le h\{d_\mu\ (GG(w, x_{2n}), GG(w, x_{2n+1}))\}$$
$$d_{2n+1} \le h\ d_{2n}$$
$$d_{2n+1}\ \le\ h\, d_{2n}$$
$$d_{2n+1}\ \le\ d_{2n}$$
$$Similarly\ \ \ d_{2n}\ \le\ d_{2n-1}$$

Thus $\{d_{2n}\}$ be monotone decreasing and hence converge to zero.

Therefore $\{GG(w,x_{2n})\}$ is a Cauchy sequence and converge to Gp and hence to point X.

Since $\{SG(w,x_{2n})\}$ and $\{TG(w,x_{2n})\}$ are subsequence of $\{GG(w,x_{2n})\}$ and so converge to same point p. Now by lemma 3.1 we obtain

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Hence pis common fixed point of S, T and G.
This completes the proof.

COROLLARY 4.4: Let $(X,\ d_\alpha : \alpha \in (0,1])$ be a Generating Polish space of quasi metric family and S,T & G are mapping from Ω × X → X are continuous random operator w.r.t. d . Suppose there is some α ϵ (0, 1) such that for x, y ϵ X and w ϵ Ω, we have the following conditions

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4.3.2) $\phi\{d_\alpha(S(w,x),T(w,y)), d_\alpha(S(w,x),G(w,y)), d_\alpha(G(w,y),T(w,y)),$
$d_\alpha(G(w,x),G(w,y))\} \leq 0 \quad \forall x,y \in X\ and\ \alpha \in (0,1],$

4.3.3) *G is continuous*

4.3.4) *The pairs* $\{S,G\}$ *and* $\{T,G\}$ *are compatible of type* (A).

*Then S,T and G have common fixed po*int.

PROOF: Similar to the proof of the corollary 4.3 by using lemma 3.2.

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