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Analytic Solutions of Parametric Type and Numerical Simulations of a Class of Nonlinear Ordinary Differential Equations [1]

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Abstract

In this paper, the mathematical model that describes the radially symmetric deformation of a micro-void centered at a compressible hyperelastic sphere is formulated as a boundary value problem (BVP) of a class of nonlinear ordinary differential equations. The analytic solutions of parametric type that describe the growth of the micro-void are obtained and the numerical simulations such as the growth of the micro-void, the stress distribution and the radial displacement are given. The physical interpretations are well shown by the numerical results.

Mathematics Subject Classification: 34A05; 34B15; 74B20

Keywords: BVP; analytic solutions of parametric type; compressible hyperelastic sphere; radial displacement

1 Formulation of mathematical model

In this paper, we are concerned with the problem of radially symmetric deformation of a micro-void centered at a compressible hyperelastic sphere, where the outer surface of the sphere is subjected to a prescribed radial stretch $\lambda > 1$, and the surface of micro-void is traction-free. Under the assumption of radially symmetric deformation, the undeformed and the deformed configurations of the sphere are respectively given by

$$D_0 = \{(R, \Theta, \Phi) | 0 < B \leq R \leq A,\ 0 \leq \Theta \leq 2\pi,\ 0 \leq \Phi \leq \pi\} \tag{1}$$

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$$D = \{(r,\theta,\varphi)|r = r(R) \geq 0, A \leq R \leq B; \theta = \Theta, \varphi = \Phi\} \tag{2}$$

where $r = r(R)$ is the radial deformation function to be determined, A and B are radii of the inner and the outer surfaces of the undeformed sphere. Moreover,

$$\lambda_1 = dr/dR = \dot{r}(R), \lambda_2 = \lambda_3 = r/R \tag{3}$$

are the principal stretches of the deformation gradient tensor, and $\lambda_i > 0, \ i = 1, 2, 3$.

The governing equations of the mathematical model which describes the radially symmetric deformation of the micro-void in the interior of the compressible hyperelastic sphere are as follows.

$$\frac{\partial^2 W}{\partial \lambda_1^2}\ddot{r}(R) + \frac{2}{R}\left[\frac{\partial^2 W}{\partial \lambda_1 \lambda_2}\left(\dot{r}(R) - \frac{r(R)}{R}\right) + \left(\frac{\partial W}{\partial \lambda_1} - \frac{\partial W}{\partial \lambda_2}\right)\right] = 0 \tag{4}$$

$$\sigma_{rr}(R) = \frac{1}{\lambda_2\lambda_3}\frac{\partial W}{\partial \lambda_1}, \sigma_{\theta\theta}(R) = \sigma_{\phi\phi}(R) = \frac{1}{\lambda_1\lambda_2}\frac{\partial W}{\partial \lambda_2} \tag{5}$$

$$r(A) = \lambda A, \sigma_{rr}(B) = 0, \tag{6}$$

Note. Eq.(4) is the differential equation with respect to $r(R)$ that describes the radially symmetric deformation of the sphere, which is under the assumption of the absence of body forces; The components in Eq.(5) are the principal stresses of stress tensor; Eq.$(6)_1$ means that the outer surface of the sphere is now subjected to a prescribed uniform radial stretch $\lambda > 1$ and Eq.$(6)_2$ implies that the surface of the micro-void is traction-free; In Eqs.(4), (5) and $(6)_2$, W is the strain energy function associated with the homogeneous compressible hyperelastic material. It is easy to see that the analytic solution of the above problem strictly depends on the form of the strain energy function W.

Similar but somewhat different investigations may be found in [1, 2, 3, 4, 5].

In this paper, we consider a class of linear approximation compressible hyperelastic materials and the corresponding strain energy function is given by[4]

$$W = C_1(\lambda_1 + \lambda_2 + \lambda_3 - 3) + C_2(\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} - 3) + C_3(\lambda_1\lambda_2\lambda_3 - 1) \tag{7}$$

where

$$C_1 = \mu\frac{1-3\nu}{1-2\nu}, \quad C_2 = \mu\frac{1-\nu}{1-2\nu}, \quad C_3 = \mu\frac{2\nu}{1-2\nu} \tag{8}$$

in which μ and ν are the infinitesimal shear modulus and the Poisson's ratio, with $\mu > 0$ and $0 < \nu < 1/3$.

Obviously, the mathematical model composed of Eqs.(4)$\sim$(7) is a boundary value problem (BVP) of a class of nonlinear ordinary differential equations that describes the radially symmetric deformation of a micro-void under the prescribed radial stretch $\lambda > 1$ at the outer surface of the sphere and the traction-free surface of the micro-void.

2 Solutions

Substituting the strain energy function (7) into Eq.(4), we have the following differential equation

$$\frac{\ddot{r}}{\lambda_1^3}+\frac{1}{R}\left(\frac{1}{\lambda_2^2}-\frac{1}{\lambda_1^2}\right)=0 \tag{9}$$

Let $t=t(R)=\lambda_1/\lambda_2=\dot{r}(R)R/r(R)$, Eq.(9) can be transformed to the following system of the first order differential equations

$$\dot{r}(1-t)(2+t)-r\dot{t}=0,\ R\dot{t}-t(1-t)(2+t)=0 \tag{10}$$

where $\dot{t}=dt/dR$. On integration of Eq.(10) yields

$$R^6=\frac{Ct^3}{(1-t)^2(2+t)},\quad r^3=\frac{D(2+t)}{1-t} \tag{11}$$

where C, D are integral constants to be determined. Thus we obtain the general solutions (11) with parameter t of Eq.(4). Next we determine the integral constants C, D in Eq.(11) by using the boundary conditions in Eq.(6).

Let $R=A$ and $R=B$ in the first equation of Eq. (11), then we have

$$A^6=\frac{Ct_A^3}{(1-t_A)^2(2+t_A)},\ B^6=\frac{Ct_B^3}{(1-t_B)^2(2+t_B)} \tag{12}$$

where t_A and t_B correspond to $R=A$ and $R=B$, respectively, and $0<t_B\leq t\leq t_A<1$.

On substitution of the boundary condition (6) into the second equation in Eq.(11) yields

$$\lambda^3A^3=\frac{D(2+t_A)}{(1-t_A)} \tag{13}$$

Substituting the strain energy function (7) into the inner boundary condition $(6)_2$, we have

$$C_1-C_2\left(\frac{C}{D^2}\right)^{1/3}\frac{1}{t_B(2+t_B)}+C_3\frac{2+t_B}{t_B}=0 \tag{14}$$

In sum, if the parameters t_A, t_B, C and D can be determined, the analytic solutions with parametric type of mathematical model Eqs. (4)∼(7) are composed of Eqs. (12)∼(14).

3 Qualitative analysis of growth of micro-void

Next we carry out the qualitative analysis and numerical simulations of the deformed micro-void.

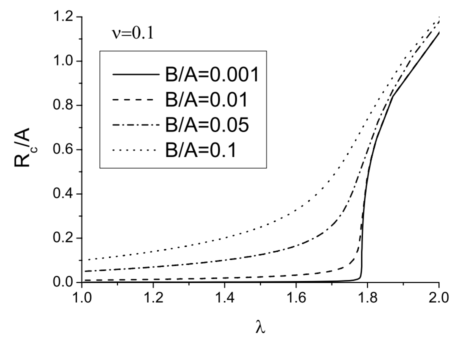

Fig 1: Relation between r_c/A and λ.

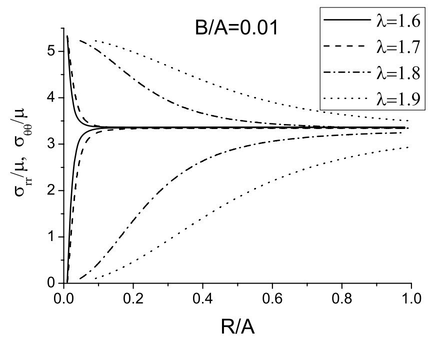

Fig 2: Relations between σ_{rr}/μ, $\sigma_{\theta\theta}/\mu$ and R/A.

3.1 Relation between growth of micro-void and stretch

On comparison of Eq. (12) and Eq. (13) yields the following expressions

$$\lambda = \left(\frac{C}{D^2}\right)^{-1/6}\left(1+\frac{2}{t_2}\right)^{1/2}, \frac{r_c}{A} = \left(\frac{C}{D^2}\right)^{-1/6}\left(1+\frac{2}{t_2}\right)^{1/2}\frac{B}{A} \tag{15}$$

where r_c denote the radius of deformed micro-void.

Interestingly, for the given radial stretch $\lambda > 1$ and the ratio of the undeformed radii of micro-void B/A, we can determine t_A, t_B, C and D, and then obtain the relations between the radii of micro-void r_c/A and the given radial stretch λ. Fig 1 shows the relation between r_c/A and λ for the given material parameter ν and for different values of B/A. As shown in Fig 1, we can see that if the value of B/A is sufficient small, the radius of the micro-void increases very slowly with the increasing λ, however, the radius of the micro-void increases suddenly as λ passes through a certain value and then grows rapidly; while if the value of B/A is relative small, the radius of the micro-void grows continuously and smoothly. This is similar to the phenomenon of cavity bifurcation of a solid sphere[3, 4, 5]. In [2], the authors pointed out that cavitation in solid spheres may be interpreted in terms of the growth of a pre-existing micro-void in the interior of solid spheres, rather than in terms of rupture.

3.2 Distribution of radial and circumferential stresses

In this subsection we investigate the distribution of the radial and circumferential stresses given by Eq.(5), namely, the radial stress and the circumferential stress. Substituting Eq. (7) into Eq.(5) we can obtain the expressions of the radial and circumferential stresses.

For the given value of B/A and for different values of λ, the relations between σ_{rr}/μ ($\sigma_{\theta\theta}/\mu$) and R/A are shown in Fig 2. As shown in Fig 2, we see that the radial stress σ_{rr} grows with the increasing R/A, however, it grows very fast near the surface of the micro-void and tend to stable near the outer surface of the sphere. Interestingly, the cases of the circumferential stress $\sigma_{\theta\theta}$ are exactly opposite to those of the radial stress.

4 Conclusions

In this paper, we investigate the finite deformations of a solid sphere with a pre-existing micro-void at its center, where the sphere is composed of a class of linear approximation compressible hyperelastic materials. We formulate the corresponding mathematical model and obtain some interesting results. Mathematically, we obtain the analytic solutions of parametric type of the mathematical model by introducing a transformation of variables. Physically, we analyze the relation between the growth of micro-void and the radial stretch, the distribution of radial and circumferential stresses, and the radial displacement, moreover, the numerical results coincide with the actual phenomena very well.

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