Applied Mathematical Sciences, Vol. 4, 2010, no. 30, 1457 - 1466

Analytical Inversion of Two Dimensional Laplace -Carson Transform by a Differential Method

B. Hasani Lichaei [1] **, H. Taheri Kate Sari** [2] **and A. Sharriffar** [3]

[1] Iran, Guilan, Rasht, Fooman Azad University
b_hasani2004@yahoo.com

[2] Iran,Tehran, Shahid Beheshti University
hosain_taheri_k@yahoo.com

[3] Iran, Guilan,Rasht, Fooman Azad University
Azin936@yahoo.com

Abstract

A differential method for recovering a function $f(t_1,t_2)$ from its two dimensional Laplace-Carson transform $pq\hat{f}(p,q)$ given as continuous or discrete data on a finite interval.The introduction of the variables $u_1 = \frac{1}{p}, u_2 = \frac{1}{q}$ converts this transform into a Mellin convolution, with a transformed kernel involving the gamma function .the truncation of the infinite product representation of $\frac{1}{\Gamma(1-s)\Gamma(1-w)}$ leads to an approximate differential expression for the solution.

Keywords: Two dimensional Laplace-Carson transform; two dimensional Mellin transform; Functional sequence; Inverse problems.

1 .Introduction

The Laplace–Carson transform provides a powerful method for analyzing linear systems.

In most problems solution inversion of Laplace-Carson transform involve numerous and complicated functions .Many interesting problems in physics lead to Laplace-Carson transforms of which inversion are not expressed in terms of classified functions .Therefore, we are interested to have analytical or numerical method to solve these problems.

C.Donalato's[1] analytical one dimensional method implying Laplace-Carson transform and Mellin transform leads to functional sequence $f_n(t)$ when $n \to \infty$ then $f_n(t)$ converge to $f(t)$.The extension of this method to two dimensional expressed in this paper without implying inversion two dimensional Laplace-Carson transform can Make a functional sequence $f_n(t_1,t_2)$ since $n \to \infty$ converge to exact $f(t_1,t_2)$.

2 .Section 2

In this section we express two lemmas and conclusion will be used in next section.

Lemma 2.1

Let $g: R \to R$ then $\forall \;\; n \ge 1$ $\frac{d^n}{du^n}\left(u^n g(u)\right) = \left[n + u\frac{d}{du}\right]\frac{d^{n-1}}{du^{n-1}}\left(u^{n-1} g(u)\right)$

Proof .According to Leibnitz Theorem if $f, g: R \to R$ and $f, g \in C^n[a,b]$ then

$$\left(f(u).g(u)\right)^{(n)} = \sum_{k=0}^{n}\binom{n}{k} f^{(n-k)}(u) g^{(k)}(u)$$

Therefore

$$\frac{d^n}{du^n}\left(u^n g(u)\right) = n\frac{d^{n-1}}{du^{n-1}}\left(u^{n-1} g(u)\right) + u\frac{d^n}{du^n}\left(u^{n-1} g(u)\right)$$

$$= \left[n + u\frac{d}{du}\right]\frac{d^{n-1}}{du^{n-1}}\left(u^{n-1} g(u)\right) \qquad \forall \;\; n \ge 1$$

Conclusion 2.1

Extension of lemma2.1 to two dimensional leads us

$$\frac{d^n}{d{u_2}^n}\left({u_2}^n g(u_1,u_2)\right) = \left[n + u_2\frac{d}{du_2}\right]\frac{d^{n-1}}{d{u_2}^{n-1}}\left({u_2}^{n-1} g(u_1,u_2)\right)$$

Lemma 2.2

Let $g: R \to R$ then

$\forall \;\; n \ge 1$ $$\frac{d^n}{du^n}\left(u^n g(u)\right) = \prod_{k=1}^{n}\left[k + u\frac{d}{du}\right] g(u)$$

Proof .According to lemma 2.1

$$\frac{d^n}{du^n}\left(u^n g(u)\right)=\left[n+u\frac{d}{du}\right]\frac{d^{n-1}}{du^{n-1}}\left(u^{n-1}g(u)\right) \qquad \forall \quad n\geq 1$$

$$=\left[n+u\frac{d}{du}\right]\left[(n-1)+u\frac{d}{du}\right]...\left[1+u\frac{d}{du}\right]g(u)$$

$$=\prod_{k=1}^{n}\left[k+u\frac{d}{du}\right]g(u)$$

Conclusion 2.2
Extension of lemma2.2 to two dimensional leads us

$$\frac{d^n}{du_2{}^n}\left(u_2{}^n g(u_1,u_2)\right)=\prod_{k=1}^{n}\left[k+u_2\frac{d}{du_2}\right]g(u_1,u_2) \quad \forall \quad n\geq 1$$

3. The inversion formula

3.1Two dimensional Laplace Carson Transform and Two dimensional Mellin Transform.

Two Dimensional Laplace-Carson Transform $\hat{f}_c(p,q)$ of a real function $f(t_1,t_2)$ $t_1,t_2\geq 0$ Is defined by Ditkin and Prudnikov [3]

$$\hat{f}_c(p,q)=pq\int_0^\infty\int_0^\infty e^{-pt_1-qt_2}f(t_1,t_2)dt_1dt_2 \qquad (1)$$

We assume that $\hat{f}_c(p,q)$ is known for real positive values of p,q in the interval $[a,b],[c,d]$ and wish to approximately reconstruct $\hat{f}_c(p,q)$ in the widest possible range of values of t_1,t_2; in most applications, $f(t_1,t_2)$ and hence $\hat{f}_c(p,q)$ are positive functions.
An interesting property of the transform of Eq.(1) is brought out by supposing that the variable and functions involved are physical quantities.then we see that $\hat{f}_c(\frac{1}{u_1},\frac{1}{u_2})$,unlike $\hat{f}(\frac{1}{u_1},\frac{1}{u_2})$,retains the physical dimensions of $f(t_1,t_2)$ .Note that $p=\frac{1}{u_1},q=\frac{1}{u_2}$ .so a new image function is written by

$$g(u_1,u_2)=\hat{f}_c(\frac{1}{u_1},\frac{1}{u_2})=\frac{1}{u_1u_2}\int_0^\infty\int_0^\infty e^{-\frac{t_1}{u_1}-\frac{t_2}{u_2}}f(t_1,t_2)dt_1dt_2 \qquad (2)$$

Thus the new transformed function $g(u_1,u_2)$ becomes physically alike to $f(t_1,t_2)$; in next section some examples will illustrate that $g(u_1,u_2)$, unlike $\hat{f}_c(p,q)$ or $\hat{f}\,(p,q)$, constitutes by itself an approximate reconstruction of $f(t_1,t_2)$.
By multiplying and dividing by t_1,t_2 the integrand, Eq(2) can be put in the form of a two dimensional mellin convolution product

$$g(u_1,u_2)=\int_0^\infty\int_0^\infty(\frac{t_1}{u_1})(\frac{t_2}{u_2})e^{-\frac{t_1}{u_1}-\frac{t_2}{u_2}}f(t_1,t_2)\frac{dt_1dt_2}{t_1t_2} \tag{3}$$

The kernel of this integral equation has the proper from $h(\frac{u_1}{t_1},\frac{u_2}{t_2})$, where

$$h(x,y)=\frac{1}{xy}e^{-\frac{1}{x}-\frac{1}{y}} \tag{4}$$

Then

$$h(\frac{u_1}{t_1},\frac{u_2}{t_2})=\frac{t_1t_2}{u_1u_2}e^{-\frac{t_1}{u_1}-\frac{t_2}{u_2}}$$

Eq.(3) can be solved by taking its two dimensional Mellin transform, which for a function $h(x,y)$ is defined by

$$M[h(x,y)]\equiv H(s,w)=\int_0^\infty\int_0^\infty h(x,y)x^{s-1}y^{w-1}dxdy \tag{5}$$

We note that

$$\Gamma(1-s)=\int_0^\infty\frac{1}{x}\,e^{-\frac{1}{x}}x^{s-1}dx\,,\quad s<1 \tag{6}$$

Where Γ denotes the usual gamma function.By Eq.(4) we can see that

$$M[h(x,y)]\equiv H(s,w)=\int_0^\infty\int_0^\infty\frac{1}{xy}e^{-\frac{1}{x}-\frac{1}{y}}x^{s-1}y^{w-1}dxdy=\Gamma(1-s)\Gamma(1-w)\ ,s<1,w<1 \tag{7}$$

Assuming the existence of the other transforms, we obtain from Eq.(3) $G(s,w)=H(s,w)F(s,w)$ and hence

$$F(s,w)=G(s,w)/H(s,w)\ =[1/(\Gamma(1-s)\Gamma(1-w))]G(s,w) \tag{8}$$

G and H have a common region of analyticity.we can obtain $f(t_1,t_2)$ from Eq.(5) which could be Mellin inverted .For this purpose instead of $\frac{1}{\Gamma}$ we employ representation of an infinite product.

2.2 .Obtaining the approximate of inversion $F(s,w)$

The infinite product representation of $\frac{1}{\Gamma}$ is (see,e,g.,[9,p,Eq.(8.322)]))

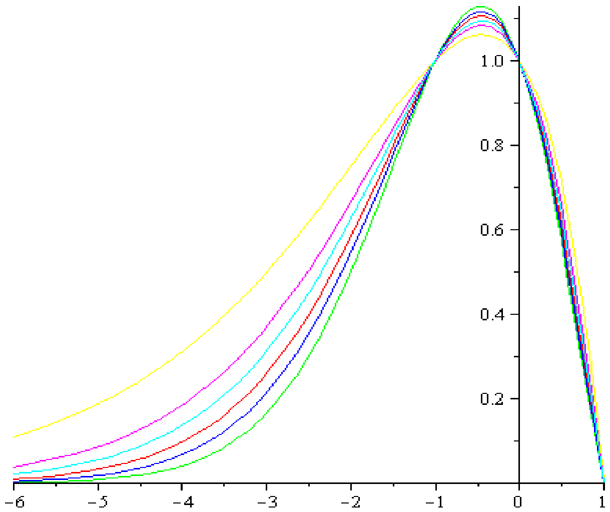
$$\frac{1}{\Gamma(z)} = z\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)\left(1+\frac{1}{k}\right)^{-z} \tag{9}$$

The truncation of this product after n terms provides an approximation for $\frac{1}{\Gamma(z)}$

The result can be simplified to

$$\frac{1}{\Gamma(1-s)} \cong (1-s)\left(1-\frac{s}{2}\right). \ . \ .\left(1-\frac{s}{n}\right)(n+1)^{s} \tag{10}$$

$$\frac{1}{\Gamma(1-w)} \cong (1-w)\left(1-\frac{w}{2}\right). \ . \ .\left(1-\frac{w}{n}\right)(n+1)^{w} \tag{11}$$



$Fig1.$The function$1/\Gamma(1-s), s<1$, and its approximation from Eq.(10) and (11)For different values of $n = 1,2,3,5,10$.The green line is exact.

The substitution of Eq.(10) and Eq.(11) into Eq.(8) yields an approximation $F_n(s,w)$ to $F(s,w)$

$$F_n(s,w) = (1-s)\left(1-\frac{s}{2}\right). \ . \ .\left(1-\frac{s}{n}\right)\ (1-w)\left(1-\frac{w}{2}\right). \ ..\left(1-\frac{w}{n}\right)(n+1)^{s+w}G(s,w) \tag{12}$$

The function $f_n(t_1,t_2)$ that is Mellin transformed into $F_n(s,w)$ constitute an approximation to the solution $f(t_1,t_2)$.now we prove that $f_n(t_1,t_2)$ can be expressed in terms of $g(u_1,u_2)$ and its first n derivatives.In fact the known properties $M\left[\frac{t_1}{k}\frac{d}{dt_1}g(t_1,t_2)\right] = -\frac{s}{k}G(s,w)$ and $M\left[\frac{t_2}{k}\frac{d}{dt_2}g(t_1,t_2)\right] = -\frac{w}{k}G(s,w)$ and $M\left[\frac{t_1t_2}{k^2}\frac{d}{dt_1}\frac{d}{dt_2}g(t_1,t_2)\right] = -\frac{sw}{k^2}G(s,w)$ show that

$$M\left[(1+\frac{t_1}{k}\frac{d}{dt_1}+\frac{t_2}{k}\frac{d}{dt_2}+\frac{t_1t_2}{k^2}\frac{d}{dt_1}\frac{d}{dt_2})g(t_1,t_2)\right]=\left(1-\frac{s}{k}\right)\left(1-\frac{w}{k}\right)G(s,w) \qquad (13)$$

We note that

$$\left[(1+\frac{t_1}{k}\frac{d}{dt_1}+\frac{t_2}{k}\frac{d}{dt_2}+\frac{t_1t_2}{k^2}\frac{d}{dt_1}\frac{d}{dt_2})g(t_1,t_2)\right]=\left[1+\frac{t_1}{k}\frac{d}{dt_1}\right]\left[1+\frac{t_2}{k}\frac{d}{dt_2}\right]g(t_1,t_2) \qquad (14)$$

Suppose that

$$L_n=\prod_{k=1}^{n}\left[1+\frac{t_1}{k}\frac{d}{dt_1}\right]\left[1+\frac{t_2}{k}\frac{d}{dt_2}\right] \qquad (15)$$

Therefore,

$$M[L_n g(t_1,t_2)]=\prod_{k=1}^{n}M\left[\left[\left[1+\frac{t_1}{k}\frac{d}{dt_1}\right]\left[1+\frac{t_2}{k}\frac{d}{dt_2}\right]\right]g(t_1,t_2)\right]=\prod_{k=1}^{n}\left(1-\frac{s}{k}\right)\left(1-\frac{w}{k}\right)G(s,w) \quad (16)$$

By recalling the property $c^{s+w}G(s,w)=M\left[g\left(\frac{t_1}{c},\frac{t_2}{c}\right)\right]$, we see that

$$M\left[\prod_{k=1}^{n}\left[\left[1+\frac{t_1}{k(n+1)}\frac{d}{dt_1}\right]\left[1+\frac{t_2}{k(n+1)}\frac{d}{dt_2}\right]\right]g\left(\frac{t_1}{(n+1)},\frac{t_2}{(n+1)}\right)\right]=\prod_{k=1}^{n}\left(1-\frac{s}{k}\right)\left(1-\frac{w}{k}\right)(n+1)^{s+w}G(s,w)$$

So we can write the function $f_n(t_1,t_2)$ in the form

$$f_n(t_1,t_2)=L_n g(u_1,u_2)=\prod_{k=1}^{n}\left[\left[1+\frac{u_1}{k}\frac{d}{du_1}\right]\left[1+\frac{u_2}{k}\frac{d}{du_2}\right]\right]g(u_1,u_2);\quad u_1=\frac{t_1}{(n+1)},u_2=\frac{t_2}{(n+1)}$$

$$=\frac{1}{(n!)^2}\prod_{k=1}^{n}\left[\left[k+u_1\frac{d}{du_1}\right]\left[k+u_2\frac{d}{du_2}\right]\right]g(u_1,u_2) \qquad (17)$$

According to the conclusion of lemma 2.2 in section.2 we have

$$f_n(t_1,t_2)=\frac{1}{(n!)}\frac{d^n}{du_1{}^n}\frac{d^n}{du_2{}^n}\left(u_1{}^n u_2{}^n g(u_1,u_2)\right);\quad u_1=\frac{t_1}{(n+1)},u_2=\frac{t_2}{(n+1)} \qquad (18)$$

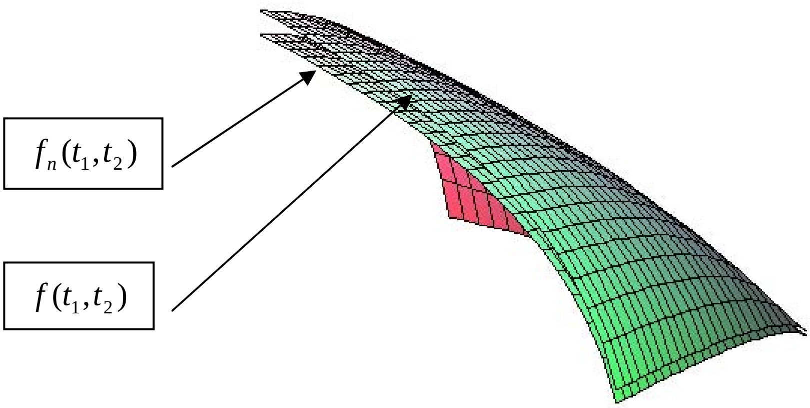
If $\hat{f}_c(p,q)$ is known for $a\le p\le b$ and $c\le q\le d$,then $g(u_1,u_2)$ is given in the rang $\frac{1}{b}\le u_1\le\frac{1}{a}$ and $\frac{1}{d}\le u_2\le\frac{1}{c}$,and Eqs.(17) or (18) yield $f_n(t_1,t_2)$ for $\frac{(n+1)}{b}\le t_1\le\frac{(n+1)}{a}$ and $\frac{(n+1)}{d}\le t_2\le\frac{(n+1)}{c}$ .since the partial products of Eq.(10) and (11) converge to $\frac{1}{\Gamma(1-s)}$ and $\frac{1}{\Gamma(1-w)}$, f_n is expected to better approximate the exact f for large n ,both for continuous and discrete input data.The examples of the next section illustrate the convergence of f_n to f when $g(u_1,u_2)$ has a known simple analytical expression**.**

4 .Analytical examples

4.1 .Example 1

For reconstructing $f(t_1,t_2)=(\ln t_1+\gamma)(\ln t_2+\gamma)$; (constant $\gamma=0.5772156...$) first we compute Laplace Carson transform $\hat{f}_c(p,q)=\ln p\ln q$ then by considering $u_1=\dfrac{1}{p}$ and $u_2=\dfrac{1}{q}$ we obtain $g(u_1,u_2)=\ln\dfrac{1}{u_1}\ln\dfrac{1}{u_2}$**.**by using Eq**.**(18) for this $g(u_1,u_2)$ we obtain

$$f_n(t_1,t_2)=\frac{1}{(n!)}\frac{d^n}{du_1^{\ n}}\frac{d^n}{du_2^{\ n}}\left(u_1^{\ n}u_2^{\ n}\ln\frac{1}{u_1}\ln\frac{1}{u_2}\right);\quad u_1=\frac{t_1}{(n+1)},u_2=\frac{t_2}{(n+1)}\,.$$



$Fig.2.$ Reconstruction of $f(t_1,t_2)=(\ln t_1+\gamma)(\ln t_2+\gamma)$ at $t_1=0..10, t_2=0..10$ from its two dimensional Laplace Carson transforms for selected values of n (in this section n=5).

4.2. Example 2

For reconstructing $f(t_1,t_2)=U(t_1-2)U(t_2-2)$ (unit step function) first we compute Laplace-Carson transform $\hat{f}_c(p,q)=e^{-2p-2q}$ then by consider $u_1=\frac{1}{p}$ and $u_2=\frac{1}{q}$ we obtain $\hat{f}_c(p,q)=e^{-\frac{2}{u_1}-\frac{2}{u_2}}$ .by using Eq .(18) for this $g(u_1,u_2)$ we obtain

$$f_n(t_1,t_2)=\frac{1}{(n!)}\frac{d^n}{du_1{}^n}\frac{d^n}{du_2{}^n}\left(u_1{}^n u_2{}^n e^{-\frac{2}{u_1}-\frac{2}{u_2}}\right); \quad u_1=\frac{t_1}{(n+1)}, u_2=\frac{t_2}{(n+1)}.$$

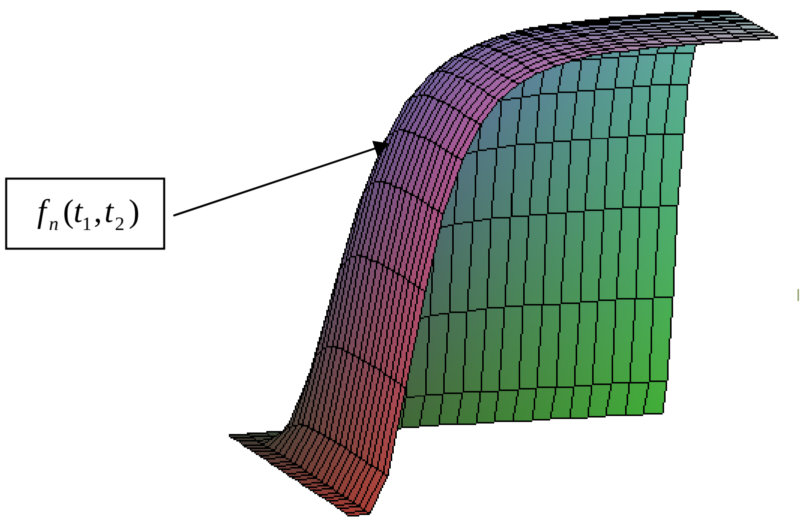


Fig.3. Reconstruction of $f(t_1,t_2)=U(t_1-2)U(t_2-2)$ at $t_1=0..10, t_2=0..10$ from its two dimensional Laplace Carson transform for selected values of n (in this section n=5).

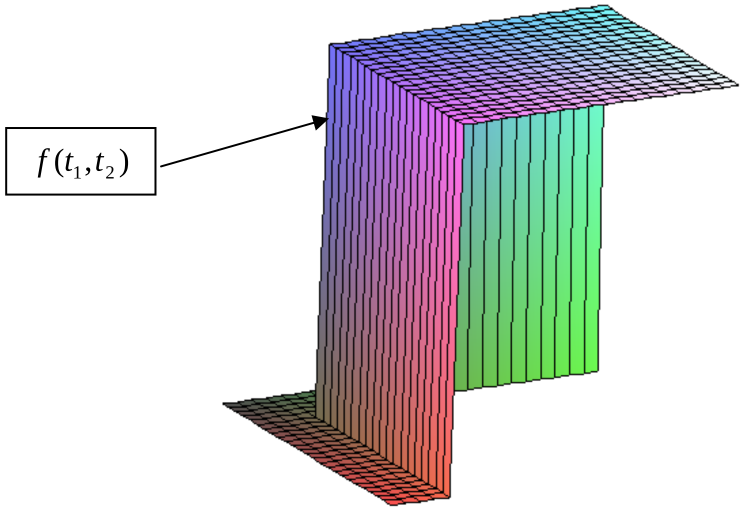


Fig.4. The exact plot of $f(t_1,t_2) = U(t_1-2)U(t_2-2)$ for $t_1 = 0..10, t_2 = 0..10$.

References

[1] C.Donalato, Analytical and numerical inversion of Laplace-Carson transform by a differential method, Italy, 2002, p298-309

[2] A.M.Makarov, Application of The Laplace-Carson method of integral transformation to the theory of unsteady visco-plastic flows, J.Engrg. Phs. Thermophys, 19(1970)94-99

[3] V.A.Ditkin and A.P.Prudnikov, Operational calculus in two variables and its applications, (English translation from Russian), Pergram Press (1962)

[4] R.S. Dahiya, computation of two-dimensional Laplace transforms, Rendiconti di atimatica, Roma 8, ser.vi (1975), 805-813.

[5] R.S. Dahia, Calculation of two-dimensional Laplace transform pairs-1,simon, A quarterly journal of Pure and Applied Mathematics 56 (1982), no. 1-2, 97-107

[6] G.E. Roberts and Kaufman,II., Table of Laplace transforms, Philadelphia:W.B. saunders co., 1966

[7] W.T. Weeks, Numerical inversion of Laplace transforms using Laguerre functions, J.ACM 13(1966) 419-429

[8] K.Singhal, J.Vlach, Numerical inversion of multi-dimensional Laplace transforms, proc. IEEE 63(1975) 1627-1628

[9] I.S. Gradshteyn, I, M.Ryzhik, Table of Integrals, series, and products, Academic, London, 1980

Received: November, 2009