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A Modified Backward Euler Scheme for the Diffusion Equation Subject to Purely Integral Conditions

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Abstract. In this paper, a modified backward Euler scheme is developped for obtaining approximate solution to the initial-boundary value problem for one-dimensional diffusion equation with purely integral conditions. Some experimental numerical results using the proposed numerical procedure are discussed.

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1. Introduction

In the recent years, evolution problems with integral condition(s) have received an increasing attention. The physical significance of integral conditions (mean, total flux, total energy, total mass, moments,...) has served as a fundamental reason for the interest carried to this type of problems.

The first investigation of such problems goes back to Cannon [3]. Later, similar problems have been studied following different approaches. A detailed survey is given in [1], [8] and [13].

Note that most of the papers on this type of problems were directed to one-dimensional second order parabolic equations which combine classical and integral boundary conditions. However, few works have been consecrated to second order parabolic equations with purely integral boundary conditions, among which, one quotes [1], [2] and [14], in which the authors proved the existence, uniqueness and continuous dependence upon the data of a solution by means of the energy-integral method, and the Rothe-time discretization method.

In this paper, we are interested in some finite difference schemes for approximating the function $u = u(x,t)$ of the following initial-boundary value problem with purely integral conditions. Find a function $u = u(x,t)$ satisfying the one-dimensional diffusion equation:

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = f(x,t), \qquad 0 \leq x \leq 1, 0 < t \leq T, \tag{1.1}$$

subject to the initial condition

$$u(x,0) = \varphi(x), \qquad 0 \leq x \leq 1, \tag{1.2}$$

and the integral conditions

$$\int_0^1 u(x,t)\,dx = E(t), \qquad 0 < t \leq T, \tag{1.3}$$

$$\int_0^1 xu(x,t)\,dx = G(t), \qquad 0 < t \leq T, \tag{1.4}$$

where f, φ, E, and G are known functions, α and T are known positive constants.

The paper is divided as follows. The mathematical model is given in Section 2. The modified backward Euler scheme for the solution of problem (1.1)-(1.4) is described in Section 3. Finally, in Section 4, we discuss some numerical computations for two test problems produced by the developed methods.

2. The mathematical model

In this Section, we follow [1] give a brief description of the mathematical model of the quasi-static flexure of a thermoelastic rod. Let us consider a rod $0 \leq x \leq 1$, the temperature $v = v(x,t)$ and the transverse displacement

$z = z(x,t)$. The quasi static flexure of thermoelastic rod can be described by the coupled partial differential equations [9]:

$$\lambda \frac{\partial^2 v}{\partial x^2} = \kappa \frac{\partial v}{\partial t} + v_0 \beta \frac{\partial^3 z}{\partial x^2 \partial t}, \tag{2.1}$$

$$\alpha \frac{\partial^4 z}{\partial x^4} = \beta \frac{\partial^2 v}{\partial x^2}, \tag{2.2}$$

where λ is the thermal conductivity, κ is the specific heat at constant strain, α is the flexural rigidity, β is a measure of the cross-coupling between thermal and mechanical effects, v_0 is a uniform reference temperature.

If we suppose that the initial temperature of the rod is $\varkappa(x)$, and the initial displacement is $\gamma(x)$; the ends $x = 0$ and $x = 1$ are clamped, namely

$$v(x,0) = \varkappa(x), \tag{2.3}$$

$$z(x,0) = \gamma(x), \tag{2.4}$$

$$z(0,t) = \frac{\partial z(0,t)}{\partial x} = z(1,t) = \frac{\partial z(1,t)}{\partial x} = 0, \tag{2.5}$$

the average temperature in the rod $0 \leq x \leq 1$ is equal to $\mu(t)$, i.e.,

$$\int_0^1 v(x,t)dx = \mu(t); \tag{2.6}$$

and the difference between the heat exchange of the atmosphere on the end $x = 0$ and the temperature on the end $x = 1$ is equal to $\zeta(t)$, then Newton's law imply

$$\frac{\partial v(0,t)}{\partial x} + v(0,t) - v(1,t) = \zeta(t). \tag{2.7}$$

We reformulate problem (2.1)-(2.7) into an equivalent form where the coupled P. D. Eqs (2.1)-(2.2) is reduced to one equation, we then introduce a new unknown function u defined as follows:

$$u(x,t) = \frac{\kappa}{v_0}(v - v_0) + \beta \frac{\partial^2 z}{\partial x^2}, \tag{2.8}$$

where u is the entropy. Then

$$\lambda \frac{\partial^2 v}{\partial x^2} = v_0 \frac{\partial u}{\partial t}, \tag{2.9}$$

and, therefore, the entropy is a solution of the heat equation:

$$\lambda \frac{\partial^2 u}{\partial x^2} = \left[\kappa + v_0 \frac{\beta^2}{\alpha}\right] \frac{\partial u}{\partial t}. \tag{2.10}$$

To deduce the initial condition on the entropy, we use the conditions (2.3) and (2.4), yields

$$u(x,0) = \frac{\kappa}{v_0}\left(\varkappa(x) - v_0\right) + \beta\gamma''(x) = u_0(x). \tag{2.11}$$

To deduce the first boundary condition on the entropy, we integrate (2.8) over $[0,1]$ with respect to x , by taking into account of (2.5) and (2.6), yields

$$\begin{aligned} &\int_0^1 u(x,t)dx \\ = &\ \frac{\kappa}{v_0}\left(\int_0^1 v(x,t)dx - \int_0^1 v_0 dx\right) + \beta\left(\frac{\partial z(1,t)}{\partial x} - \frac{\partial z(0,t)}{\partial x}\right) \\ = &\ \frac{\kappa}{v_0}\left(\mu(t) - v_0\right) = E(t), \end{aligned} \tag{2.12}$$

which is the average entropy. To conclude the second boundary condition, we multiply equation (2.9) by the weight $(1-x)$ and we integrate the result over $[0,1]$ by taking into account the condition (2.7), we obtain

$$\begin{aligned} &\int_0^1 xu(x,t)dx \\ = &\ E(t) + \frac{\lambda}{v_0}\int_0^t \zeta(\tau)d\tau + \int_0^1 xu_0(x)dx - E(0) = G(t), \end{aligned} \tag{2.13}$$

which is the weighted average entropy. Then, instead of searching for a pair (v,z) solution of (2.1)-(2.7), we search for the function u, solution of problem (2.10)-(2.13).

3. A modified backward Euler method

First, we take a positive integers N and M. We divide the intervals $[0,1]$ and $[0,T]$ into M and N subintervals of equal lengths $\Delta x = 1/M$ and $\Delta t = T/N$, respectively. The grid points (x_i, t_n) are given by

$$\begin{aligned} x_i &= i\Delta x, \qquad i = 0,1,\ldots,M, \\ t_n &= n\Delta t, \qquad n = 0,1,...,N. \end{aligned}$$

By u_i^n, we denote the approximation to u at the $i-th$ grid-point and $n-th$ time step. We also introduce the following notation:

$$\delta_x^2 u_i^k = u_{i-1}^k - 2u_i^k + u_{i+1}^k.$$

We start with the approximation of equation (1.1) by using the modified backward Euler method:

$$\frac{1}{\Delta t}\left(u_i^{n+1} - u_i^n\right) - \alpha\left(\frac{\partial^2 u}{\partial x^2}\right)_i^{n+1} = f_i^{n+1}, \tag{3.1}$$

where

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i^{n+1} \approx \frac{1}{(\Delta x)^2}\frac{\delta_x^2}{1+\frac{1}{12}\delta_x^2}u_i^{n+1}, \tag{3.2}$$

which is fourth-order accurate [15]. If we set in (3.2),

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i^{n+1} = w_i^{n+1} \tag{3.3}$$

and apply operator $\left(1+\frac{1}{12}\delta_x^2\right)$ to both sides, we get

$$\frac{1}{(\Delta x)^2}\delta_x^2 u_i^{n+1} = \left(1+\frac{1}{12}\delta_x^2\right)w_i^{n+1},$$

or

$$\begin{aligned}\frac{1}{(\Delta x)^2}\left(u_{i-1}^{n+1}-2u_i^{n+1}+u_{i+1}^{n+1}\right)\\ = \frac{1}{12}w_{i-1}^{n+1}+\frac{10}{12}w_i^{n+1}+\frac{1}{12}w_{i+1}^{n+1}.\end{aligned} \tag{3.4}$$

According to formula (3.1), we have

$$\frac{1}{\alpha\Delta t}\left(u_i^{n+1}-u_i^n-\Delta t\ f_i^{n+1}\right) = \left(\frac{\partial^2 u}{\partial x^2}\right)_i^{n+1} = w_i^{n+1}. \tag{3.5}$$

Then, by substituting (3.4) for $i-1$, i, and $i+1$ into (3.1), it yields

$$\begin{aligned}&\frac{1}{(\Delta x)^2}\left(u_{i-1}^{n+1}-2u_i^{n+1}+u_{i+1}^{n+1}\right)\\ =\ &\frac{1}{12\alpha\Delta t}\left(u_{i-1}^{n+1}-u_{i-1}^n-\Delta t\ f_{i-1}^{n+1}\right)\\ &+\frac{10}{12\alpha\Delta t}\left(u_i^{n+1}-u_i^n-\Delta t\ f_i^{n+1}\right)\\ &+\frac{1}{12\alpha\Delta t}\left(u_{i+1}^{n+1}-u_{i+1}^n-\Delta t f_{i+1}^{n+1}\right),\end{aligned}$$

giving

$$\begin{aligned}&(1-12\varrho)\,u_{i-1}^{n+1}+2\,(5+12\varrho)\,u_i^{n+1}+(1-12\varrho)\,u_{i+1}^{n+1}\\ =\ &u_{i-1}^n+10u_i^n+u_{i+1}^n+\Delta t\left(f_{i-1}^{n+1}+10f_i^{n+1}+f_{i+1}^{n+1}\right),\end{aligned} \tag{3.6}$$

for $1\le i\le M-1$ and $1\le n\le N$, where $\varrho=\alpha\Delta t/(\Delta x)^2$.
The initial condition (1.2) along $t=0$ is

$$u_i^0=\varphi_i, \qquad 0\le i\le M. \tag{3.7}$$

Owing to the fact that (3.6) represents $M-1$ linear equations with $M+1$ unknowns $u_0^{n+1}, u_1^{n+1}, \cdots, u_M^{n+1}$, we must eliminate u_0^{n+1} and u_M^{n+1} from the system which are not known a priori here. For this purpose, we approximate the integral conditions (1.3) and (1.4) by Simpson's rule which requires M to be even. Setting $M = 2m$, we have

$$\begin{aligned} &\int_0^1 \omega\left(x, t^k\right) dx \\ = \ & \frac{\Delta x}{3}\left(\omega_0^k + 4\omega_1^k + 2\omega_2^k + \cdots + 2\omega_{M-2}^k + 4\omega_{M-1}^k + \omega_M^k\right) + O\left((\Delta x)^4\right), \end{aligned}$$

so, we can write, for $k = n, n+1$,

$$u_0^k + 4u_1^k + 2u_2^k + \cdots + 2u_{M-2}^k + 4u_{M-1}^k + u_M^k = \frac{3E^k}{\Delta x}, \tag{3.8}$$

$$\begin{aligned} &4 \cdot x_1 u_1^k + 2 \cdot x_2 u_2^k + 4 \cdot x_3 u_3^k + \cdots + \\ &+2x_{M-2}u_{M-2}^k + 4x_{M-1}u_{M-1}^k + x_M u_M^k = \frac{3G^k}{(\Delta x)^2}, \end{aligned} \tag{3.9}$$

from where it comes

$$\begin{aligned} &u_M^k \\ = \ &\frac{1}{M}\left(\frac{3G^k}{(\Delta x)^2} - \left(4 \cdot 1u_1^k + 2 \cdot 2u_2^k + 4 \cdot 3u_3^k\right.\right. \\ &\left.\left. + \cdots + 2(M-2)u_{M-2}^k + 4(M-1)u_{M-1}^k\right)\right), \end{aligned} \tag{3.10}$$

$$\begin{aligned} &u_0^k \\ = \ &\frac{1}{M}\left(\frac{3ME^k}{\Delta x} - \frac{3G^k}{(\Delta x)^2}\right. \\ &-\left(4(M-1)u_1^k + 2(M-2)u_2^k\right. \\ &+4(M-3)u_3^k + \cdots + 4 \cdot 3u_{M-3}^k \\ &\left.\left.+2 \cdot 2u_{M-2}^k + 4 \cdot 1u_{M-1}^k\right)\right). \end{aligned} \tag{3.11}$$

Then, substituting (3.11) into (3.6) for $i = 1$ and (3.10) into (3.6) for $i = M-1$, we get, respectively, after some rearrangement

$$\begin{aligned}
(3.12)\qquad & \left(-\left(1-12\varrho\right)\left(6M-4\right)+12M\right)u_1^{n+1} \\
& -\left(1-12\varrho\right)\left(\left(M-4\right)u_2^{n+1}+4\left(M-3\right)u_3^{n+1}+\cdots\right. \\
& \left.+4\cdot 3u_{M-3}^{n+1}+2\cdot 2u_{M-2}^{n+1}+4\cdot 1u_{M-1}^{n+1}\right) \\
= \; & \left(-\left(6M-4\right)+12M\right)u_1^{n}-\left(M-4\right)u_2^{n}-\left(4\left(M-3\right)u_3^{n}+\cdots\right. \\
& \left.+4\cdot 3u_{M-3}^{n}+2\cdot 2u_{M-2}^{n}+4\cdot 1u_{M-1}^{n}\right) \\
& -\frac{3M}{\Delta x}\left(\left(1-12\varrho\right)E^{n+1}-E^{n}\right) \\
& +\frac{3}{\left(\Delta x\right)^2}\left(\left(1-12\varrho\right)G^{n+1}-G^{n}\right) \\
& +M\Delta t\left(f_0^{n+1}+10f_1^{n+1}+f_2^{n+1}\right),
\end{aligned}$$

$$\begin{aligned}
(3.13)\qquad & -\left(1-12\varrho\right)\left(4\cdot 1u_1^{n+1}+2\cdot 2u_2^{n+1}\right. \\
& +4\cdot 3u_3^{n+1}+\cdots+4\left(M-3\right)u_{M-3}^{n+1} \\
& \left.+\left(M-4\right)u_{M-2}^{n+1}\right)+\left(-\left(1-12\varrho\right)\left(6M-4\right)+12M\right)u_{M-1}^{n+1} \\
= \; & -\left(4\cdot 1u_1^{n}+2\cdot 2u_2^{n}+4\cdot 3u_3^{n}+\cdots+4\left(M-3\right)u_{M-3}^{n}\right. \\
& \left.+\left(M-4\right)u_{M-2}^{n}\right)+\left(6M+4\right)u_{M-1}^{n}\Big) \\
& +\frac{3}{\left(\Delta x\right)^2}\left(-\left(1-12\varrho\right)G^{n+1}+G^{n}\right) \\
& +M\Delta t\left(f_{M-1}^{n+1}+10f_M^{n+1}+f_{M+1}^{n+1}\right).
\end{aligned}$$

Therefore, we obtain a linear system of equations $(M-1)\times(M-1)$, by setting (3.12) as the first, (3.13) as the $(M-1)-th$, and (3.6) for $2 \le i \le M-2$ as the other equations. Therefore, this system can be written in the following way

$$AU^{n+1} = BU^{n} + \Delta t C^{n+1} + D^{n+1}, \tag{3.14}$$

where

$$A = -\left(1-12\varrho\right)P + 12Q, \tag{3.15}$$

$$B = -P + 12Q, \tag{3.16}$$

(3.17)

$$P=\begin{pmatrix} 6M-4 & M-4 & 4\left(M-3\right) & \cdots & \cdots & 4.3 & 2.2 & 4.1 \\ -1 & 2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & & & & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \vdots & & & & & & 0 \\ & & & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & & & & -1 & 2 & -1 \\ 4.1 & 2.2 & 4.3 & \cdots & \cdots & 4\left(M-3\right) & M-4 & 6M-4 \end{pmatrix},$$

$$Q=\begin{pmatrix} M & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & M \end{pmatrix}, \tag{3.18}$$

$$C^{n+1}=\begin{pmatrix} M\left(f_0^{n+1}+10f_1^{n+1}+f_2^{n+1}\right) \\ f_1^{n+1}+10f_2^{n+1}+f_3^{n+1} \\ \vdots \\ f_{M-3}^{n+1}+10f_{M-2}^{n+1}+f_{M-1}^{n+1} \\ M\left(f_{M-2}^{n+1}+10f_{M-1}^{n+1}+f_M^{n+1}\right) \end{pmatrix}, \tag{3.19}$$

(3.20)

$$D^{n+1}=\begin{pmatrix} -\frac{3M}{\Delta x}\left(\left(1-12\varrho\right)E^{n+1}-E^n\right)+\frac{3}{\left(\Delta x\right)^2}\left(\left(1-12\varrho\right)G^{n+1}-G^n\right) \\ 0 \\ \vdots \\ 0 \\ -\frac{3}{\left(\Delta x\right)^2}\left(\left(1-12\varrho\right)G^{n+1}-G^n\right) \end{pmatrix},$$

$$U^n=\begin{pmatrix} u_1^n \\ \vdots \\ \\ \vdots \\ u_{M-1}^n \end{pmatrix}, \qquad U^{n+1}=\begin{pmatrix} u_1^{n+1} \\ \vdots \\ \\ \vdots \\ u_{M-1}^{n+1} \end{pmatrix}. \tag{3.21}$$

4. Numerical test results

In this section, we report some results of numerical computations using finite difference schemes proposed in the previous section. These techniques are applied to solve three test problems.

Test 1. The test problem is given as follows:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-\exp-(x+t)\,, \tag{4.1}$$

$$u(x,0)=\exp(-x)\,, \tag{4.2}$$

$$\int_0^1 u(x,t)\,dx=\left(1-e^{-1}\right)\cosh(t)\,, \tag{4.3}$$

$$\int_0^1 xu(x,t)\,dx=\left(1-2e^{-1}\right)\cosh(t)\,. \tag{4.4}$$

The analytical solution of this problem is:

$$u(x,t)=\exp(-x)\cosh(t)\,. \tag{4.5}$$

The relative error computed by $\frac{U_{approx}-U_{exact}}{U_{exact}}$ are shown in Table I.

x	$U_{approx.}$	U_{exact}	$Error$
0.20	0.885107197	0.885123767	-0.000018721
0.40	0.724664482	0.724198494	0.000643039
0.60	0.593305097	0.592891910	0.000696416
0.80	0.485757129	0.485839980	-0.000170560

Table I

2. The second problem is given as follows

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}, \tag{4.6}$$

$$u(x,0)=\sin(\pi x)\,, \tag{4.7}$$

$$\int_0^1 u(x,t)\,dx=\frac{2}{\pi}\exp\left(-\pi^2 t\right)\,, \tag{4.8}$$

$$\int_0^1 xu(x,t)\,dx=\frac{1}{\pi}\exp\left(-\pi^2 t\right)\,. \tag{4.9}$$

The analytical solution is:

$$u(x,t)=\exp\left(-\pi^2 t\right)\sin(\pi x)\,. \tag{4.10}$$

The computed results are shown in Table II.

x	$U_{approx.}$	U_{exact}	$Error$
0.20	0.88505949	0.88510720	0.00005390
0.40	0.72591464	0.72466448	-0.00172515
0.60	0.59443146	0.59330510	-0.00189845
0.80	0.48559736	0.48575713	0.00032890

Table II

3. The third problem is given as follows

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = -\frac{2\left(x^2+1+t\right)}{\left(1+t\right)^3}, \tag{4.11}$$

$$u\left(x,0\right) = x^2, \tag{4.12}$$

$$\int_0^1 u\left(x,t\right)dx = \frac{1}{3\left(1+t\right)^2}, \tag{4.13}$$

$$\int_0^1 xu\left(x,t\right)dx = \frac{1}{4\left(1+t\right)^2}. \tag{4.14}$$

The analytical solution of this problem is

$$u(x,t) = \left(\frac{x}{1+t}\right)^2. \tag{4.15}$$

The computed results are shown in Table III:

x	$U_{approx.}$	U_{exact}	$Error$
0.20	0.885107197	0.885123767	-0.000018721
0.40	0.724664482	0.724198494	0.000643039
0.60	0.593305097	0.592891910	0.000696416
0.80	0.485757129	0.485839980	-0.000170560

Table III

5. Conclusion

In this article, a modified backward Euler schemes is developed for one-dimensional heat equation with purely integral conditions. This method is first-order accurate in time and fourth-order accurate in space, and it is scheme even more implicit: an implicit scheme plus implicit boundary conditions. They are a highly accurate numerical integration, which is stable and absorbent of error. Numerical examples are provided to confirm the accuracy. The authors plan to generalize this method to fourth-order accurate in time and space.

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