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A Modified Mann Iteration Process for Common Fixed Points of an Infinite Family of Nonexpansive Mappings in Banach Spaces

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Abstract

Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings from K to itself with $F := \{x \in K : T_i x = x, \forall i \geq 1\} \neq \emptyset$. For an arbitrary initial point $x_1 \in K$, the modified Mann iteration scheme $\{x_n\}$ is defined as follows: $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n^* x_n, n \geq 1$, where $T_n^* = T_i$ with i satisfying: $n = [(k-i+1)(i+k)/2] + [1+(i-1)(i+2)/2], k \geq i-1$, and $\{\alpha_n\}$ is a sequence in $[a, 1-a]$ for some $a \in (0,1)$. Under some suitable conditions, the strong and weak convergence theorems of $\{x_n\}$ to a common fixed point of the nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$ are obtained.The results presented in this paper extend those of the authors whose research areas are limited to the situation where the operators in the iteration procedure is just a finite family of nonexpansive mappings. Furthermore, for each T_i, a faster rate of convergence of $\{x_n\}$ to the common fixed point of $\{T_i\}_{i=1}^{\infty}$ is obtained.

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1 Introduction

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. A self-mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.

Iterative techniques for approximating fixed points of nonexpansive mappings have been studied by various authors(see, e.g.,[1–4])who used the Mann iteration process or the Ishikawa process. In 2001, Xu and Ori [5] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$, where $I = \{1, 2, ..., N\}$. For any initial point $x_0 \in K$:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n, \tag{1.1}$$

where $T_n = T_{n(mod\ N)}$, the mod function N takes values in I, and $\{\alpha_n\}$ is a real sequence in $(0, 1)$. They proved the weak convergence of the process above to a common fixed point of the finite family of nonexpansive mappings.

Inspired and motivated by those work mentioned above, in this paper, we will construct a modified Mann iteration scheme for approximating common fixed points of an infinite family of nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$ and to prove some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces.

2 Preliminaries

A Banach space E is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \tag{2.1}$$

for all $y \in E$ with $y \neq x$, where $x_n \rightharpoonup x$ denotes that $\{x_n\}$ converges weakly to x.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

We need the following lemmas for our main results.

Lemma 2.1. [6] *Let $\{a_n\}, \{\delta_n\}$, and $\{b_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \quad \forall n \ge 1, \tag{2.2}$$

if $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.2. [7] *Let E be a real uniformly convex Banach space and let a, b be two constant with $0 < a < b < 1$. Suppose that $\{t_n\} \subset [a, b]$ is a real sequence and $\{x_n\}, \{y_n\}$ are two sequences in E. Then the conditions*

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = d, \limsup_{n\to\infty} \|x_n\| \le d, \limsup_{n\to\infty} \|y_n\| \le d \tag{2.3}$$

imply that $lim_{n\to\infty}\|x_n - y_n\| = 0$, *where* $d \geq 0$ *is a constant.*

Lemma 2.3. [8] *Let E be a real uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let $T : K \to K$ be a nonexpansive mapping. Then $I - T$ is demi-closed at zero.*

3 Main Results

Lemma 3.1. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings from K to itself. For an arbitrary initial point $x_1 \in K$, the modified Mann iteration scheme $\{x_n\}$ is defined as follows:*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n^* x_n, n \geq 1, \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $[a, 1-a]$ for some $a \in (0,1)$ and $T_n^* = T_i$ with i satisfying the following equation:

$$n = [(k - i + 1)(i + k)/2] + [1 + (i - 1)(i + 2)/2], k \geq i - 1, \tag{3.2}$$

where k is a nonnegative integer. That is

$$T_1^* = T_1, T_2^* = T_1, T_3^* = T_2, T_4^* = T_1, T_5^* = T_2, T_6^* = T_3, T_7^* = T_1, T_8^* = T_2, ...$$

If $F := \{x \in K : T_i x = x, \forall i \geq 1\} \neq \emptyset$, then

(1) $\lim_{n\to\infty} \|x_n - q\| exists, \forall q \in F$;

(2) $\lim_{n\to\infty} d(x_n, F) exists, where\ d(x_n, F) = \inf_{q\in F} \|x_n - q\|$;

(3) $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \forall i \geq 1$.

Proof.(1) For any $q \in F$, by (3.1), we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1 - \alpha_n)(x_n - q) + \alpha_n(T_n^* x_n - q)\| \\ &\leq (1 - \alpha_n)\|x_n - q\| + \alpha_n\|x_n - q\| \\ &= \|x_n - q\|. \end{aligned} \tag{3.3}$$

It follows from Lemma 2.1 that $\lim_{n\to\infty} \|x_n - q\|$ exists.

(2) This conclusion can be easily shown by taking the infimum in (3.3) for all $q \in F$.

(3) Assume, by the conclusion of (1), $\lim_{n\to\infty} \|x_n - q\| = d$. That is

$$\lim_{n\to\infty} \|(1 - \alpha_n)(x_n - q) + \alpha_n(T_n^* x_n - q)\| = d. \tag{3.4}$$

In addition,

$$\limsup_{n\to\infty} \|T_n^* x_n - q\| \leq \limsup_{n\to\infty} \|x_n - q\| = d, \tag{3.5}$$

and hence, it follows from (3.4), (3.5) and Lemma 2.2 that

$$\lim_{n\to\infty} \|x_n - T_n^* x_n\| = 0. \tag{3.6}$$

On the other hand, since $\|x_{n+1} - x_n\| = \alpha_n \|x_n - T_n^* x_n\|$, we obtain, by (3.6),

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}$$

By induction, for any nonnegative integer j, we also have

$$\lim_{n\to\infty} \|x_{n+j} - x_n\| = 0. \tag{3.8}$$

For each $i \geq 1$, since

$$\begin{aligned} \|x_n - T_{n+i}^* x_n\| &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}^* x_n\| \\ &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}^* x_{n+i}\| \\ &\quad + \|T_{n+i}^* x_{n+i} - T_{n+i}^* x_n\| \\ &\leq 2\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}^* x_{n+i}\|, \end{aligned} \tag{3.9}$$

it follows from (3.6) and (3.8) that

$$\lim_{n\to\infty} \|x_n - T_{n+i}^* x_n\| = 0. \tag{3.10}$$

Now, for each $i \geq 1$, we claim that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0. \tag{3.11}$$

As a matter of fact, setting

$$n = N(k,i) + i,$$

where $N(k,i) = [(k-i+1)(i+k)/2 + (1 + (i-1)(i+2)/2)] - i$, we obtain that

$$\begin{aligned} \|x_n - T_i x_n\| &\leq \|x_n - x_{N(k,i)}\| + \|x_{N(k,i)} - T_i x_n\| \\ &\leq \|x_n - x_{N(k,i)}\| + \|x_{N(k,i)} - T_{N(k,i)+i}^* x_{N(k,i)}\| \\ &\quad + \|T_{N(k,i)+i}^* x_{N(k,i)} - T_i x_n\| \\ &= \|x_n - x_{N(k,i)}\| + \|x_{N(k,i)} - T_{N(k,i)+i}^* x_{N(k,i)}\| \\ &\quad + \|T_i x_{N(k,i)} - T_i x_n\| \\ &\leq 2\|x_n - x_{N(k,i)}\| + \|x_{N(k,i)} - T_{N(k,i)+i}^* x_{N(k,i)}\| \\ &= 2\|x_n - x_{n-i}\| + \|x_{N(k,i)} - T_{N(k,i)+i}^* x_{N(k,i)}\|. \end{aligned} \tag{3.12}$$

Then, since $N(k,i) \to \infty$ as $n \to \infty$, it follows from (3.8) and (3.10) that (3.11) holds obviously. This completes the proof.

Theorem 3.2. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings from K to itself. Suppose that $\{x_n\}$ is a sequence defined by (3.1). If $F := \{x \in K : T_i x = x, \forall i \geq 1\} \neq \emptyset$ and there exist $T_{i_0} \in \{T_i\}_{i=1}^{\infty}$ and a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $f(d(x_n, F)) \leq \|x_n - T_{i_0} x_n\|$ for all $n \geq 1$, then $\{x_n\}$ converges strongly to some common fixed point of $\{T_i\}_{i=1}^{\infty}$.*

Proof. Since

$$f(d(x_n, F)) \leq \|x_n - T_{i_0} x_n\|,$$

by taking *limsup* as $n \to \infty$ on both sides in the inequality above, we get

$$\lim_{n\to\infty} f(d(x_n, F)) = 0,$$

which implies $\lim_{n\to\infty} d(x_n, F) = 0$ by the definition of the function f.

Now we will show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, F) = 0$, then for any $\epsilon > 0$, there exists a positive integer N such that $d(x_n, F) < \epsilon/2$ for all $n \geq N$. On the other hand, there exists a $p \in F$ such that $\|x_N - p\| = d(x_N, F) < \epsilon/2$, because $d(x_N, F) = \inf_{q\in F} \|x_N - q\|$ and F is closed.

Thus, for any $n, m \geq N$, it follows from (3.3) that

$$\|x_n - x_m\| \leq \|x_n - p\| + \|x_m - p\| \leq 2\|x_N - p\| < \epsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Thus, there exists a $x \in K$ such that $x_n \to x$ as $n \to \infty$, since E is complete. Then, $\lim_{n\to\infty} d(x_n, F) = 0$ yields that $d(x, F) = 0$. Further, it follows from the closedness of F that $x \in F$. This completes the proof.

Theorem 3.3. *Let E be a real uniformly convex Banach space satisfying Opial's condition and K a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings from K to itself. Suppose that $\{x_n\}$ is a sequence defined by (3.1). If $F := \{x \in K : T_i x = x, \forall i \geq 1\} \neq \emptyset$, then $\{x_n\}$ converges weakly to some common fixed point of $\{T_i\}_{i=1}^{\infty}$.*

Proof. For any $q \in F$, by Lemma 3.1, we know that $\lim_{n\to\infty} \|x_n - q\|$ exists. We now prove that $\{x_n\}$ has a unique weakly subsequential limit in F. First of all, Lemmas 2.3 and 3.1 guarantee that each weakly subsequential limit of $\{x_n\}$ is a common fixed point of $\{T_i\}_{i=1}^{\infty}$. Secondly, Opial's condition guarantees that the weakly subsequential limit of x_n is unique. Consequently, $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^{\infty}$. This completes the proof.

Remark 3.4. *The results presented in this paper extend those of the authors whose research areas are limited to the situation where the operators used in* *the iteration procedure is just a finite family of nonexpansive mappings. Furthermore, for each T_i, the rate of convergence of $\{x_n\}$ to the common fixed point of $\{T_i\}_{i=1}^{\infty}$ is greatly enhanced because of the gradual increase of span of the operator.*

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