# On the Output Stabilizability
# of the Diffusion Equation

**Faouzi Haddouchi**

Department of Physics, University of Sciences and Technology
El M'naouar, BP 1505, Oran, 31000, Algeria
f_haddouchi@yahoo.fr

### Abstract

This note is devoted to study the output stabilizability of a simplified and a one-dimensional diffusion equation. Necessary and sufficient conditions for the system to be output stabilizable will be given. These conditions are given in terms of the eigenvalues of the infinitesimal generator and the Fourier coefficients of input and output operators.

**Mathematics Subject Classification:** 93D15, 93C25

**Keywords:** Infinite dimensional systems, controllability, state stabilizability, output stabilizability, diffusion equation

## 1 Introduction

In this note, we consider the output stabilizability of the diffusion equation on the interval $(0, 1)$:

$$
\begin{cases}
\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial \xi^2} - \alpha \frac{\partial z}{\partial \xi} + kz + b(\xi)u(t) \\
\qquad\qquad z(\xi, 0) = z_0(\xi) \\
\qquad z(0, t) = z(1, t) = 0
\end{cases}
\tag{1}
$$

where $b \in L^2(0, 1)$, $\alpha > 0$ and $k > 0$.

With the output function given by

$$
y(t) = \int_0^1 \exp(-\alpha\xi)c(\xi)z(\xi, t)d\xi.
\tag{2}
$$

Take $H = L^2(0, 1)$ to be the Hilbert space with the weighted inner product

$$
\langle f, g \rangle = \int_0^1 \exp(-\alpha\xi)f(\xi)g(\xi)d\xi.
\tag{3}
$$

The system (1),(2) can be rewritten in the abstract form with state space $H$

$$\dot{x}(t) = Ax(t) + Bu(t), \ x(0) = x_0 \tag{4}$$

where $B = b, \ y(t) = \langle \ c, z(.,t) \rangle_H = Cz(.,t)$.

$$A = A_0 + kI, \ \text{and} \ A_0 h = \frac{d^2 h}{d\xi^2} - \alpha \frac{dh}{d\xi} \tag{5}$$

for $h$ in the domain of $A_0$ given by

$$D(A_0) = \left\{ \begin{array}{c} h : \ h, \frac{dh}{d\xi} \ \text{are absolutely continuous} \\ \text{and} \ \frac{d^2 h}{d\xi^2} \in H, \ h(0) = h(1) = 0 \end{array} \right\} \tag{6}$$

It is no hard to show that $A$ is self-adjoint with eigenvalues $\lambda_n = -\frac{\alpha^2}{4} - n^2 \pi^2 + k$ and normalized eigenvectors $\phi_n(\xi) = \sqrt{2} \exp(\alpha\xi/2) \sin(n\pi\xi), \ n \in \mathbb{N}$, which form an orthonormal basis for $L^2(0,1)$.

A focus of this paper is to give a criterion for the output stabilization by a linear bounded feedback $u = Fx, F \in L(H, \mathbb{R})$. The motivation for considering this class of systems is given by the work of [2], that gave a result on state stabilizability for a class of distributed parameter systems.

The paper is structured as follows. In second section, we shall review some well-known concepts of approximate controllability, state and output stabilizability for infinite dimensional systems defined in Hilbert spaces.

The third section deals with controllability and stabilization for the class of systems studied here. A fully explicit description of the controllable and uncontrollable subspaces for this class of systems is given in section 3. We also give a criterion for output stabilizability. Finally, we shall conclude the paper with some examples.

## 2 Preliminary Notes

In the beginning of this section let us recall some definitions. Consider the abstract system (S) with the state given by

$$\dot{x}(t) = Ax(t) + Bu(t), \ x(0) = x_0 \tag{7}$$

and the output given by

$$y(t) = Cx(t) \tag{8}$$

with the following hypothesis:

(*i*) $x(t) \in H$ (the state space), $u(t) \in U$ (the input space) and $y(t) \in Y$ (the output space), where $H$, $U$ and $Y$ are always intended infinite dimensional Hilbert spaces unless otherwise stated;

(*ii*) $B$ and $C$ are linear and continuous operators, i.e., $B \in L(U,H)$, $C \in L(H,Y)$;

(*iii*) The operator $A$ is an infinitesimal generator of a $C_0$-semigroup $S(t)$ on the state space $H$. As usual $u$, $x$, $y$ represent respectively the input, state and output of the system (7) and (8).

**Definition 2.1** *The system (7) (or the pair $(A,B)$) is approximately controllable if $N = \{0\}$.*

*Where $N = \bigcap_{t \geq 0} \ker B^* S^* (t)$ .*

*$L = N^\perp$ and $N$ are called, the controllable and uncontrollable subspaces of the system (7), respectively.*

Following [6], we can decompose the state space $H$ as $L \oplus N$ and then $A$, $B$ and $C$ are represented by the operators matrix

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \tag{9}$$

Using these operators, we arrive at the split case:

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + B_1 u \\ \dot{x}_2 = A_{22}x_2 \\ y = y_1 + y_2 \end{cases} \tag{10}$$

where $y_i = C_i x_i$ , for $i = 1, 2$.

**Definition 2.2** *The pair $(A, B)$ is called (exponentially) stabilizable if there is an $F \in L(H,U)$ such that the semigroup $S_{A+BF}(t)$ is (exponentially) asymptotically stable.*

*Where $S_{A+BF}(t)$ is the semigroup generated by $A + BF$.*

It follows immediately that if the control is given by the feedback $u = Fx$, for all $x_0 \in H$ there exists positive $M$ and $\omega$ such that

$$\|x(t)\| \leq M \exp(-\omega t) \|x_0\|$$

and therefore $x(t) \to 0$, if $t \to \infty$.

**Definition 2.3** *The system (7), (8) is output stabilizable by a bounded feedback if there is an $F \in L(H,U)$ such that the output $y(t)$ of the closed system*

$$\dot{x}(t) = (A + BF)x(t), \ x(0) = x_0 \tag{11}$$

*is exponentially stable, i.e., $y(t)$ converges to zero when $t \to \infty$ , for every $x_0 \in H$.*

See e.g.,[1],[5], [6].

## 3 Main Results

Under assumption about our system operator $A$, $A$ and $S(t)$ have the spectral decompositions

$$Ax = \sum_{n=1}^{\infty} \lambda_n . E(\lambda_n) x \quad \text{for } x \in D(A) \tag{12}$$

$$S(t) = \sum_{n=1}^{\infty} \exp(\lambda_n t) E(\lambda_n) \tag{13}$$

where $E(\lambda_n)$ are the spectral projections associated with the eigenvalues $\lambda_n$ of $A$ and are given by

$$E(\lambda_n) = \langle . , \phi_n \rangle \phi_n. \tag{14}$$

Furthermore, $x \in H$ also has the decomposition

$$x = \sum_{n=1}^{\infty} E(\lambda_n) x. \tag{15}$$

**Proposition 3.1** *The system* (4) *(or the pair* $(A, b)$*) is (exponentially) stabilizable if and only if the operator* $A_{22}$ *is (exponentially) stable.*

**Proof:** Since $(A_{11}, B_1)$ is approximately controllable by construction. Then, by [5] it follows that the pair $(A_{11}, B_1)$ is exponentially stabilizable. From [6] we can get directly the desired result.

Before we shall prove our main result, we need some technical lemmas.

**Lemma 3.2** *The uncontrollable subspace $N$ of the system* (4) *is of the following form*

$$N = \overline{span} \{ \phi_n, \ n \in J \subset \mathbb{N} \ / \ B^* \phi_n = 0 \} \tag{16}$$

*where $B^* = \langle b, . \rangle_H$ and $\overline{span} \{ e_n, \ n \in I \}$ denotes the closed subspace generated by the vectors $e_n, \ n \in I$.*

**Proof:** By the definition of $N$ and according to [4], this subspace is closed and is invariant for $S^*(t) = S(t)$. Then by the proof of theorem IV.6 in [3], $N$ is of the following form

$$N = \sum_{n \in J} E(\lambda_n) N \quad \text{and} \quad E(\lambda_n) N \subset N \quad \text{for all } n \text{ in } J$$

where $J = \{n \ / \ E(\lambda_n) \ N \neq \{0\}\}$. We have $B^* S^* (t) x = 0$ if and only if for all $t \geq 0$

$$\sum_{n=1}^{\infty} \exp(\lambda_n t)\langle x, \ \phi_n\rangle \ \langle \ b, \phi_n\rangle = 0$$

First let $x \in E(\lambda_{n_0})N$, $x \neq 0$, for a certain $n_0 \in J$. Then, since $E(\lambda_{n_0})N \subset N$, it follows from [7] that

$$\langle x, \ \phi_{n_0}\rangle\langle \ b, \phi_{n_0} \ \rangle = 0 \tag{17}$$

Rewriting equation (17) gives

$$B^* \phi_{n_0} = 0.$$

This shows that

$$N \subset \overline{span} \ \{\phi_n, \ n \in J \subset \mathbb{N} \ / \ \langle \ b, \phi_n\rangle_H \ = 0\}$$

Now it remains to verify that $\phi_n \in N$, where $\langle \ b, \phi_n\rangle_H \ = 0$ for $n \in J$. But the proof of this part is easy and will be omitted here.

Using the precise description of $N$ and the fact that $L = N^\perp$ one can immediately get.

**Lemma 3.3** *The controllable subspace L of the system (4) is given by*

$$L = \overline{span} \ \{\phi_n \ / \ \langle \ b, \phi_n\rangle_H \ \neq 0\}. \tag{18}$$

As a main result of this paper we establish the following proposition:

**Proposition 3.4** *The system (4) is output stabilizable if and only if*

$$\lambda_n < 0 \ \ for \ all \ \ n \ in \ \ K, \tag{19}$$

*where* $K = \{ \ n \ / \ \langle \ c, \phi_n\rangle \neq 0 \ and \ \langle \ b, \phi_n\rangle = 0 \ \}.$

**Proof:** From [6] we have that $A_{ii}$ is the infinitesimal generator of a $C_0$-semigroup $S_i \ (t)$ on $H_i$, for $i = 1, 2$. $H_1 = L$, $H_2 = N$.

Furthermore, it follows that with respect to the spectral decomposition of $A$ we have

$$S_1(t) = \sum_{n \in I} \exp(\lambda_n t)E(\lambda_n) \ , \ \ S_2(t) = \sum_{n \in \mathbb{N}-I} \exp(\lambda_n t)E(\lambda_n)$$

where $I = \{n \ / \ \langle \ b, \phi_n \rangle \neq 0\}$.

According to the proof of proposition 3.1, it follows that the output $y$ of the system (4) is exponentially stabilizable if and only if the output $y_2$ is exponentially stable.

In order to study the stability of the output $y_2(t) = C_2 x_2(t)$ on $N$, we again consider the subsystem

$$\begin{cases} \dot{x}_2 = A_{22} x_2, & x_2(0) = x_{02} \\ y_2 = C_2 x_2 \end{cases} \tag{20}$$

where $x(0) = x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \in L \oplus N$ .

The output $y_2(t) = C_2 S_2(t) x_{02}$ of the subsystem (20) is given by

$$y_2(t) = \sum_{n \in \mathbb{N} - I} \exp(\lambda_n t) \langle x_0, \ \phi_n \rangle \langle \ c, \phi_n \rangle \tag{21}$$

Using a similar argument as above one can decompose the state space $N$ of the subsystem (20) as $M \oplus W$, where $M = \bigcap_{t \geq 0} \ker C_2 S_2(t)$ is the unobservable subspace of the pair $(C_2, A_{22})$ and $W = M^\perp$ is the observable subspace of the subsystem (20).

The operators $A_{22}$, $C_2$ may be written in the form

$$A_{22} = \begin{pmatrix} A_{22}^1 & 0 \\ 0 & A_{22}^2 \end{pmatrix}, C_2 = \begin{bmatrix} 0 & C_2^2 \end{bmatrix} \tag{22}$$

Subsystem (20) can then be written as:

$$\begin{cases} \dot{x}_2^1 = A_{22}^1 x_2^1 \\ \dot{x}_2^2 = A_{22}^2 x_2^2 \\ y_2 = C_2^2 x_2^2 \end{cases} \tag{23}$$

where $x_{02} = \begin{bmatrix} x_{02}^1 \\ x_{02}^2 \end{bmatrix} \in M \oplus W$ .

The stability of the output $y_2$ on $N$ can then be analyzed by studying it on the observable subspace $W$ of the subsystem (20). A similar argument as that used above can be used to show that the observable subspace of the pair $(C_2, A_{22})$ is given by

$$W = \overline{span} \{\phi_n \ / \ \langle \ c, \phi_n \rangle \neq 0\} \tag{24}$$

and the output $y_2(t) = C_2^2 S_2^2(t) x_{02}^2$ of the subsystem (20) is given by

$$y_2(t) = \sum_{n \in K} \exp(\lambda_n t) \langle x_0, \ \phi_n \rangle \langle \ c, \phi_n \rangle \tag{25}$$

where $K = \{ n \ / \ \langle c, \phi_n \rangle \neq 0 \text{ and } \langle b, \phi_n \rangle = 0 \}, S_2^{i}(t)$ being the semi-group generated by $A_{22}^{i}$ for $i = 1, 2$.

The necessary condition is straightforward. So we concentrate on the Sufficiency. From [7] and [5] it follows that if $\lambda_n < 0$ for all $n$ in $K$, then the output $y_2(t)$ is exponentially stable. Hence the output $y(t)$ of the system (4) is exponentially stabilizable.

# 4    Examples

**Example 4.1** *By choosing*

$$b(\xi) = \chi_{[p_1, q_1]}(\xi), \ \ c(\xi) = \chi_{[p_2, q_2]}(\xi), \tag{26}$$

*where $\chi_{[a, b]}$ denotes the characteristic function of the interval $[a, b]$. Straightforward calculations show that*

$$\begin{cases} b_n = \frac{-2\sqrt{2}\alpha}{\alpha^2 + 4n^2\pi^2} [e^{\frac{-\alpha q_1}{2}} A_{n,q_1} - e^{\frac{-\alpha p_1}{2}} A_{n,p_1}] \\ c_n = \frac{-2\sqrt{2}\alpha}{\alpha^2 + 4n^2\pi^2} [e^{\frac{-\alpha q_2}{2}} A_{n,q_2} - e^{\frac{-\alpha p_2}{2}} A_{n,p_2}] \end{cases} \tag{27}$$

*where $c_n = \langle c, \phi_n \rangle$ , $b_n = \langle b, \phi_n \rangle$, $n \in \mathbb{N}$*
*$A_{n,m} = (\sin(n\pi m) + (2n\pi/\alpha)\cos(n\pi m))$.*

*Take $p_1 = p_2 = 1/4, q_1 = 1/2$ and $q_2 = 3/4$. Since $(A, b)$ is controllable it is clear that the output of the system (4) is exponentially stabilizable.*

**Example 4.2** *In this example we take $\alpha = 0$ and*

$$b(\xi) = \chi_{\left[\frac{1}{4}, \frac{3}{4}\right]}(\xi), \ \ c(\xi) = \chi_{\left[\frac{1}{4}, \frac{1}{2}\right]}(\xi). \tag{28}$$

*Elementary calculations show then that*

$$b_n = \langle b(\xi), \phi_n(\xi) \rangle_{L_2(\frac{1}{4}, \frac{3}{4})} = -\frac{2\sqrt{2}}{n\pi} \sin\left[\frac{n\pi}{2}\right] \sin\left[\frac{n\pi}{4}\right], \tag{29}$$

$$c_n = \langle c(\xi), \phi_n(\xi) \rangle_{L_2(\frac{1}{4}, \frac{1}{2})} = -\frac{2\sqrt{2}}{n\pi} \sin\left[\frac{n\pi}{8}\right] \sin\left[\frac{3n\pi}{8}\right]. \tag{30}$$

*A simple calculation show that the index set $K$ takes the form*

$$K = \{8p + 2, \ 8p + 4, \ 8p + 6; \ p \in \mathbb{N}\}. \tag{31}$$

*Thus concerning proposition 3.4, we have that for $k = \pi^2$ the stabilizability of the output $y(t) = \langle c(\xi), z \rangle_{L^2(0,1)}$ is achieved.*

# References

[1] A.V. Balakrishnan, *Applied Functional Analysis,* Springer-Verlag, Second Edition, Berlin, 1981.

[2] A. El Jai, A.J. Pritchard, Sensors and actuators in distributed systems, Int. J. Control., Vol.46, No.4, (1987), 1139-1153.

[3] H.J. Zwart, *Geometric Theory for Infinite Dimensional Systems,* Lecture Notes in Control and Information Sciences, Springer- Verlag, Berlin, Vol.105, 1988.

[4] H.J. Zwart, Characterization of all Controlled Invariant Subspaces For Spectral Systems, SIAM. J. Control. Optim., Vol.26, No.2, (1988), 369-387.

[5] R. Triggiani, On the Stabilizability Problem in Banach Space, J. Math. Anal. Appl., **52** (1975), 383-403.

[6] R. Rabah, D. Ionescu, Stabilization Problem in Hilbert Spaces, Int. J. Control., Vol.46, No.6, (1987), 2035-2042.

[7] R.F. Curtain, A.J. Pritchard, *Infinite Dimensional Linear Systems Theory,* Lecture Notes in Control and Information Sciences, Springer- Verlag, Berlin, 1978.