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A Note on the Euclidean Norms of Matrices with Arithmetic-Geometric-Harmonic Means

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Abstract

In this paper, we define the matrices with arithmetic-geometric-harmonic means of forms $A=\left(\frac{i+j}{2}\right)_{nn}$, $G=\left(\sqrt{ij}\right)_{nn}$, $A=\left(\frac{2ij}{i+j}\right)_{nn}$. After we study Euclidean norms of these matrices and their Hadamard inverses.

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1 Introduction

For positive real numbers *a* and *b*, arithmetic-geometric-harmonic mean inequality

$$\frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2}.$$

Generally,

$$A.M.(x_1, x_2, ..., x_n) = \frac{x_1 + x_2 + ... + x_n}{n} \quad (x_1, x_2, ..., x_n \in \mathbb{R}),$$

$$G.M.(x_1, x_2, ..., x_n) = \sqrt[n]{x_1 x_2 ... x_n} \quad (x_1, x_2, ..., x_n \in \mathbb{R}^+),$$

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$$H.M.(x_1, x_2, ..., x_n) = n\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \quad (x_1, x_2, ..., x_n \in \mathbb{R}^+),$$

where *A.M.*, *G.M.* and *H.M.* denote arithmetic, geometric, harmonic mean of numbers $x_1, x_2, ..., x_n$ respectively.

In [7], Heinz and Heron means that interpolate between the geometric and arithmetic mean are considered. Comparison inequalities between them are established. Operator versions of there inequalities are obtained. In [3], for positive semi-definite *n*x*n* matrices the inequality $4\|AB\| \le \|(A+B)^2\|$ is shown to hold for every unitarily invariant norm. Sagae and Tanabe give new upper and lower bounds for the arithmetic, geometric and harmonic means of a set of positive definite matrices [2]. As well as, some inequalities related to arithmetic-geometric were proved in [1,5, 6].

Mathias show that if $x_i > 0$ and $q \ge p \ge 0$ the *n*x*n* matrices

$$\left(\frac{\sqrt{x_i x_j}}{x_i + x_j}\right), \quad \left(\frac{x_i^{-1} + x_j^{-1}}{\sqrt{x_i x_j}}\right) \text{ and } \left(\frac{x_i^p + x_j^p}{x_i^q + x_j^q}\right)$$

are positive definite and relate these facts to some matrix valued arithmetic-geometric- harmonic mean inequalities-some of which involve Hadamard products and others unitarily invariant norms [4].

In this study, firstly we define the matrices which entries are consist of arithmetic, geometric and harmonic means. After, we study Euclidean norms of these matrices and give some numerical examples. Now, we give some preliminaries.

Gamma(*n*) function is defined as:

$$gamma(n) = \left[\sum_{k=1}^{m} \frac{\ln k^n}{k} - \ln\frac{m^{n+1}}{n+1}\right].$$

If *n*=0 then gamma(0)=gamma is known as Euler's constant, which is approximately 0,577 .

Psi(*x*) is digamma function, which is given by the logaritmic derivative of the GAMMA function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, that is Psi(*x*)= $\Psi(x) = \frac{d}{dx}\left[\Gamma(x)\right]$. Psi(*n*,*x*) is n*th* polygamma function, which is the n*th* derivative of digamma function: Psi(*n*,*x*)= $\Psi(n,x) = \frac{d}{dx^n}\left[Psi(x)\right]$.

Let $A = (a_{ij})_{mn}$ be any matrix. The Hadamard inverse of *A* is $A^{\circ^{-1}} = (\frac{1}{a_{ij}})_{mn}$.

Let $A = (a_{ij})_{mn}$ be any complex matrix. The Euclidean norm of *A* is

$$\|A\|_E = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} .$$

2 The Main Results

Theorem 2.1. Let be matrix $A = \left(\frac{i+j}{2} \right)_{nn}$ .Then $\|A\|_E = n\sqrt{\frac{7n^2+12n+5}{24}}$.

Proof. From the define of Euclidean norm,

$$\|A\|_E^2 = \sum_{s=1}^{n} s\left(\frac{s+1}{2} \right)^2 + \sum_{s=1}^{n-1} (n-s)\left(\frac{n+s+1}{2} \right)^2 .$$

Where,

$$\sum_{s=1}^{n} s\left(\frac{s+1}{2} \right)^2 = \frac{(n+1)^4}{16} + \frac{(n+1)^3}{24} - \frac{(n+1)^2}{16} - \frac{(n+1)}{24},$$

$$\sum_{s=1}^{n-1} (n-s)\left(\frac{n+s+1}{2} \right)^2 = \frac{11n^4}{48} + \frac{5n^3}{24} - \frac{11n^2}{48} - \frac{5}{24} .$$

Thus $\|A\|_E^2 = \frac{n^2(7n^2+12n+5)}{24}$.

Theorem 2.2. Let be matrix $G = \left(\sqrt{ij} \right)_{nn}$ .Then $\|G\|_E = \frac{n(n+1)}{2}$.

Proof. Matrix G is of form:

$$G = \begin{pmatrix} \sqrt{1} & \sqrt{2} & \cdots & \sqrt{n} \\ \sqrt{2} & \sqrt{4} & \cdots & \sqrt{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \sqrt{n} & \sqrt{2n} & \cdots & \sqrt{n^2} \end{pmatrix}_{nn} .$$

Hence, Euclidean norm of G,

$$\begin{aligned} \|G\|_E^2 &= \sum_{s=1}^{n} \left((\sqrt{s})^2 + (\sqrt{2s})^2 + \ldots + (\sqrt{ns})^2 \right) \\ &= \sum_{s=1}^{n} (s + 2s + .. + ns) = \sum_{s=1}^{n} s(1 + 2 + .. + n) \\ &= \left(\frac{n(n+1)}{2} \right)^2 . \end{aligned}$$

Thus the proof is completed.

Conjecture2.1. $\|G\|_E = \|G\|_2$ where $\|G\|_2$ is denote spectral norm of matrix *G*.

Theorem 2.3. Let be matrix $H = \left(\frac{2ij}{i+j}\right)_{nn}$ .Then

$$\|H\|_E = 2\sqrt{\begin{array}{l}\frac{1}{15}(6n^5+15n^4+10n^3-n)[\Psi(1,n+2)-\Psi(1,2n+1)] \\ +(n^4+2n^3+n^2)[\Psi(n+2)-\Psi(2n+1)] \\ -\frac{1}{15}\Psi(n+2)+\frac{1}{30}\Psi(2n+1)-\frac{\gamma}{30}+\frac{33n^4+18n^3-29n^2-8n+4}{60}\end{array}}.$$

Proof. Euclidean norm of matrix *H*

$$\|H\|_E^2 = 4\left(\sum_{p=2}^{n+1}\sum_{s=1}^{p-1}\left(\frac{s}{p}\right)^2(p-s)^2+\sum_{p=n+2}^{2n}\sum_{s=1}^{n}\left(\frac{s}{p}\right)^2(p-s)^2-\sum_{p=2}^{n}\sum_{s=1}^{p-1}s^2\left(\frac{n+p-s}{n+p}\right)^2\right). \qquad (2.1)$$

The values of sums in (2.1):

$$\sum_{p=2}^{n+1}\sum_{s=1}^{p-1}\left(\frac{s}{p}\right)^2(p-s)^2 = \frac{(n+2)^4}{120}-\frac{(n+2)^3}{60}+\frac{(n+2)^2}{120}-\frac{\Psi(n+2)}{30}-\frac{\gamma}{30},$$

$$\sum_{p=n+2}^{2n}\sum_{s=1}^{n}\left(\frac{s}{p}\right)^2(p-s)^2 = -\frac{(6n^5+15n^4+10n^3-n)[\Psi(1,2n+1)-\Psi(1,n+2)]}{30}$$

$$-\frac{(n^4+2n^3+n^2)[\Psi(2n+1)-\Psi(n+2)]}{2}+\frac{4n^4+8n^3+5n^2+n}{6}$$

And

$$\sum_{p=2}^{n}\sum_{s=1}^{p-1}s^2\left(\frac{n+p-s}{n+p}\right)^2 = \frac{(6n^5+15n^4+10n^3-n)[\Psi(1,2n+1)-\Psi(1,n+2)]}{30}$$

$$+\frac{(15n^4+30n^3+15n^2-1)[\Psi(2n+1)-\Psi(n+2)]}{30}+\frac{-31n^4+62n^3+17n^2+8n-4}{120}.$$

If we write these values in (2.1), the proof is completed.

Example 2.1. The values of norms of matrices *A*,*G* and *H* is given in Table 2.1 :

Table 2.1

n	$\|A\|_E$	$\|G\|_E$	$\|H\|_E$
1	1	1	1
2	3.08	3	2.92
3	6.24	6	5.79
4	10.45	10	9.61
5	15.81	15	14.39
10	58.63	55	52.54
20	225.28	210	200.37
30	499.94	465	443.53

The figure of Table 2.1 :

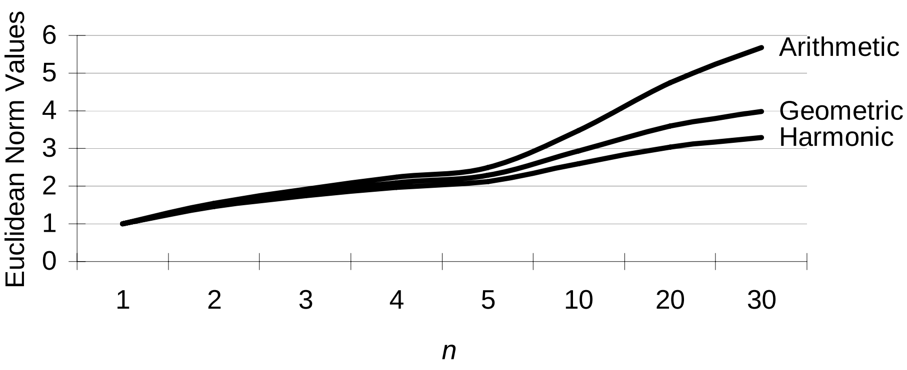

Figure 2.1

From Table 2.1 and Figure 2.1 the inequality $\|A\|_E \geq \|G\|_E \geq \|H\|_E$ is validity.

Theorem 2.4. The Euclidean norm of Hadamard inverse of matrix *A* by Theorem 2.1. is

$$\left\|A^{\circ^{-1}}\right\|_E = \sqrt{8\Psi(n+2)+8(n+1)\Psi(1,n+2)-4\Psi(2n+1)-(8n+4)\Psi(1,2n+1)+4\gamma-\frac{2\pi^2}{3}}.$$

Proof. From the define Euclidean norm,

$$\left\|A^{\circ^{-1}}\right\|_E^2 = \sum_{s=1}^{n} s\left(\frac{2}{s+1}\right)^2 + \sum_{s=1}^{n-1}(n-s)\left(\frac{2}{n+s+1}\right)^2.$$

where,

$$\sum_{s=1}^{n} s\left(\frac{2}{s+1}\right)^2 = 4\Psi(n+2)+4\Psi(1,n+2)+4\gamma-\frac{2\pi^2}{3},$$

$$\sum_{s=1}^{n-1}(n-s)\left(\frac{2}{n+s+1}\right)^2 = -(8n+4)\Psi(1,2n+1)-4\Psi(2n+1)+(8n+4)\Psi(1,n+2)+4\Psi(n+2)$$

.

Thus the proof is completed.

Theorem 2.5. The Euclidean norm of Hadamard inverse of matrix G is

$$\left\|G^{\circ^{-1}}\right\|_E = H_n,$$

where H_n is denote n *th* harmonic number(i.e. $H_n = 1+\frac{1}{2}+...+\frac{1}{n}$).

Proof. Euclidean norm of matrix $G^{\circ^{-1}}$,

$$\left\|G^{\circ^{-1}}\right\|_E^2 = \sum_{s=1}^{n}\left(1+\frac{1}{2}+...+\frac{1}{n}\right)\frac{1}{s}$$
$$=\left(\sum_{s=1}^{n}\frac{1}{s}\right)^2$$
$$=(H_n)^2.$$

Conjecture2.2. $\left\|G^{\circ^{-1}}\right\|_E = \left\|G^{\circ^{-1}}\right\|_2$ where $\left\|G^{\circ^{-1}}\right\|_2$ is denote spectral norm of matrix $G^{\circ^{-1}}$.

Theorem 2.6. The Euclidean norm of Hadamard inverse of matrix H is

$$\left\|H^{\circ^{-1}}\right\|_E = \sqrt{\frac{-n\Psi(1,n+1)}{2}+\frac{n\pi^2}{12}+\frac{(H_n)^2}{2}}$$

where H_n is denote n *th* harmonic number.

Proof.

$$\left\|H^{\circ^{-1}}\right\|_E^2 = \sum_{s=1}^{n}\left(\frac{s+1}{2s}\right)^2+\sum_{s=1}^{n}\left(\frac{s+2}{4s}\right)^2+...+\sum_{s=1}^{n}\left(\frac{s+n}{2ns}\right)^2$$

$$=\frac{1}{4}\sum_{s=1}^{n}\left(\frac{s+1}{s}\right)^2+\frac{1}{16}\sum_{s=1}^{n}\left(\frac{s+2}{s}\right)^2+...+\frac{1}{4n^2}\sum_{s=1}^{n}\left(\frac{s+n}{s}\right)^2$$

the value of each sum in this equality,

$$\sum_{s=1}^{n}\left(\frac{s+1}{s}\right)^2 = n-\Psi(1,n+1)+2\Psi(n+1)+\frac{\pi^2}{6}+2\gamma,$$

$$\sum_{s=1}^{n}\left(\frac{s+2}{s}\right)^2 = n - 4\Psi(1,n+1) + 4\Psi(n+1) + \frac{2\pi^2}{3} + 4\gamma\ ,$$
$$\vdots$$
$$\sum_{s=1}^{n}\left(\frac{s+n}{s}\right)^2 = n - n^2\Psi(1,n+1) + 2n\Psi(n+1) + \frac{n^2\pi^2}{6} + 2n\gamma\ .$$

Then

$$\left\|H^{\circ -1}\right\|_E^2 = \sum_{s=1}^{n}\frac{1}{(2s)^2}[n - s^2\Psi(1,n+1) + 2s\Psi(n+1) + \frac{s^2\pi^2}{6} + 2s\gamma]$$

$$= \frac{-n\Psi(1,n+1)}{2} + \frac{n\pi^2}{12} + \frac{[\Psi(n+1)+\gamma]^2}{2}$$

$$= \frac{-n\Psi(1,n+1)}{2} + \frac{n\pi^2}{12} + \frac{(H_n)^2}{2}.$$

Example 2.2. The values of norms of matrices $A^{\circ -1}, G^{\circ -1}$ and $H^{\circ -1}$ H is given in Table 2.2. :

Table 2.2

n	$\left\|A^{\circ -1}\right\|_E$	$\left\|G^{\circ -1}\right\|_E$	$\left\|H^{\circ -1}\right\|_E$
1	1	1	1
2	1.46	1.5	1.54
3	1.75	1.83	1.92
4	1.96	2.08	2.24
5	2.12	2.28	2.50
10	2.60	2.93	3.47
20	3.04	3.60	4.74
30	3.28	3.99	5.67

The figure of Table 2.2 :

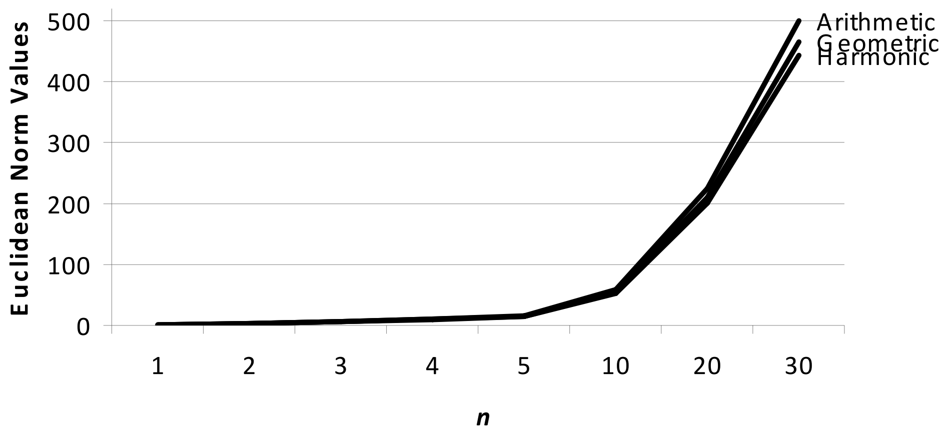

Figure 2.2

From Table 2.2. and Figure 2.2. the inequality $\left\|H^{\circ^{-1}}\right\|_E \geq \left\|G^{\circ^{-1}}\right\|_E \geq \left\|A^{\circ^{-1}}\right\|_E$ is validity.

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